

NON-ISOTHERMAL CYCLIC FATIGUE IN AN OSCILLATING ELASTOPLASTIC BEAM

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(Communicated by Stefano Bianchini)

ABSTRACT. We propose a temperature dependent model for fatigue accumulation in an oscillating elastoplastic beam. The full system consists of the momentum and energy balance equations, and an evolution equation for the fatigue rate. The main modeling hypothesis is that the fatigue accumulation rate is proportional to the dissipation rate. In nontrivial cases, the process develops a thermal singularity in finite time. The main result consists in proving the existence and uniqueness of a strong solution in a time interval depending on the size of the data.

Introduction. Elastoplastic materials subject to cyclic loading exhibit increasing *fatigue*, which is manifested by *material softening*, and *material failure in finite time with strong heat release*. The analysis of the so-called *rainflow method* for cyclic fatigue accumulation in uniaxial processes [2] has discovered a qualitative and quantitative relationship between accumulated fatigue and dissipated energy. Indeed, the rainflow algorithm counts closed hysteresis loops in the loading history, and with each closed loop associates a number depending on its amplitude – the contribution of the loop to total damage – taken from an experimental diagram (the *Wöhler line*). This corresponds to the mechanism of energy dissipation: The number associated with a closed loop is its area in this case.

In multiaxial loading processes, the concept of closed loop is meaningless, and no counterpart of the rainflow algorithm is known to the authors. On the other hand, the notion of energy dissipation is independent of the experimental setting. In [5], we have proposed a model for fatigue accumulation based on the hypothesis that

2000 *Mathematics Subject Classification.* Primary: 74R20; Secondary: 74K10, 74H35, 74N30.

Key words and phrases. Elastoplastic beam, Prandtl-Ishlinskii operator, material fatigue, thermo-elasto-plasticity, blow-up.

Supported by GAČR Grant P201/10/2315 and by the project MSM4781305904 of the Czech Ministry of Education. The work of M.E. was partially supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase).

there exists a proportionality between fatigue and dissipated energy in the multiaxial case as well, and the model was demonstrated on the example of a transversally oscillating thermoelastoplastic plate. This idea is perhaps not completely unrealistic. Plastic deformations are manifested on the microscale by moving dislocations which, at the same time, damage the material cohesion, reduce the stiffness, and dissipate mechanical energy into heat. Note that in engineering practice, e.g. in aircraft industry, temperature tests for fatigue estimation are commonly used, and similar considerations appear also in theoretical engineering literature, going back e.g. to [11].

In the whole generality, a rigorous existence and uniqueness theory is still missing. Here, we make a first step in this direction and investigate the well-posedness of the system describing non-isothermal fatigue accumulation in a transversally oscillating elastoplastic beam. The equation for transversal oscillation of an Euler-Bernoulli beam with the single yield von Mises plasticity law was shown in [15] to give rise to a 1D beam equation with a multiyield Prandtl-Ishlinskii hysteresis operator as a result of dimensional reduction. Here, we allow the hardening modulus to depend on the accumulated fatigue, couple the momentum balance equation with the full energy balance involving temperature, and prove that the resulting PDE system admits a unique solution in the whole existence time interval until a thermal shock occurs. As the primary goal of the model is to describe material failure in finite time, we cannot expect to have solutions that are global in time. For some details about modeling issues, we refer the interested reader to [5]. More generally, the analysis of non-isothermal problems in mathematical modelling of advanced materials is gaining more and more importance in the recent years. We refer for instance, without aiming at completeness, to [1, 7, 8, 18, 19] in the setting of elastic materials, plastic materials, shape memory alloys and liquid crystals.

Some extensions of our analysis to a higher-dimensional setting and further generalizations are in progress. A thermodynamic model for fatigue accumulation in an oscillating elastoplastic Kirchhoff plate based on the same hypothesis, namely that the fatigue accumulation rate is proportional to the dissipation rate, is considered in [6]. The existence of a unique solution in the whole time interval before a singularity (material failure) occurs is proved there under the simplifying hypothesis that the temperature history is a priori given.

In Section 1, we derive the governing system of equations from the thermodynamic principles. The main existence and uniqueness results are stated in Section 2. Section 3 is devoted to the analysis of a spatially semi-discrete scheme and estimates for the approximate solution, and the proof of the main theorem is carried out in Section 4.

1. The model.

1.1. Transversal oscillations of a beam. In [15], it was shown that the Euler-Bernoulli dimensional reduction, applied to the 3D momentum balance for transversal oscillations of a thin elastoplastic body with a single yield von Mises plasticity law, leads to the system

$$u_t = \frac{Bh^2}{3}w_{xx} + P[w_{xx}], \quad (1.1)$$

$$\varrho w_t - \frac{\varrho h^2}{3}w_{xxt} = -u_{xx} + \frac{1}{2h}f \quad (1.2)$$

for $x \in (0, \ell)$ and $t \in [0, T]$, where the subscripts $_x$ and $_t$ denote partial derivatives with respect to x and t , the function w is the transversal displacement, $2h$ is the thickness of the beam, $B > 0$ is the hardening modulus, P is the multiyield elastoplastic Prandtl-Ishlinskii operator (1.12) with a specific form of the density function γ , ϱ is the mass density and f is the time integral of the external load. In fact, (1.1) results from the strain-stress law, as $\varepsilon = w_{xx}$ is, up to the factor $-z$, the only nontrivial component ε_{11} of the strain tensor at distance z from the midsurface, and $\sigma = u_t$ is a weighted average over the thickness of the σ_{11} stress component. Note that the multiyield character of the Prandtl-Ishlinskii constitutive law in the beam or plate bending problem, cf. also [9], reflects the fact that eccentric layers are subject to larger deformations than the central ones, so that plastic yielding propagates gradually from the outer surface towards the midsurface as on Figure 1. The hysteresis memory state of the Prandtl-Ishlinskii operator P keeps the information about the time dependent stress distribution across the profile after elimination of the transversal variable z , see Subsection 1.2 below.

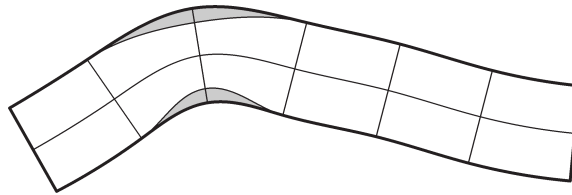


FIGURE 1. Bending of an elastoplastic beam with gray plasticized zones.

In the present paper, we want to include fatigue and temperature effects by extending the constitutive law into

$$\sigma = \frac{B(m)h^2}{3}\varepsilon + P[\varepsilon] + \nu\varepsilon_t - \beta(\theta - \theta^{ref}), \quad (1.3)$$

where $m > 0$ is a scalar time and space dependent fatigue parameter, ν is the viscosity, β represents thermal expansion, $\theta > 0$ is the absolute temperature, and θ^{ref} is a fixed referential temperature. The function $B(m)$ accounts for material softening by the action of fatigue.

In fact, physically, $\varepsilon = w_{xx}$ is the curvature and σ in (1.3) is the bending moment. The hypothesis that one component of the bending moment is proportional to the temperature with factor β can be justified in case that the beam has a layered structure with mass densities, elasticity moduli, and thermal expansion coefficients depending on the transversal variable z (e.g. bimetal, say, with two different values of the material constants below ($z < 0$) and above ($z > 0$) the midsurface). A discussion about the form of the Prandtl-Ishlinskii operator in the case of layered materials can be found in [15, Remark 2.2].

With the constitutive law (1.3), we associate the *free energy operator*

$$\mathcal{F}(\varepsilon, \theta) = c\theta(1 - \log(\theta/\theta^{ref})) + \frac{B(m)h^2}{6}\varepsilon^2 + V[\varepsilon] - \beta(\theta - \theta^{ref})\varepsilon, \quad (1.4)$$

where V is the Prandtl-Ishlinskii potential (1.15), and c (the specific heat capacity) is a given constant. The *entropy operator* \mathcal{S} and *internal energy operator* \mathcal{U} then

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$$\mathcal{S}(\varepsilon, \theta) = -\frac{\partial \mathcal{F}}{\partial \theta} = c \log(\theta/\theta^{ref}) + \beta \varepsilon, \quad (1.5)$$

$$\mathcal{U}(\varepsilon, \theta) = \mathcal{F}(\varepsilon, \theta) + \theta \mathcal{S}(\varepsilon, \theta) = c\theta + \frac{B(m)h^2}{6} \varepsilon^2 + V[\varepsilon] + \beta \theta^{ref} \varepsilon. \quad (1.6)$$

We consider the first and the second principles of thermodynamics in the form

$$\mathcal{U}(\varepsilon, \theta)_t + q_x = \varepsilon_t \sigma, \quad (1.7)$$

$$\mathcal{S}(\varepsilon, \theta)_t + \left(\frac{q}{\theta}\right)_x \geq 0, \quad (1.8)$$

where $q = -\kappa \theta_x$ is the heat flux with a constant heat conductivity $\kappa > 0$. Note that (1.7) is the energy conservation law, (1.8) is the Clausius-Duhem inequality. We rewrite (1.7) using (1.3), (1.6), and (1.17) as

$$c\theta_t - \kappa \theta_{xx} = -\frac{B'(m)h^2}{6} \varepsilon^2 m_t + |D[\varepsilon]_t| + \nu \varepsilon_t^2 - \beta \theta \varepsilon_t, \quad (1.9)$$

where $B'(m)$ is the derivative of $B(m)$ with respect to m , and $D[\varepsilon]$ is the dissipation operator (1.16). Inequality (1.8) can be formally written in the form

$$\frac{1}{\theta} \left(-\frac{B'(m)h^2}{6} \varepsilon^2 m_t + |D[\varepsilon]_t| + \nu \varepsilon_t^2 + \kappa \frac{\theta_x^2}{\theta} \right) \geq 0.$$

This is certainly satisfied provided the fatigue rate m_t is positive, i.e. fatigue is increasing in time, and $B'(m) \leq 0$, i.e. the elasticity modulus decreases with increasing fatigue in agreement with experimental observations.

The system has to be complemented with an evolution equation for m . Motivated by the rainflow method of cyclic fatigue evaluation, we assume, as mentioned in the Introduction, that the fatigue rate is proportional to the elastoplastic dissipation rate

$$-\frac{B'(m)h^2}{6} \varepsilon^2 m_t + |D[\varepsilon]_t|. \quad (1.10)$$

We assume that the fatigue rate at a point x is proportional to the bulk elastoplastic dissipation rate in a neighborhood of the point x . This means, for a nonnegative function λ with (small) compact support we set

$$m_t(x, t) = \int_0^\ell \lambda(x-y) \left(-\frac{B'(m)h^2}{6} \varepsilon^2 m_t + |D[\varepsilon]_t| \right)(y, t) dy. \quad (1.11)$$

The modeling requirement that the system should develop a thermal singularity at the moment of failure, is clearly satisfied in (1.11). In nontrivial cases, if the external load is large, $\varepsilon^2(x, t)$ also becomes large on a large set, and m_t blows up to $+\infty$. Consequently, also the heat supply in the energy balance (1.9) becomes infinite, producing thus a thermal shock in the system.

1.2. The Prandtl-Ishlinskii operator. Assuming that a nonnegative function $\gamma \in L^1(0, \infty)$ is given, we define the Prandtl-Ishlinskii operator $P : W^{1,1}(0, T) \rightarrow W^{1,1}(0, T)$ by the integral

$$P[\varepsilon](t) = \int_0^\infty \gamma(r) \mathfrak{s}_r[\varepsilon](t) dr, \quad (1.12)$$

where \mathfrak{s}_r is the *stop operator* with threshold r . The modeling idea goes back to [10, 17]. In the literature, e.g. [16], it is also called *generalized Saint-Venant model*. Let us first recall the definition of the stop operator.

Definition 1.1. Let $\varepsilon \in W^{1,1}(0, T)$ and $r > 0$ be given. The variational inequality

$$\left. \begin{aligned} \varepsilon(t) &= \sigma(t) + \xi(t) && \forall t \in [0, T], \\ |\sigma(t)| &\leq r && \forall t \in [0, T], \\ \dot{\xi}(t)(\sigma(t) - z) &\geq 0 \text{ a.e.} && \forall |z| \leq r, \\ \sigma(0) &= Q_r(\varepsilon(0)), \end{aligned} \right\} \quad (1.13)$$

where Q_r is the projection of \mathbb{R} onto the interval $[-r, r]$, defines the stop and play operators \mathfrak{s}_r and \mathfrak{p}_r by the formula, see Figure 2,

$$\sigma(t) = \mathfrak{s}_r[\varepsilon](t), \quad \xi(t) = \mathfrak{p}_r[\varepsilon](t). \quad (1.14)$$

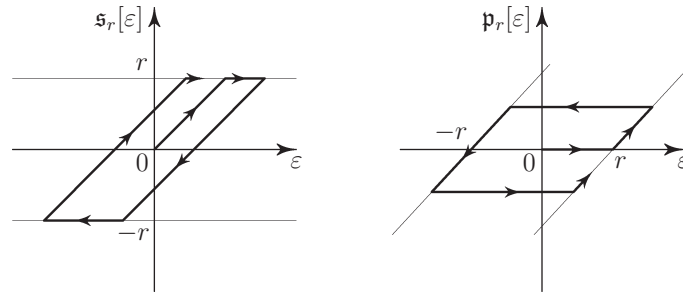


FIGURE 2. A typical hysteresis diagram of the stop and play for a cyclic input.

The stop and play operators were introduced in [12]. The parameter r is the *memory variable*, and for each given time t_0 , the functions $r \mapsto \mathfrak{p}_r[\varepsilon](t_0)$, $r \mapsto \mathfrak{s}_r[\varepsilon](t_0)$ represent the memory state of the system. They keep the information about important local minima and maxima of the input ε in the time interval $[0, t_0]$, see [3, Sections 2.3, 2.4]. Let us illustrate the Prandtl-Ishlinskii memory structure on the example

$$\mathfrak{s}_r[\varepsilon](t_0) = \begin{cases} -r, & r \in [0, a), \\ r - 2a, & r \in [a, b), \\ -r - 2a + 2b, & r \in [b, c), \\ -2a + 2b - c = \varepsilon(t_0), & r \geq c, \end{cases}$$

corresponding to an input history of ε with an absolute minimum $\varepsilon(t_2) = -c$, absolute maximum $\varepsilon(t_1) = 2b - c$, and $\varepsilon(t_1) \geq \varepsilon(t) \geq \varepsilon(t_0) = -2a + 2b - c$ for $t \in [t_1, t_0]$, $0 \leq t_2 < t_1 < t_0$, $0 < a < b < c$. The memory curves are represented in the left part of Figure 3.

The distribution of the stress σ at time t_0 across the beam profile along the transversal variable $z \in [0, h]$ from the midsurface $z = 0$ to the upper boundary $z = h$ in case that the elasticity modulus E is constant in $(0, h)$, is given by the

formula [15, Eq. (44)], that is,

$$\sigma(z, t_0) = -Ez \mathfrak{s}_{\frac{R}{Ez}}[\varepsilon](t_0) = \begin{cases} R, & z \geq \frac{R}{Ea}, \\ -R + 2aEz, & z \in \left[\frac{R}{Eb}, \frac{R}{Ea}\right), \\ R + (2a - 2b)Ez, & z \in \left[\frac{R}{Ec}, \frac{R}{Eb}\right), \\ -Ez\varepsilon(t_0), & z \in \left[0, \frac{R}{Ec}\right), \end{cases}$$

where R is the von Mises yield limit and E is the Young elasticity modulus, see the right part of Figure 3. The interval $[R/Ea, h]$ corresponds to the plasticized zone near the outer boundary $z = h$.

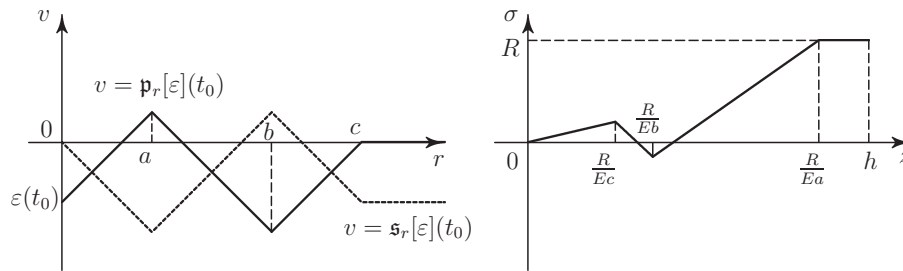


FIGURE 3. Memory distribution at time t_0 .

Let us list here some basic properties of the play and stop operators. The proofs are elementary and can be found e.g. in [3, 13].

Proposition 1.2. *Let $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0, T)$ and $r > 0$ be given, $\sigma_i = \mathfrak{s}_r[\varepsilon_i]$, $\xi_i = \varepsilon_i - \sigma_i = \mathfrak{p}_r[\varepsilon_i]$, $i = 1, 2$. Then*

- (i) $(\sigma_1(t) - \sigma_2(t))(\dot{\varepsilon}_1(t) - \dot{\varepsilon}_2(t)) \geq \frac{1}{2} \frac{d}{dt} (\sigma_1(t) - \sigma_2(t))^2 \quad a. e.;$
- (ii) $|\dot{\xi}_1(t) - \dot{\xi}_2(t)| + \frac{d}{dt} |\sigma_1(t) - \sigma_2(t)| \leq |\dot{\varepsilon}_1(t) - \dot{\varepsilon}_2(t)| \quad a. e.;$
- (iii) $|\sigma_1(t) - \sigma_2(t)| \leq 2 \max_{0 \leq \tau \leq t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)| \quad \forall t \in [0, T];$
- (iv) $\dot{\xi}_i(t)\dot{\varepsilon}_i(t) = \dot{\xi}_i(t)^2 \quad a. e.$

It is easy to see that the variational inequality (1.13) can be equivalently written in the form

$$\dot{\varepsilon}(t)\sigma(t) = \frac{d}{dt} \left(\frac{1}{2} \sigma^2(t) \right) + r|\dot{\xi}(t)|,$$

representing the energy balance. Indeed, $\dot{\varepsilon}(t)\sigma(t)$ is the power supplied to the system, which is partly used for the increase of the potential $\frac{1}{2}\sigma^2(t)$, and the rest $r|\dot{\xi}(t)|$ is dissipated. This enables us to establish the energy balance for the Prandtl-Ishlinskii operator (1.12). Indeed, if we define the Prandtl-Ishlinskii potential

$$V[\varepsilon](t) = \frac{1}{2} \int_0^\infty \gamma(r) \mathfrak{s}_r^2[\varepsilon](t) \, dr, \tag{1.15}$$

and the dissipation operator

$$D[\varepsilon](t) = \int_0^\infty r\gamma(r)\mathfrak{p}_r[\varepsilon](t) \, dr, \tag{1.16}$$

we can write the Prandtl-Ishlinskii energy balance in the form

$$\dot{\varepsilon}(t)P[\varepsilon](t) = \frac{d}{dt}V[\varepsilon](t) + \left| \frac{d}{dt}D[\varepsilon](t) \right| \quad \text{a.e.} \tag{1.17}$$

As a consequence of Proposition 1.2 (iv), we have

$$\left| \frac{d}{dt}D[\varepsilon](t) \right| \leq |\dot{\varepsilon}(t)| \int_0^\infty r\gamma(r) \, dr.$$

We see that Hypothesis 2.1 (i) below enables us to estimate the dissipation from above in terms of the input velocity. Furthermore, for $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0, T)$, we estimate the difference of the dissipation rates using (1.16) and Proposition 1.2 (ii) as follows:

$$\begin{aligned} & \int_0^t \left| \left| \frac{d}{dt}D[\varepsilon_1](\tau) \right| - \left| \frac{d}{dt}D[\varepsilon_2](\tau) \right| \right| \, d\tau \\ & \leq \int_0^\infty r\gamma(r) \, dr \int_0^t |\dot{\varepsilon}_1(\tau) - \dot{\varepsilon}_2(\tau)| \, d\tau + \int_0^\infty r\gamma(r) |\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2]|(0) \, dr. \end{aligned} \tag{1.18}$$

2. Statement of the problem. The mathematical analysis of the problem derived in Subsection 1.1 is independent of the exact values of the material constants. For any $T > 0$, we denote $\Omega_T := (0, 1) \times (0, T)$. In an a priori unknown domain Ω_{T^*} , we consider the system

$$u_t = B(m)w_{xx} + P[w_{xx}] + w_{xxt} - (\theta - \theta^{ref}), \tag{2.1}$$

$$w_t - w_{xxt} = -u_{xx} + f \tag{2.2}$$

$$\theta_t = \theta_{xx} - \frac{1}{2}B'(m)w_{xx}^2 m_t + |D[w_{xx}]_t| + w_{xxt}^2 - \theta w_{xxt}, \tag{2.3}$$

$$m_t(x, t) = \int_0^1 \lambda(x - y) \left(-\frac{1}{2}B'(m)w_{xx}^2 m_t + |D[w_{xx}]_t| \right) (y, t) \, dy \tag{2.4}$$

for unknown functions u, w, θ, m , with initial and boundary conditions

$$\left. \begin{aligned} w(x, 0) &= u(x, 0) = m(x, 0) = 0, \\ \theta(x, 0) &= \theta^0(x), \end{aligned} \right\} \tag{2.5}$$

$$\left. \begin{aligned} w(0, t) &= u(0, t) = w(1, t) = u(1, t) = 0, \\ \theta_x(0, t) &= \theta_x(1, t) = 0, \end{aligned} \right\} \tag{2.6}$$

under the following hypothesis:

Hypothesis 2.1.

- (i) P is a Prandtl-Ishlinskii operator (1.12) and D is its associated dissipation operator (1.16). We assume that its distribution function $\gamma \in L^1(0, \infty)$ is such that $\gamma \geq 0$ a.e., and $\int_0^\infty r\gamma(r) \, dr < \infty$.
- (ii) $B : [0, \infty) \rightarrow (0, \infty)$ is a C^2 function, $B'(0) = 0$, $-1 \leq B''(m) \leq 0$ for all $m > 0$.
- (iii) $\lambda : \mathbb{R} \rightarrow [0, \infty)$ is a C^1 function with compact support, $L := \max\{\lambda(x) + |\lambda'(x)|, x \in \mathbb{R}\}$.
- (iv) $f \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $f_{tt}, f_{xx} \in L^2(\Omega_T)$.

- (v) $\theta^0 \in L^\infty(0, 1)$ is such that $\theta^0 \geq \theta_* > 0$, $\theta_{xx}^0 \in L^2(0, 1)$.
- (vi) θ^{ref} is a given positive constant.

The main result reads as follows.

Theorem 2.2. *Let Hypothesis 2.1 hold. Then there exists T^* such that system (2.1)–(2.6) has a unique solution in Ω_{T^*} with the regularity*

- $w_{xxxt}, w_{xxtt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(\Omega_{T^*})$,
- $m_t \in L^\infty(\Omega_{T^*})$.

Remark 2.3. As mentioned in Section 1, a global solution cannot be expected to exist in general. A blowup of m_t to $+\infty$ in Eq. (2.4) certainly occurs in finite time as soon as w_{xx} becomes large. If for example the external excitation $f(x, t)$ is very strong, w_{xx} reaches the critical value faster and the blowup occurs earlier. In Subsection 3.2, we derive an explicit lower bound for the lifetime of the solution in terms of the $L^1(0, T; L^2(0, 1))$ -norm of f_t . In particular, if the external load f_t is sufficiently small, the solution will exist globally in $[0, T]$.

Remark 2.4. We restrict ourselves to zero initial conditions for u, w , and m . The motivation for this choice is twofold. First, in practice, it is impossible to determine the initial fatigue if we do not know the full loading history. Hence, the only realistic hypothesis for the problem of estimating the lifetime of a material subject to cyclic fatigue, is to assume that it is at rest, and completely undamaged at the beginning. The second reason is that general speculative initial conditions would require complicated compatibility conditions involving derivatives of hysteresis operators, which do not bring any added value, while the resulting additional technical difficulties in the proof would make the presentation less transparent.

3. Approximation. We choose a cutoff parameter $R > 1$ and an integer $n \in \mathbb{N}$, and consider the truncated space discrete approximations of (2.1)–(2.4) for $k = 1, \dots, n - 1$

$$\dot{u}_k = B(m_k)\varepsilon_k + P[\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta^{ref}, \tag{3.1}$$

$$\dot{w}_k - \dot{\varepsilon}_k = -n^2(u_{k+1} - 2u_k + u_{k-1}) + f_k, \tag{3.2}$$

$$\varepsilon_k = n^2(w_{k+1} - 2w_k + w_{k-1}), \tag{3.3}$$

$$\dot{\theta}_k = n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) - \frac{1}{2}B'(m_k)\varepsilon_k^2\dot{m}_k + |D[\varepsilon_k]_t| + \dot{\varepsilon}_k^2 - \theta_k\dot{\varepsilon}_k, \tag{3.4}$$

$$m_k(t) = m^{(n)}(k/n, t), \tag{3.5}$$

$$m_t^{(n)}(x, t) = \int_0^1 \lambda(x - y) \left(-\frac{1}{2}B'(m^{(n)})(\bar{\varepsilon}^{(n)})^2 m_t^{(n)} \right)(y, t) dy + Q_R \left(\int_0^1 \lambda(x - y) |D[\bar{\varepsilon}^{(n)}]_t|(y, t) dy \right) \quad \text{for } x \in [0, 1], \tag{3.6}$$

where we denote by $Q_R(z) = \min\{R, z\}$ the projection of $[0, \infty)$ onto $[0, R]$,

$$\bar{\varepsilon}^{(n)}(x, t) = \begin{cases} \varepsilon_k(t) & \text{for } x \in [(k - 1)/n, k/n), \quad k = 1, \dots, n - 2, \\ \varepsilon_{n-1}(t) & \text{for } x \in [(n - 2)/n, 1], \end{cases} \tag{3.7}$$

and

$$f_k(t) = n \int_{(k-1)/n}^{k/n} f(x, t) dx.$$

We prescribe initial conditions for $k = 1, \dots, n - 1$

$$\left. \begin{aligned} w_k(0) &= u_k(0) = 0, \\ \theta_k(0) &= \theta^0(k/n), \\ m^{(n)}(x, 0) &= 0, \end{aligned} \right\} \tag{3.8}$$

and “boundary conditions”

$$\left. \begin{aligned} w_0 &= w_n = u_0 = u_n = 0, \\ \theta_0 &= \theta_1, \theta_n = \theta_{n-1}. \end{aligned} \right\} \tag{3.9}$$

This is a system of ODEs for u_k, w_k, θ_k with locally Lipschitz continuous right hand sides, and one integrodifferential equation for $m^{(n)}$. We proceed as follows: We first check that the system admits a local solution on a time interval $[0, T_n)$, and then establish a lower bound for $0 < T^R \leq T_n$ independent of n . On the interval $[0, T^R)$, we derive estimates for the approximate solutions which enable us, on the one hand, to show that the truncation in (3.5) never becomes active if R is sufficiently large, and can be removed. On the other hand, we select a convergent subsequence indexed by n and pass to the limit as $n \rightarrow \infty$ to obtain a solution to (2.1)–(2.6).

3.1. Existence of local solutions to (3.1)–(3.6). For given functions $a, r \in L^1(\Omega_T)$, such that

$$a(x, t) \leq 0, \quad \left| \int_0^1 a(x, t) \, dx \right| \leq A, \quad 0 \leq r(x, t) \leq R \quad \text{a.e.}, \tag{3.10}$$

consider the problem

$$m_t(x, t) = \int_0^1 \lambda(x - y)(a(y, t)(B'(m)m_t)(y, t)) \, dy + r(x, t), \quad m(x, 0) = 0. \tag{3.11}$$

We define an auxiliary function μ as the maximal solution to the differential equation

$$\dot{\mu}(t)(1 - LA\mu(t)) = R, \quad \mu(0) = 0, \tag{3.12}$$

that is,

$$\mu(t) = \frac{1}{LA} - \left(\frac{1}{L^2A^2} - \frac{2R}{LA}t \right)^{1/2}, \quad \dot{\mu}(t) = \frac{R}{LA} \left(\frac{1}{L^2A^2} - \frac{2R}{LA}t \right)^{-1/2} \tag{3.13}$$

We see that $\dot{\mu}(t)$ blows up to $+\infty$ for $t \nearrow 1/(2LAR)$. We plan to show that the solution of (2.1)–(2.6) can be continued until the blowup of μ occurs. To this end, we choose an arbitrarily small $b \in (0, 1)$ that we keep fixed throughout the paper, and set

$$T_{A,b}^R := \frac{1 - b^2}{2LAR}. \tag{3.14}$$

Then

$$\dot{\mu}(t) \in \left[R, \frac{R}{b} \right], \quad \mu(t) \in \left[0, \frac{1 - b}{LA} \right] \quad \text{for } t \in [0, T_{A,b}^R]. \tag{3.15}$$

We define the set

$$M = \{m \in L^\infty(\Omega_{T_{A,b}^R}) : m_t \in L^\infty(\Omega_{T_{A,b}^R}), m(x, 0) = 0, 0 \leq m_t(x, t) \leq \dot{\mu}(t) \text{ a.e.}\}. \tag{3.16}$$

We prove the following statement.

Lemma 3.1. *Let Hypotheses 2.1 (iii) and (3.10) hold. Then (3.11) admits a unique solution $m \in M$. Moreover, if $(a_1, r_1), (a_2, r_2)$ are two sets of data satisfying (3.10), then the corresponding solutions m^1, m^2 satisfy the inequality*

$$\int_0^{T_{A,b}^R} |m_t^1 - m_t^2|_\infty(t) dt \leq \frac{(1-b)R}{b^2 A} \int_0^{T_{A,b}^R} |a_1 - a_2|_1(t) dt + \frac{1}{b} \int_0^{T_{A,b}^R} |r_1 - r_2|_\infty(t) dt. \tag{3.17}$$

Here and in the sequel, we denote by $|\cdot|_p$ for $p \in [1, \infty]$ the norm in $L^p(0, 1)$.

Proof. We choose $\bar{m}^1, \bar{m}^2 \in M$, and define $\hat{m}^1, \hat{m}^2 \in M$ by the formula

$$\hat{m}_t^i(x, t) = \int_0^1 \lambda(x-y) a_i(y, t) (B'(\bar{m}^i) \bar{m}_t^i)(y, t) dy + r_i(x, t), \quad \hat{m}^i(x, 0) = 0, \quad i = 1, 2. \tag{3.18}$$

We have $\hat{m}_t^i(x, t) \geq 0, |\hat{m}_t^i(x, t)| \leq LA\mu(t)\dot{\mu}(t) + R = \dot{\mu}(t)$ a.e., hence $\hat{m}^i \in M$. We further have

$$\begin{aligned} |\hat{m}_t^1 - \hat{m}_t^2|_\infty(t) &\leq LA\mu(t) |\bar{m}_t^1 - \bar{m}_t^2|_\infty(t) + LA\dot{\mu}(t) |\bar{m}^1 - \bar{m}^2|_\infty(t) \\ &\quad + L\mu(t)\dot{\mu}(t) |a_1 - a_2|_1(t) + |r_1 - r_2|_\infty(t) \\ &\leq LA \frac{d}{dt} \left(\mu(t) \int_0^t |\bar{m}_\tau^1 - \bar{m}_\tau^2|_\infty(\tau) d\tau \right) \\ &\quad + L\mu(t)\dot{\mu}(t) |a_1 - a_2|_1(t) + |r_1 - r_2|_\infty(t). \end{aligned} \tag{3.19}$$

For $a_1 = a_2$ and $r_1 = r_2$ we obtain

$$\int_0^{T_{A,b}^R} |\hat{m}_t^1 - \hat{m}_t^2|_\infty(t) dt \leq (1-b) \int_0^{T_{A,b}^R} |\bar{m}_t^1 - \bar{m}_t^2|_\infty(t) dt, \tag{3.21}$$

so that existence and uniqueness of m is established by the Banach fixed point argument. Inequality (3.17) follows immediately from (3.19) with $\hat{m}^i = \bar{m}^i = m^i$. \square

This result enables us to conclude that (3.1)–(3.6) admits a local $W^{1,\infty}$ solution in an interval $[0, T_n]$ for every n and R . First, denoting by \mathbf{w} the vector (w_1, \dots, w_{n-1}) , and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n-1})$, we have $-\boldsymbol{\varepsilon} = S\mathbf{w}$ with a positive definite matrix S , which has the form

$$S = n^2 \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Hence, the left hand side of (3.2) reads $(I + S)\dot{\mathbf{w}}$. Furthermore, the dissipation rates $|D[\varepsilon_k]_t|$ depend Lipschitz continuously on $\dot{\boldsymbol{\varepsilon}}$ by virtue of (1.18). Hence, by Lemma 3.1, Eq. (3.6) defines a locally Lipschitz continuous mapping that with $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$ associates the solution $m^{(n)}$ and $m_t^{(n)}$. By (3.2), $\dot{\boldsymbol{\varepsilon}}$ itself is a Lipschitz continuous mapping of $\mathbf{u} = (u_1, \dots, u_{n-1})$. We see that (3.1)–(3.4) can be considered as an ODE system in u_k, w_k, θ_k , with a right hand side, which is locally Lipschitz in $L^1(0, t)$ for every $t \in [0, T]$, and the existence of a local absolutely continuous

solution in an interval $[0, T_n]$ follows from the standard theory of ODEs. Consequently, the right hand side is bounded, and we conclude that the solution belongs to $W^{1,\infty}(0, T_n)$.

In the sequel, we will systematically use the “summation by parts formula”

$$\sum_{k=1}^{n-1} \xi_k(\eta_{k+1} - 2\eta_k + \eta_{k-1}) + \sum_{k=1}^n (\xi_k - \xi_{k-1})(\eta_k - \eta_{k-1}) = \xi_n(\eta_n - \eta_{n-1}) - \xi_0(\eta_1 - \eta_0). \tag{3.22}$$

At the end of this subsection, we prove that θ_k remain positive in the whole range of existence. As a first step, we test (3.4) by $-\theta_k^-$ (the negative part of θ_k). We obtain from (3.22) that

$$\frac{d}{dt} \sum_{k=1}^{n-1} (\theta_k^-)^2 + n^2 \sum_{k=1}^n (\theta_k^- - \theta_{k-1}^-)^2 + \sum_{k=1}^{n-1} (\theta_k^-)^2 \dot{\varepsilon}_k \leq 0,$$

and Gronwall’s argument yields $\theta_k^-(t) = 0$ for all k and t , hence $\theta_k(t) \geq 0$ for all k and t . Furthermore, we have

$$\dot{\theta}_k - n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) \geq -\frac{1}{4}\theta_k^2.$$

Let p be the solution of the differential equation

$$\dot{p} + \frac{1}{4}p^2 = 0, \quad p(0) = \theta_*.$$

Then

$$(\dot{p} - \dot{\theta}_k) - n^2((p - \theta_{k+1}) - 2(p - \theta_k) + (p - \theta_{k-1})) + \frac{1}{4}(p^2 - \theta_k^2) \leq 0. \tag{3.23}$$

Testing (3.23) by $(p - \theta_k)^+$ and using (3.22), we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{k=1}^{n-1} ((p - \theta_k)^+)^2 + n^2 \sum_{k=1}^n ((p - \theta_k)^+ - (p - \theta_{k-1})^+)^2 + \frac{1}{4}((p - \theta_k)^+)^2(p + \theta_k) \leq 0, \tag{3.24}$$

hence

$$\theta_k(t) \geq p(t) = \frac{4\theta_*}{4 + t\theta_*} \tag{3.25}$$

for all k and all $t \in [0, T_n]$.

In the remaining four subsections of Section 3, we derive three types of estimates for the approximate solutions of (2.1)–(2.4) defined by (3.1)–(3.6). The first estimate in Subsection 3.2 is the energy estimate. Its formal “continuous” counterpart consists in testing Eq. (2.1) by w_{xxt} , differentiating Eq. (2.2) in t and testing by w_t , testing (2.3) by 1, and summing up. This yields estimates for θ in $L^\infty(0, T_n; L^1(0, 1))$, w_t, w_{xx} in $L^\infty(0, T_n; L^2(0, 1))$, and w_{xt} in $L^2(0, T_n; L^2(0, 1))$. Furthermore, it implies that T_n are bounded from below by some $T^R > 0$.

This is indeed related to the energy balance (1.7). The equations of motion (2.1)–(2.2) can be formally written as $w_{tt} - w_{xxtt} = -\sigma_{xx} + f_t$, and the term $\varepsilon_t \sigma$ on the right hand side of (1.7), when integrated by parts, yields $w_t(-w_{tt} + w_{xxtt} + f_t)$. Denoting by $\mathcal{K}(w) = \frac{1}{2}(w_t^2 + w_{xt}^2)$, we obtain from (1.7) the equation

$$\frac{d}{dt} \int_0^1 (\mathcal{U}(\varepsilon, \theta) + \mathcal{K}(w)) \, dx = \int_0^1 f_t w_t \, dx,$$

which has the usual interpretation that the time increment of the total (potential plus kinetic) energy equals the energy supply.

The second estimate in Subsection 3.4 is the so-called Dafermos estimate. It consists in testing (2.3) by $\theta^{-1/3}$ and getting upper bounds for w_{xxt} in $L^2(0, T_n; L^2(0, 1))$ and θ in $L^{8/3}(0, T_n; L^{8/3}(0, 1))$.

Higher order estimates are derived in Subsection 3.5. We first formally differentiate (2.1) in x and square both sides of the resulting equation to obtain

$$\begin{aligned} & \int_0^1 ((B(m)w_{xx} + P[w_{xx}])_x + w_{xxxt} - \theta_x)^2 dx = \int_0^1 u_{xt}^2 dx \\ & = \int_0^1 (B(m)w_{xx} + P[w_{xx}] + w_{xxt} - (\theta - \theta^{ref}))(w_{tt} - w_{xxtt} - f_t) dx. \end{aligned}$$

In the right hand side, we have integrated by parts and replaced u_t from (2.1) and u_{xx} from (2.2). This enables us to estimate the L^2 -norm of w_{xxtt} by the L^2 -norms of θ_t and θ_x , which are in turn estimated by testing the heat equation (2.3) by θ_t and using the Gagliardo-Nirenberg inequality. Finally, we differentiate (2.1) in t and test by w_{xxtt} , differentiate (2.2) twice in t and test by w_{tt} , sum up the results, eliminating the terms in u_{tt} by integrating by parts, and obtain an L^2 -bound for w_{xxtt} .

We now carry out this programme more rigorously.

3.2. Energy estimate. We test (3.1) by $\dot{\varepsilon}_k$, differentiate (3.2) in t and test by \dot{w}_k , and sum up over $k = 1, \dots, n-1$. From (3.3), with a repeated use of (3.9) and (3.22), we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2n} \sum_{k=1}^{n-1} \dot{w}_k^2 + \frac{n}{2} \sum_{k=1}^n (\dot{w}_k - \dot{w}_{k-1})^2 \right) \\ & + \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k (B(m_k)\varepsilon_k + P[\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta^{ref}) = \frac{1}{n} \sum_{k=1}^{n-1} \dot{f}_k \dot{w}_k. \end{aligned} \quad (3.26)$$

We add (3.26) to (3.4), which yields, by virtue of (1.17),

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{n} \sum_{k=1}^{n-1} \left(\theta_k + \frac{1}{2} \dot{w}_k^2 + \frac{1}{2} B(m_k) \varepsilon_k^2 + V[\varepsilon_k] + \theta^{ref} \varepsilon_k \right) + \frac{n}{2} \sum_{k=1}^n (\dot{w}_k - \dot{w}_{k-1})^2 \right) \\ & = \frac{1}{n} \sum_{k=1}^{n-1} \dot{f}_k \dot{w}_k. \end{aligned} \quad (3.27)$$

By hypothesis, $B(m_k)$ is bounded from below by a constant. Furthermore,

$$\frac{1}{n} \sum_{k=1}^{n-1} \dot{f}_k^2(t) \leq \int_0^1 f_t^2(x, t) dx,$$

hence,

$$\frac{1}{n} \sum_{k=1}^{n-1} \left(\theta_k + \dot{w}_k^2 + \varepsilon_k^2 \right)(t) + n \sum_{k=1}^n (\dot{w}_k - \dot{w}_{k-1})^2(t) \leq C, \quad (3.28)$$

with the convention that here and in the sequel, C denotes any constant independent of n and R . Here, C depends only on the lower bound for $B(m_k)$, and on $\|f_t\|_{L^1(0, T; L^2(0, 1))}$.

We see that (3.6) is of the form (3.10)–(3.11), with $a(x, t) = -\frac{1}{2}(\bar{\varepsilon}^{(n)})^2(x, t)$ and with

$$\left| \int_0^1 a(y, t) \, dy \right| = \frac{1}{2} \int_0^1 (\bar{\varepsilon}^{(n)})^2(y, t) \, dy \leq \frac{1}{2n} \sum_{k=1}^{n-1} \varepsilon_k^2(t) \leq C =: A. \tag{3.29}$$

We conclude that the solution to (3.1)–(3.6) exists in the whole interval $[0, T^R]$ with $T^R := T_{A,b}^R$.

3.3. Discrete Gagliardo-Nirenberg inequality. Let $p, q, s \in [1, \infty]$ be such that $q > s$, and let $|\cdot|_p$ denote the norm in $L^p(0, 1)$. The *Gagliardo-Nirenberg inequality* states that there exists a constant $C > 0$ such that for every $v \in W^{1,p}(0, 1)$ we have

$$|v|_q \leq C (|v|_s + |v|_s^{1-\varrho} |v'|_p^\varrho), \quad \varrho = \frac{\frac{1}{s} - \frac{1}{q}}{1 + \frac{1}{s} - \frac{1}{p}}. \tag{3.30}$$

In fact, (3.30) is straightforward: If we introduce an auxiliary parameter $r = 1 + s(1 - \frac{1}{p})$ and use the chain rule $\frac{d}{dx}|v(x)|^r \leq r|v(x)|^{r-1}|v'(x)|$ a.e., we obtain from Hölder’s inequality the estimate

$$|v|_\infty \leq |v|_r + C|v|_s^{1-(1/r)}|v'|_p^{1/r}.$$

Combined with the obvious interpolation inequality $|v|_h \leq |v|_\infty^{1-(s/h)}|v|_s^{s/h}$ for $h = r$ and $h = q$, this yields (3.30).

Let now $\mathbf{v} = (v_0, v_1, \dots, v_n)$ be a vector, and let us denote

$$|\mathbf{v}|_p = \left(\frac{1}{n} \sum_{k=0}^n |v_k|^p \right)^{1/p}, \quad |\mathbf{D}\mathbf{v}|_p = \left(n^{p-1} \sum_{k=1}^n |v_k - v_{k-1}|^p \right)^{1/p}. \tag{3.31}$$

The discrete counterpart of (3.30) reads

$$|\mathbf{v}|_q \leq C (|\mathbf{v}|_s + |\mathbf{v}|_s^{1-\varrho} |\mathbf{D}\mathbf{v}|_p^\varrho), \quad \varrho = \frac{\frac{1}{s} - \frac{1}{q}}{1 + \frac{1}{s} - \frac{1}{p}}, \tag{3.32}$$

and can be easily derived from (3.30) by defining v e.g. as equidistant piecewise linear interpolations of v_k .

3.4. The Dafermos estimate. Following the idea of [4], we test (3.4) by $\theta_k^{-1/3}$. The two dissipation terms on the right hand side of (3.4) corresponding to (1.10) are nonnegative and can be omitted. To deal with the heat flux term $\theta_{k+1} - 2\theta_k + \theta_{k-1}$, we use (3.22) with $\xi_k = \theta_k^{-1/3}$, $\eta_k = \theta_k$, and the elementary inequality

$$-(x - y)(x^{-1/3} - y^{-1/3}) \geq 3(x^{1/3} - y^{1/3})^2$$

with $x = \theta_k$, $y = \theta_{k-1}$. We obtain for all $t \in [0, T^R]$, after summing up from $k = 1, \dots, n - 1$ and integrating in time, that

$$\begin{aligned} & \int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3} \dot{\varepsilon}_k^2 + 3n \sum_{k=1}^n \left(\theta_k^{1/3} - \theta_{k-1}^{1/3} \right)^2 \right) \, d\tau \\ & \leq \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{2/3} |\dot{\varepsilon}_k| \, d\tau + \frac{3}{2n} \sum_{k=1}^{n-1} \theta_k^{2/3}(t). \end{aligned} \tag{3.33}$$

The last term on the right hand side is bounded by virtue of (3.28). By Hölder’s inequality,

$$\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{2/3} |\dot{\varepsilon}_k| \, d\tau \leq \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{5/3} \, d\tau \right)^{1/2} \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3} \dot{\varepsilon}_k^2 \, d\tau \right)^{1/2},$$

hence,

$$\int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3} \dot{\varepsilon}_k^2 + 3n \sum_{k=1}^n \left(\theta_k^{1/3} - \theta_{k-1}^{1/3} \right)^2 \right) \, d\tau \leq C \left(1 + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{5/3} \, d\tau \right). \tag{3.34}$$

We now apply the inequality (3.32) to the case $v_k = \theta_k^{1/3}$, $s = 3$, $p = 2$, $q = 5$, $\varrho = 4/25$, that is (note that $|\mathbf{v}(t)|_3 \leq C$ by virtue of (3.28) again),

$$\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{5/3} = |\mathbf{v}|_5^5 \leq C(1 + |\mathbf{D}\mathbf{v}|_2^{4/5}) = C \left(1 + \left(n \sum_{k=1}^n \left(\theta_k^{1/3} - \theta_{k-1}^{1/3} \right)^2 \right)^{2/5} \right), \tag{3.35}$$

which implies that

$$\int_0^t \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{-1/3} \dot{\varepsilon}_k^2 + n \sum_{k=1}^n \left(\theta_k^{1/3} - \theta_{k-1}^{1/3} \right)^2 \right) \, d\tau \leq C. \tag{3.36}$$

Using (3.32) again with $s = 3$, $p = 2$, $q = 8$, $\varrho = 1/4$, and with \mathbf{v} as above, we obtain

$$\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{8/3}(\tau) \, d\tau \leq C, \tag{3.37}$$

and, as a consequence of (3.26),

$$\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(\tau) \, d\tau \leq C. \tag{3.38}$$

3.5. Higher order estimates. We define $\varepsilon_0, \varepsilon_n$ as solutions to the differential equations

$$\left. \begin{aligned} B(m_0)\varepsilon_0 + P[\varepsilon_0] + \dot{\varepsilon}_0 - \theta_0 + \theta^{ref} &= 0, \\ B(m_n)\varepsilon_n + P[\varepsilon_n] + \dot{\varepsilon}_n - \theta_n + \theta^{ref} &= 0, \end{aligned} \right\} \tag{3.39}$$

with initial conditions $\varepsilon_0(0) = \varepsilon_n(0) = 0$. Then (3.1) holds for all $k = 0, \dots, n$, and we have

$$\dot{u}_k - \dot{u}_{k-1} = B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1} + P[\varepsilon_k] - P[\varepsilon_{k-1}] + \dot{\varepsilon}_k - \dot{\varepsilon}_{k-1} - \theta_k + \theta_{k-1} \tag{3.40}$$

for all $k = 1, \dots, n$. By (3.22) we have

$$n \sum_{k=1}^n (\dot{u}_k - \dot{u}_{k-1})^2 = -n \sum_{k=1}^{n-1} \dot{u}_k (\dot{u}_{k+1} - 2\dot{u}_k + \dot{u}_{k-1}),$$

hence, by (3.1)–(3.2) and (3.40),

$$\begin{aligned} & n \sum_{k=1}^n (B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1} + P[\varepsilon_k] - P[\varepsilon_{k-1}] + \dot{\varepsilon}_k - \dot{\varepsilon}_{k-1} - \theta_k + \theta_{k-1})^2 \\ &= \frac{1}{n} \sum_{k=1}^{n-1} (\ddot{u}_k - \ddot{\varepsilon}_k - \dot{f}_k)(B(m_k)\varepsilon_k + P[\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta^{ref}). \end{aligned} \tag{3.41}$$

This yields

$$\begin{aligned} & \frac{d}{dt} \left(\frac{n}{2} \sum_{k=1}^n (\dot{w}_k - \dot{w}_{k-1})^2 + \frac{1}{2n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2 \right. \\ & \left. - \frac{1}{n} \sum_{k=1}^{n-1} (\dot{w}_k - \dot{\varepsilon}_k)(B(m_k)\varepsilon_k + P[\varepsilon_k] - \theta_k + \theta^{ref}) \right) + n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2 \quad (3.42) \\ & \leq C \left(\frac{1}{n} \sum_{k=1}^{n-1} |\dot{w}_k - \dot{\varepsilon}_k| |B'(m_k)\varepsilon_k \dot{m}_k + B(m_k)\dot{\varepsilon}_k + P[\varepsilon_k]_t - \dot{\theta}_k| \right. \\ & \quad \left. + \frac{1}{n} \sum_{k=1}^{n-1} |\dot{f}_k| |B(m_k)\varepsilon_k + P[\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta^{ref}| \right. \\ & \quad \left. + n \sum_{k=1}^n ((B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1})^2 + (P[\varepsilon_k] - P[\varepsilon_{k-1}])^2 + (\theta_k - \theta_{k-1})^2) \right). \quad (3.43) \end{aligned}$$

We estimate the terms on the right hand side of (3.43) as follows:

$$\begin{aligned} & \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} |\dot{w}_k - \dot{\varepsilon}_k| |B'(m_k)\varepsilon_k \dot{m}_k + B(m_k)\dot{\varepsilon}_k + P[\varepsilon_k]_t - \dot{\theta}_k| d\tau \\ & \leq C \left(R + \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2 d\tau \right)^{1/2} \right), \\ & \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} |\dot{f}_k| |B(m_k)\varepsilon_k + P[\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta^{ref}| d\tau \leq C, \end{aligned}$$

where we have used (3.15), (3.28), (3.37), and (3.38). To estimate in (3.43) the “fatigue” term $(B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1})^2$, we notice that

$$|B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1}| \leq C|\varepsilon_k - \varepsilon_{k-1}| + |\varepsilon_{k-1}||m_k - m_{k-1}|.$$

We have

$$\begin{aligned} & |\dot{m}_k(t) - \dot{m}_{k-1}(t)| \\ & \leq \int_{(k-1)/n}^{k/n} \int_0^1 |\lambda'(x-y)| \left(-\frac{1}{2} B'(m^{(n)})(\bar{\varepsilon}^{(n)})^2 m_t^{(n)} + |D[\bar{\varepsilon}^{(n)}]_t| \right) (y,t) dy dx \\ & \leq \frac{C}{n} \left(\dot{\mu}(t) + \int_0^1 |\bar{\varepsilon}_t^{(n)}|(y,t) dy \right), \quad (3.44) \end{aligned}$$

hence

$$|m_k(t) - m_{k-1}(t)| \leq \frac{C}{n}(\mu(t) + 1) \leq C \quad (3.45)$$

by virtue of (3.38). This yields

$$\begin{aligned} & \int_0^t n \sum_{k=1}^n (B(m_k)\varepsilon_k - B(m_{k-1})\varepsilon_{k-1})^2(\tau) \, d\tau \\ & \leq C \int_0^t \left(n \sum_{k=1}^n (\varepsilon_k - \varepsilon_{k-1})^2 + \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^2 \right)(\tau) \, d\tau \\ & \leq C \left(1 + \int_0^t \int_0^\tau n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau') \, d\tau' \, d\tau \right). \end{aligned}$$

Moreover, by Proposition 1.2,

$$|P[\varepsilon_k](t) - P[\varepsilon_{k-1}](t)| \leq \max_{\tau \in [0,t]} |\varepsilon_k(\tau) - \varepsilon_{k-1}(\tau)| \leq \int_0^t |\dot{\varepsilon}_k(\tau) - \dot{\varepsilon}_{k-1}(\tau)| \, d\tau.$$

From the above considerations and from (3.43) we conclude that

$$\begin{aligned} & \frac{n}{2} \sum_{k=1}^n (\dot{w}_k - \dot{w}_{k-1})^2(t) + \frac{1}{2n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \int_0^t n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau) \, d\tau \\ & \leq C \left(R + \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2(t) + \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2 \, d\tau \right)^{1/2} + \int_0^t n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(\tau) \, d\tau \right. \\ & \quad \left. + \int_0^t \int_0^\tau n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau') \, d\tau' \, d\tau \right). \end{aligned}$$

Gronwall’s argument then yields

$$\begin{aligned} & \frac{1}{2n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \int_0^t n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau) \, d\tau \\ & \leq C \left(R + \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2(t) + \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2 \, d\tau \right)^{1/2} + \int_0^t n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(\tau) \, d\tau \right). \end{aligned} \tag{3.46}$$

We now test (3.4) by θ_k and obtain

$$\frac{d}{dt} \left(\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2 \right) + n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2 \leq C \left(\frac{1}{n} \sum_{k=1}^{n-1} (R\varepsilon_k^2 + \dot{\varepsilon}_k^2 + |\dot{\varepsilon}_k|(1 + \theta_k))\theta_k \right),$$

where, by (3.37),

$$\begin{aligned} \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} R\varepsilon_k^2 \theta_k \, d\tau & \leq R \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau \right)^{1/2} \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2 \, d\tau \right)^{1/2} \\ & \leq CR \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau \right)^{1/2}, \end{aligned}$$

$$\begin{aligned} \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2 \theta_k \, d\tau &\leq \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/2} \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2 \, d\tau \right)^{1/2} \\ &\leq C \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/2}, \\ \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} |\dot{\varepsilon}_k| \theta_k^2 \, d\tau &\leq \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/4} \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{8/3} \, d\tau \right)^{3/4} \\ &\leq C \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/4}, \end{aligned}$$

hence

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n-1} \theta_k^2(t) + \int_0^t n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(\tau) \, d\tau \\ &\leq C \left(1 + R \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau \right)^{1/2} + \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/2} \right). \end{aligned} \tag{3.47}$$

Finally, we test (3.4) by $\dot{\theta}_k$ and obtain from Hölder’s inequality that

$$\begin{aligned} &\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2(\tau) \, d\tau + n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(t) \\ &\leq C \left(1 + R^2 \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^4 \, d\tau \right). \end{aligned} \tag{3.48}$$

We have

$$\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^4 \, d\tau \leq \max_{\tau \in [0,t]} \max_{k=1,\dots,n} \theta_k^{5/3}(\tau) \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \theta_k^{8/3} \, d\tau \leq C \max_{\tau \in [0,t]} \max_{k=1,\dots,n} \theta_k^{5/3}(\tau)$$

and, by (3.32) with $q = \infty, s = 1, p = 2, \varrho = 2/3$,

$$\max_{k=1,\dots,n} \theta_k(\tau) \leq C \left(1 + \left(n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(\tau) \right)^{1/3} \right),$$

and we can infer from (3.48) that

$$\begin{aligned} &\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\theta}_k^2(\tau) \, d\tau + n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(t) \\ &\leq C \left(1 + R^2 \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right). \end{aligned} \tag{3.49}$$

Combining (3.46) with (3.47) and (3.49) yields

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \int_0^t n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau) \, d\tau \\ &\leq CR \left(1 + \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \varepsilon_k^4 \, d\tau \right)^{1/2} \right) + C \left(\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^4 \, d\tau \right)^{1/2}. \end{aligned} \tag{3.50}$$

Using the vector notation (3.31), we have by (3.39) and (3.37) that

$$\begin{aligned} |\dot{\varepsilon}(t)|_2^2 &= \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \frac{1}{n} (\dot{\varepsilon}_0^2(t) + \dot{\varepsilon}_n^2(t)) \\ &\leq \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \frac{C}{n} \left(1 + \sum_{k=1}^{n-1} \theta_k^2(t) \right) \leq C + \frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t), \end{aligned}$$

and we rewrite (3.50) as

$$\max_{\tau \in [0,t]} |\dot{\varepsilon}(\tau)|_2^2 + \int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \leq CR \left(1 + \int_0^t |\varepsilon(\tau)|_4^4 d\tau \right)^{1/2} + C \left(\int_0^t |\dot{\varepsilon}(\tau)|_4^4 d\tau \right)^{1/2}. \tag{3.51}$$

We estimate the right hand side of (3.51) using (3.32) as follows:

$$|\dot{\varepsilon}(\tau)|_4 \leq C (|\dot{\varepsilon}(\tau)|_2 + |\dot{\varepsilon}(\tau)|_2^{3/4} |\mathbf{D}\dot{\varepsilon}(\tau)|_2^{1/4}).$$

We have $\int_0^t |\dot{\varepsilon}(\tau)|_2^2 d\tau \leq C$ by virtue of (3.38), hence

$$\begin{aligned} \int_0^t |\dot{\varepsilon}(\tau)|_4^4 d\tau &\leq C \left(\int_0^t |\dot{\varepsilon}(\tau)|_2^4 d\tau + \left(\int_0^t |\dot{\varepsilon}(\tau)|_2^6 d\tau \right)^{1/2} \left(\int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \right)^{1/2} \right) \\ &\leq C \left(\max_{\tau \in [0,t]} |\dot{\varepsilon}(\tau)|_2^2 + \max_{\tau \in [0,t]} |\dot{\varepsilon}(\tau)|_2^2 \left(\int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \right)^{1/2} \right), \end{aligned}$$

and

$$\left(\int_0^t |\dot{\varepsilon}(\tau)|_4^4 d\tau \right)^{1/2} \leq C \left(1 + \max_{\tau \in [0,t]} |\dot{\varepsilon}(\tau)|_2^{4/3} \right) + \delta \int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \tag{3.52}$$

with some $\delta > 0$ sufficiently small. From the inequality

$$|\varepsilon(\tau)|_4 \leq C (|\varepsilon(\tau)|_2 + |\varepsilon(\tau)|_2^{3/4} |\mathbf{D}\varepsilon(\tau)|_2^{1/4})$$

and from (3.28) we then obtain

$$\max_{\tau \in [0,t]} |\varepsilon(\tau)|_4 \leq C \left(1 + \left(\int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \right)^{1/8} \right) \tag{3.53}$$

Combining (3.51) with (3.52)–(3.53) yields

$$|\dot{\varepsilon}(t)|_2^2 + \int_0^t |\mathbf{D}\dot{\varepsilon}(\tau)|_2^2 d\tau \leq C(1 + R^{4/3}). \tag{3.54}$$

In (3.6), we thus have in particular

$$\int_0^1 \lambda(x - y) |D[\bar{\varepsilon}^{(n)}]_t|(y, t) dy \leq C(1 + R^{2/3}).$$

We thus may fix R sufficiently large depending only on the data of the problem such that

$$Q_R \left(\int_0^1 \lambda(x - y) |D[\bar{\varepsilon}^{(n)}]_t|(y, t) dy \right) = \int_0^1 \lambda(x - y) |D[\bar{\varepsilon}^{(n)}]_t|(y, t) dy$$

for all x and t , so that the truncation in (3.6) is never active, we may set $T^* = T^R$, and all the above estimates are thus independent of R . In particular, there exist a

constant $C > 0$ such that

$$\frac{1}{n} \sum_{k=1}^{n-1} \dot{\varepsilon}_k^2(t) + \int_0^t n \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau) \, d\tau + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} (\dot{\varepsilon}_k^4 + \varepsilon_k^4)(\tau) \, d\tau \leq C, \quad (3.55)$$

$$\int_0^t \frac{1}{n} \sum_{k=1}^{n-1} (\theta_k^4 + \dot{\theta}_k^2)(\tau) \, d\tau + n \sum_{k=1}^n (\theta_k - \theta_{k-1})^2(t) \leq C \quad (3.56)$$

for $t \in [0, T^*]$. By comparison, we also have

$$\int_0^t n^3 \sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})^2(\tau) \, d\tau \leq C, \quad (3.57)$$

and similarly for u_k . Finally, we differentiate (3.1) once in t and test by $\ddot{\varepsilon}_k$, (3.2) twice in t and test by \ddot{w}_k , and sum the two equations up. Using (3.55)–(3.56), we immediately get the estimate

$$\frac{1}{n} \sum_{k=1}^{n-1} \ddot{w}_k^2(t) + n \sum_{k=1}^n (\ddot{w}_k - \ddot{w}_{k-1})^2(t) + \int_0^t \frac{1}{n} \sum_{k=1}^{n-1} \ddot{\varepsilon}_k^2(\tau) \, d\tau \leq C. \quad (3.58)$$

Notice that the initial conditions for $\ddot{w}_k, \ddot{\varepsilon}_k$ are estimated by the second spatial derivatives of θ^0 and f .

4. Proof of the main Theorem 2.2.

4.1. Existence. For a generic sequence $\{\varphi_k : k = 0, 1, \dots, n\}$ we put $\Delta_k \varphi = n(\varphi_k - \varphi_{k-1})$, and $\Delta_k^2 \varphi = n^2(\varphi_{k+1} - 2\varphi_k + \varphi_{k-1})$, and define piecewise constant, piecewise linear, and piecewise quadratic interpolations

$$\bar{\varphi}^{(n)}(x) = \begin{cases} \varphi_k & \text{for } x \in [\frac{k-1}{n}, \frac{k}{n}), \quad k = 1, \dots, n-1, \\ \varphi_{n-1} & \text{for } x \in [\frac{n-1}{n}, 1], \end{cases} \quad (4.1)$$

$$\hat{\varphi}^{(n)}(x) = \varphi_{k-1} + (x - \frac{k-1}{n}) \Delta_k \varphi \quad \text{for } x \in [\frac{k-1}{n}, \frac{k}{n}), \quad k = 1, \dots, n, \quad (4.2)$$

$$\tilde{\varphi}^{(n)}(x) = \begin{cases} \frac{1}{2}(\varphi_{k-1} + \varphi_k) + (x - \frac{k-1}{n}) \Delta_k \varphi + \frac{1}{2}(x - \frac{k-1}{n})^2 \Delta_k^2 \varphi & \text{for } x \in [\frac{k-1}{n}, \frac{k}{n}), \quad k = 1, \dots, n-1, \\ \frac{1}{2}(\varphi_{n-1} + \varphi_n) + (x - \frac{n-1}{n}) \Delta_n \varphi + \frac{1}{2}(x - \frac{n-1}{n})^2 \Delta_{n-1}^2 \varphi & \text{for } x \in [\frac{n-1}{n}, 1]. \end{cases} \quad (4.3)$$

For functions $\bar{\varepsilon}^{(n)}, \bar{\theta}^{(n)}, \bar{u}^{(n)}, \bar{w}^{(n)}, \hat{\varepsilon}^{(n)}, \hat{\theta}^{(n)}, \tilde{u}^{(n)}, \tilde{w}^{(n)}$, we have derived estimates (3.55)–(3.58) that we rewrite in the form

$$|\bar{\varepsilon}_t(t)|_2^2 + \int_0^t |\hat{\varepsilon}_{xt}(\tau)|_2^2 \, d\tau + \int_0^t (|\bar{\varepsilon}_t(\tau)|_4^4 + |\bar{\varepsilon}(\tau)|_4^4) \, d\tau \leq C, \quad (4.4)$$

$$\int_0^t (|\bar{\theta}_t(\tau)|_2^2 + |\bar{\theta}(\tau)|_4^4) \, d\tau + |\hat{\theta}_x(t)|_2^2 \leq C, \quad (4.5)$$

$$\int_0^t |\tilde{\theta}_{xx}(\tau)|_2^2 \, d\tau \leq C, \quad (4.6)$$

$$|\bar{w}_{tt}(t)|_2^2 + |\hat{w}_{xtt}(t)|_2^2 + \int_0^t |\bar{\varepsilon}_{tt}(\tau)|_2^2 d\tau \leq C, \tag{4.7}$$

and by (3.2)–(3.3),

$$\int_0^t (|\tilde{w}_{xxt}(\tau)|_2^2 + |\tilde{u}_{xxt}(\tau)|_2^2) d\tau \leq C. \tag{4.8}$$

System (3.1)–(3.4) has the form

$$\bar{u}_t^{(n)} = B(\bar{m}^{(n)})\bar{\varepsilon}^{(n)} + P[\bar{\varepsilon}^{(n)}] + \bar{\varepsilon}_t^{(n)} - (\bar{\theta}^{(n)} - \theta^{ref}), \tag{4.9}$$

$$\bar{w}_t^{(n)} - \bar{\varepsilon}_t^{(n)} = -\tilde{u}_{xx}^{(n)} + \bar{f}^{(n)}, \tag{4.10}$$

$$\bar{\varepsilon}^{(n)} = \tilde{w}_{xx}^{(n)}, \tag{4.11}$$

$$\bar{\theta}_t^{(n)} = \tilde{\theta}_{xx}^{(n)} - \frac{1}{2}B'(\bar{m}^{(n)})(\bar{\varepsilon}^{(n)})^2 \bar{m}_t^{(n)} + |D[\bar{\varepsilon}^{(n)}]_t| + (\bar{\varepsilon}_t^{(n)})^2 - \bar{\theta}^{(n)} \bar{\varepsilon}_t^{(n)}. \tag{4.12}$$

We further have

$$\int_0^t |\hat{\varepsilon}_{tt}(\tau)|_2^2 d\tau \leq \int_0^t \frac{2}{n} \sum_{k=0}^n \ddot{\varepsilon}_k^2(\tau) d\tau \leq \int_0^t \left(\frac{2}{n} \sum_{k=1}^{n-1} \ddot{\varepsilon}_k^2(\tau) + \frac{2}{n} (\ddot{\varepsilon}_0^2(\tau) + \ddot{\varepsilon}_n^2(\tau)) \right) d\tau.$$

By (3.39), we have for $k = 0$ and $k = n$

$$\ddot{\varepsilon}_k^2(\tau) \leq C(1 + \theta_k^2(\tau) + \dot{\theta}_k^2(\tau)),$$

hence

$$\int_0^t |\hat{\varepsilon}_{tt}(\tau)|_2^2 d\tau \leq C \left(1 + \int_0^t \frac{2}{n} \sum_{k=1}^{n-1} (\ddot{\varepsilon}_k^2 + \theta_k^2 + \dot{\theta}_k^2)(\tau) d\tau \right) \leq C. \tag{4.13}$$

From (4.4), (4.13), and from Sobolev embedding theorems it follows that there exists $\varepsilon \in W^{1,2}(\Omega_{T^*})$ such that $\varepsilon_{xt}, \varepsilon_{tt} \in L^2(\Omega_{T^*})$, and a subsequence of $\{\hat{\varepsilon}^{(n)}\}$, still indexed by n , such that

$$\hat{\varepsilon}^{(n)} \rightarrow \varepsilon \text{ strongly in } C(\overline{\Omega_{T^*}}), \quad \hat{\varepsilon}_t^{(n)} \rightarrow \varepsilon_t \text{ strongly in } L^p(\Omega_{T^*})$$

for all $p > 1$. Furthermore,

$$|\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x, t) \leq |\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}|^2(t)$$

for $x \in [(k-1)/n, k/n]$, hence

$$\int_0^t \int_0^1 |\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x, \tau) dx d\tau \leq \int_0^t \frac{1}{n} \sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau) d\tau \leq \frac{C}{n^2},$$

so that

$$\bar{\varepsilon}_t^{(n)} \rightarrow \varepsilon_t \text{ strongly in } L^2(\Omega_{T^*}).$$

Similarly,

$$|\bar{\varepsilon}^{(n)} - \hat{\varepsilon}^{(n)}|^2(x, t) \leq |\varepsilon_k - \varepsilon_{k-1}|^2(t) \leq \sum_{k=1}^n (\varepsilon_k - \varepsilon_{k-1})^2(t) \leq \frac{C}{n},$$

hence

$$\bar{\varepsilon}^{(n)} \rightarrow \varepsilon \text{ strongly in } C(\overline{\Omega_{T^*}}).$$

We check in the same way that there exist $u, w, \theta \in C(\overline{\Omega_{T^*}})$ such that, selecting again a subsequence if necessary,

$$\begin{aligned} \tilde{w}_{xxt}^{(n)} &\rightarrow \varepsilon_t = w_{xxt}, \quad \tilde{u}_{xx}^{(n)} \rightarrow u_{xx} \quad \text{strongly in } L^2(\Omega_{T^*}), \\ \tilde{\theta}_t^{(n)} &\rightarrow \theta_t, \quad \tilde{\theta}_{xx}^{(n)} \rightarrow \theta_{xx} \quad \text{weakly in } L^2(\Omega_{T^*}), \quad \tilde{\theta}^{(n)} \rightarrow \theta \quad \text{strongly in } C(\overline{\Omega_{T^*}}). \end{aligned}$$

Finally, by Lemma 3.1, using also the estimate $|m_x^{(n)}(x, t)| \leq C$ analogous to (3.45) to control the difference $\bar{m}^{(n)} - m^{(n)}$, we obtain

$$\bar{m}_t^{(n)} \rightarrow m_t \quad \text{strongly in } L^1(0, T^*; L^\infty(0, 1)),$$

where m satisfies (2.4). This enables us to pass to the limit in (3.6) and (4.9)–(4.12) and conclude that (u, w, θ) is a strong solution to (2.1)–(2.4) with the regularity indicated in Theorem 2.2 and satisfying the initial conditions (2.5). It remains to check that the boundary conditions (2.6) hold. We have $w_n(t) = 0$, hence

$$\begin{aligned} |\tilde{w}^{(n)}(1, t)| &= |2w_n(t) - \frac{3}{2}w_{n-1}(t) + \frac{1}{2}w_{n-2}(t)| \\ &= |w_n(t) - w_{n-1}(t) - \frac{1}{2}(w_{n-1}(t) - w_{n-2}(t))| \\ &\leq 2 \left(\sum_{k=1}^n |w_k - w_{k-1}|^2(t) \right)^{1/2} \leq \frac{C}{\sqrt{n}}, \end{aligned}$$

and similarly for $w(0, t), u(1, t), u(0, t)$. To complete the existence proof, we only have to verify the homogeneous Neumann boundary condition for θ . In other words, we have to check that for every $\psi \in C^1(\overline{\Omega_{T^*}})$ we have

$$\int_0^{T^*} \int_0^1 (\theta_x \psi_x + \theta_{xx} \psi)(x, t) \, dx \, dt = 0. \tag{4.14}$$

A straightforward computation yields

$$\int_0^{T^*} \int_0^1 (\tilde{\theta}_x^{(n)} \psi_x + \tilde{\theta}_{xx}^{(n)} \psi)(x, t) \, dx \, dt = - \int_0^t \psi(1, t) n (\theta_{n-1} - \theta_{n-2})(t) \, dt. \tag{4.15}$$

We have

$$\begin{aligned} \int_0^t n^2 (\theta_{n-1} - \theta_{n-2})^2(t) \, dt &= \int_0^t n^2 (\theta_n - 2\theta_{n-1} + \theta_{n-2})^2(t) \, dt \\ &\leq \int_0^t n^2 \sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})^2(t) \, dt \leq \frac{C}{n}, \end{aligned}$$

hence

$$\lim_{n \rightarrow \infty} \int_0^{T^*} \int_0^1 (\tilde{\theta}_x^{(n)} \psi_x + \tilde{\theta}_{xx}^{(n)} \psi)(x, t) \, dx \, dt = 0$$

and (4.14) follows.

4.2. Uniqueness. Let $(u, w, \theta), (\tilde{u}, \tilde{w}, \tilde{\theta})$ be two solutions of (2.1)–(2.6). We integrate the difference of (2.3) for θ and $\tilde{\theta}$ in time and test by $\theta - \tilde{\theta}$. Taking into

account the estimates derived in the previous section and Proposition 1.2, we obtain

$$\begin{aligned}
& \int_0^1 |\theta - \tilde{\theta}|^2(x, t) dx + \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\int_0^t (\theta_x - \tilde{\theta}_x)(x, \tau) d\tau \right)^2 dx \\
& \leq C \int_0^1 \left(\int_0^t (|m - \tilde{m}| + |m_t - \tilde{m}_t| + |w_{xx}^2 - \tilde{w}_{xx}^2|)(x, \tau) d\tau \right)^2 dx \\
& \quad + C \int_0^1 \left(\int_0^t |w_{xxt} - \tilde{w}_{xxt}|(x, \tau) d\tau \right)^2 dx + \int_0^1 \left(\int_0^t |w_{xxt}^2 - \tilde{w}_{xxt}^2|(x, \tau) d\tau \right)^2 dx \\
& \quad + \int_0^1 \left(\int_0^t |\theta - \tilde{\theta}| |w_{xxt}|(x, \tau) d\tau \right)^2 dx. \tag{4.16}
\end{aligned}$$

We estimate the terms on the right hand side as follows. Using Lemma 3.1, Proposition 1.2, and the fact that w_{xx} belongs to $L^\infty(\Omega_{T^*})$, we have

$$\int_0^t |m - \tilde{m}|_\infty d\tau \leq C \int_0^t |m_t - \tilde{m}_t|_\infty d\tau \leq C \int_0^1 \int_0^t |w_{xxt} - \tilde{w}_{xxt}|(x, \tau) d\tau dx.$$

For the last two integrals in (4.16), we obtain from Hölder's inequality

$$\begin{aligned}
& \int_0^1 \left(\int_0^t |w_{xxt}^2 - \tilde{w}_{xxt}^2|(x, \tau) d\tau \right)^2 dx \\
& \leq \int_0^1 \left(\int_0^t |w_{xxt} - \tilde{w}_{xxt}|^2(x, \tau) d\tau \right) \left(\int_0^t |w_{xxt} + \tilde{w}_{xxt}|^2(x, \tau) d\tau \right) dx, \\
& \quad \int_0^1 \left(\int_0^t |\theta - \tilde{\theta}| |w_{xxt}|(x, \tau) d\tau \right)^2 dx \\
& \leq \int_0^1 \left(\int_0^t |\theta - \tilde{\theta}|^2(x, \tau) d\tau \right) \left(\int_0^t |w_{xxt}|^2(x, \tau) d\tau \right) dx. \tag{4.17}
\end{aligned}$$

We have $w_{xxt}, w_{xxxt} \in L^2(\Omega_{T^*})$. Hence,

$$\max_{x \in [0,1]} \int_0^t |w_{xxt}|^2(x, \tau) d\tau \leq C,$$

and we finally obtain

$$\begin{aligned}
& \int_0^1 |\theta - \tilde{\theta}|^2(x, t) dx + \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\int_0^t (\theta_x - \tilde{\theta}_x)(x, \tau) d\tau \right)^2 dx \\
& \leq C \int_0^t \int_0^1 (|w_{xxt} - \tilde{w}_{xxt}|^2 + |\theta - \tilde{\theta}|^2)(x, \tau) dx d\tau. \tag{4.18}
\end{aligned}$$

In the next step, we test the difference of the time derivatives of (2.2) for w and \tilde{w} by $w_t - \tilde{w}_t$, the difference of (2.1) for u and \tilde{u} by $w_{xxt} - \tilde{w}_{xxt}$, and sum up. Arguing as above, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 ((w_t - \tilde{w}_t)^2 + (w_{xt} - \tilde{w}_{xt})^2)(x, t) dx + \int_0^1 |w_{xxt} - \tilde{w}_{xxt}|^2(x, t) dx \\
& \leq C \left(\int_0^1 |\theta - \tilde{\theta}|^2(x, t) dx + \int_0^t \int_0^1 |w_{xxt} - \tilde{w}_{xxt}|^2(x, \tau) dx d\tau \right). \tag{4.19}
\end{aligned}$$

From (4.18)–(4.19) it follows that

$$\begin{aligned} & \int_0^1 (|w_{xxt} - \tilde{w}_{xxt}|^2 + |\theta - \tilde{\theta}|^2)(x, t) dx \\ & + \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\left(\int_0^t (\theta_x - \tilde{\theta}_x)(x, \tau) d\tau \right)^2 + (w_t - \tilde{w}_t)^2 + (w_{xt} - \tilde{w}_{xt})^2 \right)(x, t) dx \\ & \leq C \int_0^t \int_0^1 (|w_{xxt} - \tilde{w}_{xxt}|^2 + |\theta - \tilde{\theta}|^2)(x, \tau) dx d\tau. \end{aligned}$$

Gronwall's argument now yields that $w = \tilde{w}$, $\theta = \tilde{\theta}$, and the proof of Theorem 2.2 is complete.

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Received July 2012; revised February 2013.

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