Guaranteed error upper bounds for singularly perturbed problems

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Outline

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega \qquad\qquad \kappa = \text{const.}$$

- 3D model problem
- Computable upper bound
- Robust flux equilibration
- Explicit and robust flux reconstruction
- Local efficiency
- Numerical examples

M. Ainsworth, T. Vejchodský: *Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems*, Numer. Math. 119 (2), 2011, 219–243.

Model problem

Classical formulation:

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^3$$

$$u = 0 \quad \text{on } \partial\Omega \qquad\qquad \kappa = \text{const.}$$

Model problem

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$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^3$$
$$u = 0 \quad \text{on } \partial\Omega \qquad\qquad \kappa = \text{const.}$$

Weak formulation:

$$u \in V : \quad \mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V$$

Notation:

$V = H_0^1(\Omega)$
$\mathcal{B}(u, v) = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)_\Omega + \kappa^2 (u, v)_\Omega$
$(u, v)_\Omega = \int_\Omega uv \, \mathrm{d}\mathbf{x}$

Model problem

Classical formulation:

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^3$$
$$u = 0 \quad \text{on } \partial\Omega \qquad\qquad \kappa = \text{const.}$$

Weak formulation:

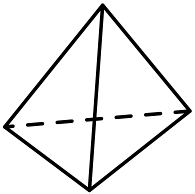
$$u \in V: \quad \mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V$$

Linear tetrahedral FEM:

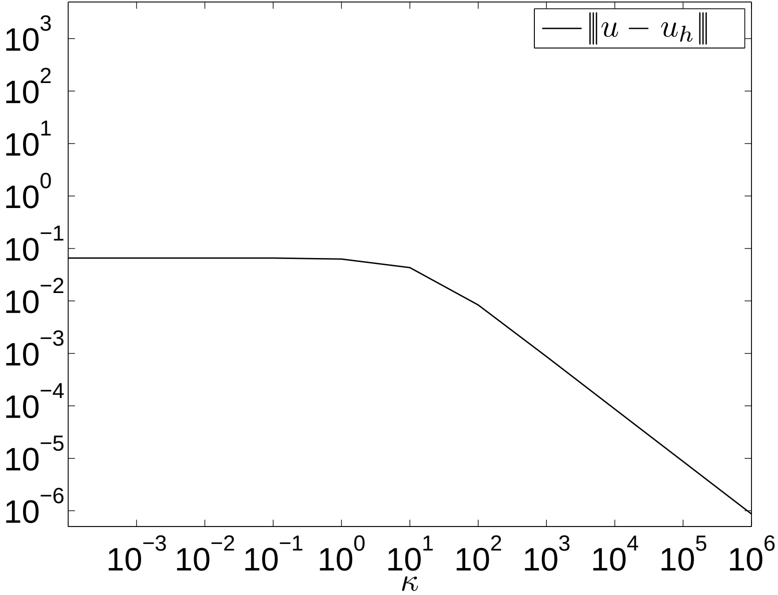
$$u_h \in V_h: \quad \mathcal{B}(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h$$

Notation:

$V = H_0^1(\Omega)$
$\mathcal{B}(u, v) = (\nabla u, \nabla v)_\Omega + \kappa^2 (u, v)_\Omega$
$(u, v)_\Omega = \int_\Omega uv \,\mathrm{d}\mathbf{x}$
$V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$



Error vs. κ (fixed mesh)



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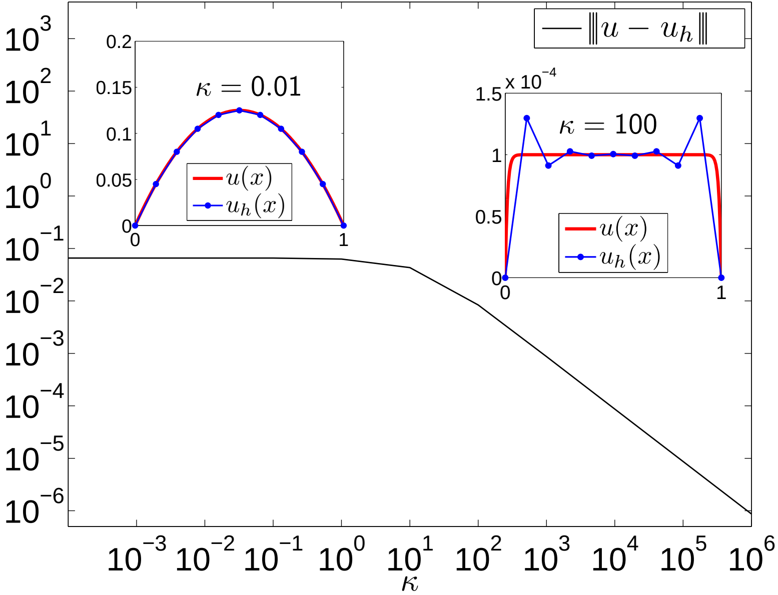

Upper bound

Theorem

$$|||u - u_h|||^2 \le \sum_{K \in \mathcal{T}_h} \left[\eta_K(\mathbf{y}) + \operatorname{osc}_K(f)\right]^2 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

where

- $\eta_K^2(\mathbf{y}) = \|\mathbf{y} - \boldsymbol{\nabla} u_h\|_{0,K}^2 + \kappa^{-2} \left\|\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}\right\|_{0,K}^2$
- $\operatorname{osc}_K(f) = \min\left\{h_K/\pi, \kappa^{-1}\right\} \|f - \Pi_K f\|_{0,K}$
- $\Pi_K f \in P^1(K) : \quad (f - \Pi_K f, \varphi)_K = 0 \quad \forall \varphi \in P^1(K)$

Notation: $|||v|||^2 = \mathcal{B}(v, v) \qquad \|v\|_{0,K}^2 = (v, v)_K$

Equilibration of fluxes

Equilibrated flux reconstruction:

$$\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega): \quad \left(f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}, 1\right)_K = 0 \quad \forall K \in \mathcal{T}_h$$

Equilibration of fluxes

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$$\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : \quad \big(f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}, 1\big)_K = 0 \quad \forall K \in \mathcal{T}_h$$

First-order equilibration: $\forall \theta \in P^1(K)$

$$\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : \quad \big(f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}, \theta\big)_K + (\mathbf{y} - \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} \theta)_K = 0$$

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$\Leftrightarrow$

$$\big(f - \kappa^2 u_h, \theta\big)_K - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} \theta)_K + \langle \mathbf{y} \cdot \mathbf{n}_K, \theta \rangle_{\partial K} = 0 \quad \forall \theta \in P^1(K)$$

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$\Leftrightarrow$

$$\left(f - \kappa^2 u_h, \theta\right)_K - (\nabla u_h, \nabla \theta)_K + \langle g_K, \theta \rangle_{\partial K} = 0 \qquad \forall \theta \in P^1(K)$$

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$\Leftrightarrow$

(*) $$\left(f - \kappa^2 u_h, \theta\right)_K - (\nabla u_h, \nabla\theta)_K + \langle g_K, \theta\rangle_{\partial K} = 0 \qquad \forall \theta \in P^1(K)$$

Fast algorithm [Ainsworth, Oden, 2000]:

- Linearity: $g_K \in P^1(\gamma), \quad \gamma \in \partial\mathcal{T}_h$
- Consistency: $g_K + g_{K'} = 0$ on $\gamma = \partial K \cap \partial K'$
- First-order equilibration: (*)

Reconstruction:

$$\mathbf{y}_K \in \mathbf{H}(\mathrm{div},K): \quad \mathbf{y}_K \cdot \mathbf{n}_K = g_K \text{ on } \partial K \quad \Rightarrow \quad \text{First-order equilibration}$$

Equilibration of fluxes

Equilibrated flux reconstruction:

$$\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega): \quad \left(f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}, 1\right)_K = 0 \quad \forall K \in \mathcal{T}_h$$

First-order equilibration: $\forall \theta \in P^1(K)$

$$\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega): \quad \left(f - \kappa^2 u_h + \mathrm{div}\, \mathbf{y}, \theta\right)_K + (\mathbf{y} - \nabla u_h, \nabla \theta)_K = 0$$

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(*) $$\left(f - \kappa^2 u_h, \theta\right)_K - (\nabla u_h, \nabla \theta)_K + \langle g_K, \theta \rangle_{\partial K} = 0 \qquad \forall \theta \in P^1(K)$$

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Reconstruction:

$\mathbf{y}_K \in \mathbf{H}(\mathrm{div}, K): \quad \mathbf{y}_K \cdot \mathbf{n}_K = g_K$ on $\partial K \quad \Rightarrow \quad$ First-order equilibration

NOT ROBUST FOR $\kappa \to \infty$ [Babuška, Ainsworth, 1999]

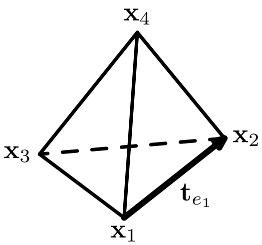
Flux reconstruction #1

Definition:

$$\mathbf{y}_K^{(1)} = \nabla u_h + \boldsymbol{\tau}_K^L + \boldsymbol{\tau}_K^Q$$

Properties:

- $\mathbf{y}_K^{(1)} \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K$
- $\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K^{(1)} = 0$



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Properties:

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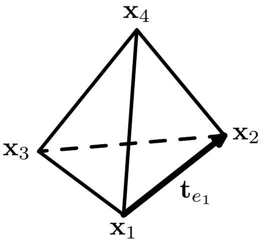
- $\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K^{(1)} = 0$

$$\boldsymbol{\tau}_K^L = 2 \sum_{n=1}^{4} \mathbf{c}_n^{(K)} \lambda_n \qquad \begin{aligned} \mathbf{c}_1^{(K)} &= |\gamma_2| R_{|\gamma_2}(\mathbf{x}_1) \nabla\lambda_3 \times \nabla\lambda_4 \\ &+ |\gamma_3| R_{|\gamma_3}(\mathbf{x}_1) \nabla\lambda_4 \times \nabla\lambda_2 \\ &+ |\gamma_4| R_{|\gamma_4}(\mathbf{x}_1) \nabla\lambda_2 \times \nabla\lambda_3 \end{aligned} \qquad \begin{aligned} \mathbf{c}_2^{(K)} &= \dots \\ \mathbf{c}_3^{(K)} &= \dots \\ \mathbf{c}_4^{(K)} &= \dots \end{aligned}$$

$$R = g_K - \nabla u_h \cdot \mathbf{n}_K$$

$$\boldsymbol{\tau}_K^Q = \frac{1}{4} \sum_{e \subset \partial K} \beta_e \mathbf{t}_e \mathbf{t}_e^T \nabla(\Pi_K f - \kappa^2 u_h)(\bar{\mathbf{x}}_K)$$



Notation:

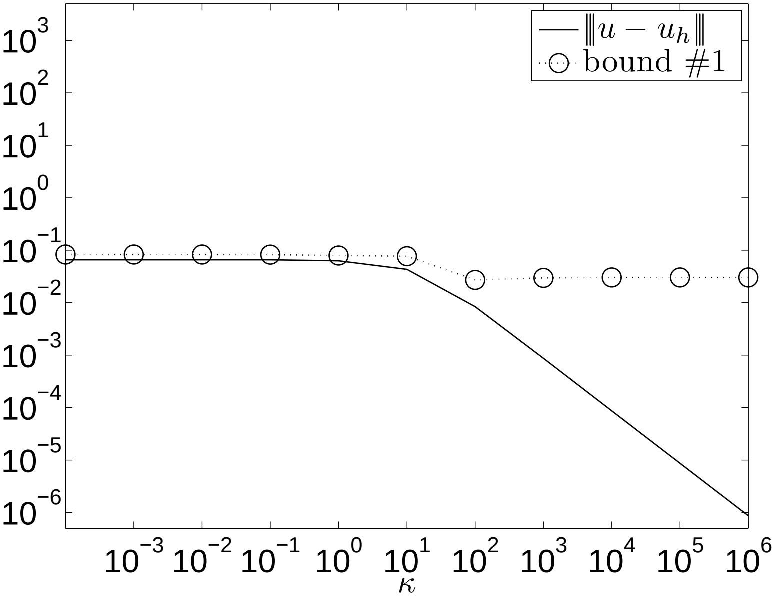
λ_n ... barycentric coords in K

$\beta_{e_1} = \lambda_1 \lambda_2$ ... edge bubble

$\mathbf{t}_{e_1} = \mathbf{x}_2 - \mathbf{x}_1$ ... edge vector

$\bar{\mathbf{x}}_K$ ... barycentre of K

Example



$\|u - u_h\|$
bound #1
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-3} 10^{-2} 10^{-1} 10^0 10^1 10^2 10^3 10^4 10^5 10^6
κ

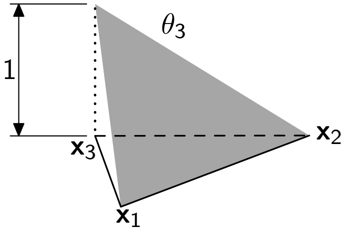
Robust equilibration of fluxes

Robust first-order equilibration:

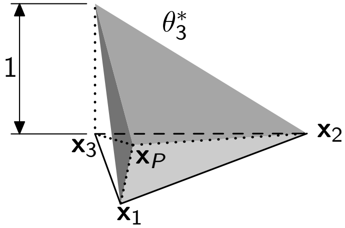
$$\left(f - \kappa^2 u_h, \theta_n^*\right)_K - (\nabla u_h, \nabla \theta_n^*)_K + \langle g_K, \theta_n^* \rangle_{\partial K} = 0 \quad \forall n \in \mathcal{N}(K)$$

If $\kappa \rho_K \leq 1$

$\Rightarrow \quad \theta_n^* = \theta_n$



If $\kappa \rho_K > 1$



$\mathbf{x}_P = \delta \mathbf{x}_1 + \delta \mathbf{x}_2 + (1 - 2\delta)\mathbf{x}_3$

$\delta = \frac{1}{2} \min\{1, 1/(\kappa \rho_K)\}$

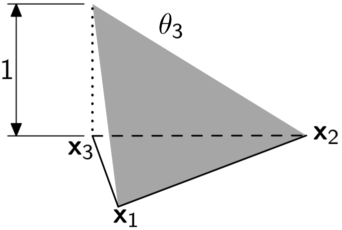
Robust equilibration of fluxes

Robust first-order equilibration:

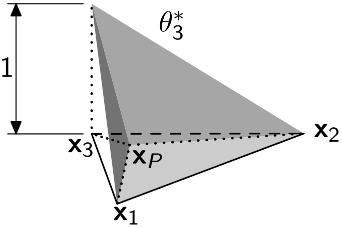
$$\left(f-\kappa^2 u_h,\theta_n^*\right)_K - (\nabla u_h, \nabla\theta_n^*)_K + \langle g_K,\theta_n^*\rangle_{\partial K} \overset{!!}{\approx} 0 \quad \forall n\in\mathcal{N}(K)$$

If $\kappa\rho_K \le 1$

$\Rightarrow \quad \theta_n^* = \theta_n$



If $\kappa\rho_K > 1$



$\mathbf{x}_P = \delta\mathbf{x}_1 + \delta\mathbf{x}_2 + (1-2\delta)\mathbf{x}_3$

$\delta = \frac{1}{2}\min\{1, 1/(\kappa\rho_K)\}$

Flux reconstruction #2

Definition:

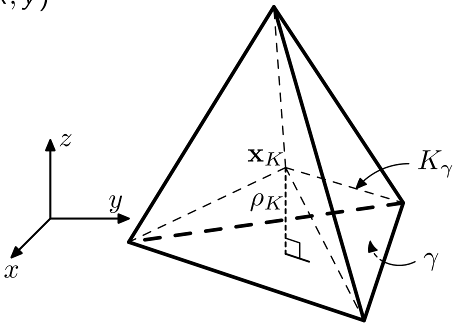
$$\mathbf{y}_K^{(2)} = \nabla u_h + \boldsymbol{\tau}_K^O$$

Properties:

- $\boldsymbol{\tau}_K^O \cdot \mathbf{n}_{K_\gamma} = 0 \quad \text{on } \partial K_\gamma \setminus \gamma$
- $\mathbf{y}_K^{(2)} \cdot \mathbf{n}_K = g_K \quad \forall K \in \mathcal{T}_h$

$$\boldsymbol{\tau}_K^O|_{K_\gamma} = \frac{1}{\rho_K} \mathrm{e}^{-\kappa z} (\mathbf{x} - \mathbf{x}_K) R(x,y)$$

$$R = g_K - \nabla u_h \cdot \mathbf{n}_K \text{ on } \gamma$$

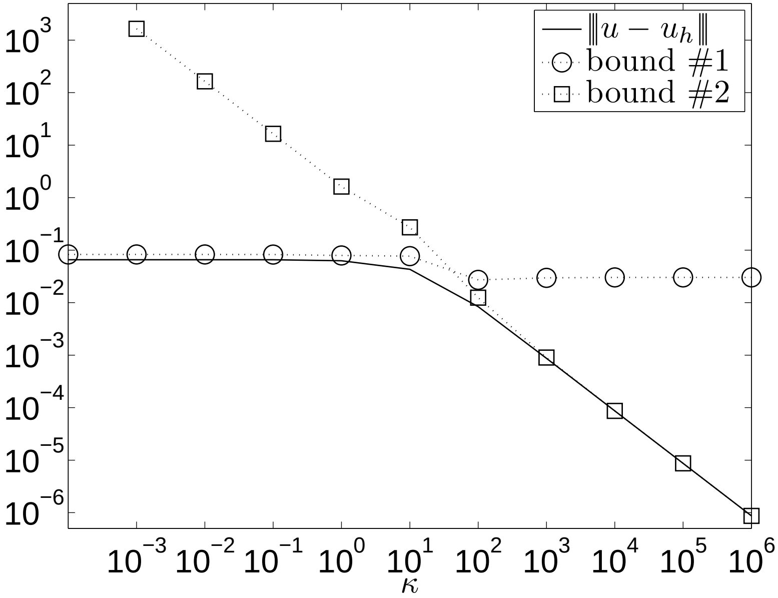


Notation:

$\mathbf{x}_K$ . . . incentre of K

ρ_K . . . inradius of K

Example


$\|u - u_h\|$
bound #1
bound #2
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-3} 10^{-2} 10^{-1} 10^{0} 10^{1} 10^{2} 10^{3} 10^{4} 10^{5} 10^{6}
κ

Local efficiency

$$\bar{\eta}_K = \min\left\{\eta_K(\mathbf{y}_K^{(1)}), \eta_K(\mathbf{y}_K^{(2)})\right\}$$

Upper bound:

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left[\bar{\eta}_K + \mathrm{osc}_K(f)\right]^2$$

Theorem

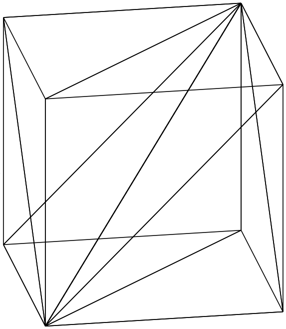
$\exists C > 0$ independent of h_K and κ:

$$\bar{\eta}_K \leq C |||u - u_h|||_{\widetilde{K}} + C\,\mathrm{osc}_{\widetilde{K}}(f)$$

Example 1

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\Omega = (-1/2, 1/2)^3$
- $f = \cos(\pi x_1)\cos(\pi x_2)\cos(\pi x_3)$
- $u = \dfrac{\cos(\pi x_1)\cos(\pi x_2)\cos(\pi x_3)}{3\pi^2 + \kappa^2}$

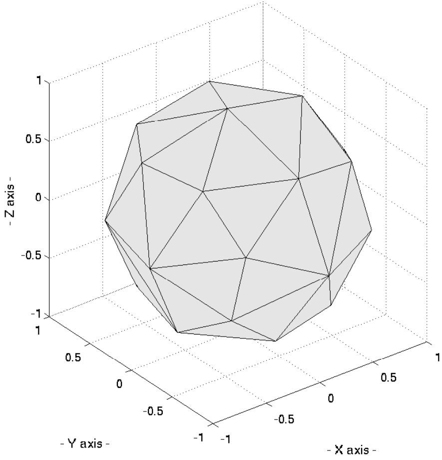


Example 2

$$-\Delta u + \kappa^2 u = 1 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

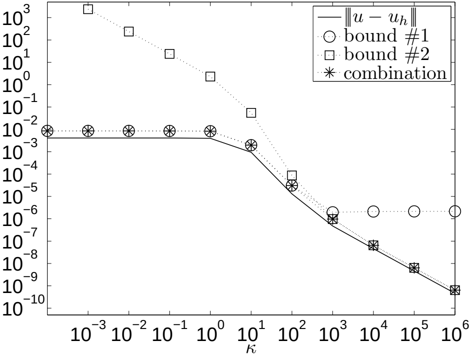
- $\Omega = \{(x_1, x_2, x_3) : r < 1\}$
- $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$
- $u(r, \theta, \phi) = \dfrac{1 - \sinh(\kappa r)}{\kappa^2 r \sinh \kappa}$

$$u = \frac{1 - r^2}{6} \qquad \text{for } \kappa = 0$$

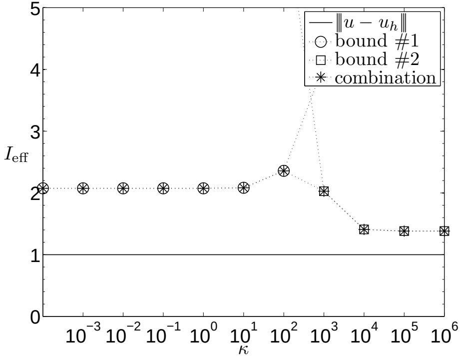


Example 1 (fixed mesh)

Error estimators



Effectivity indices

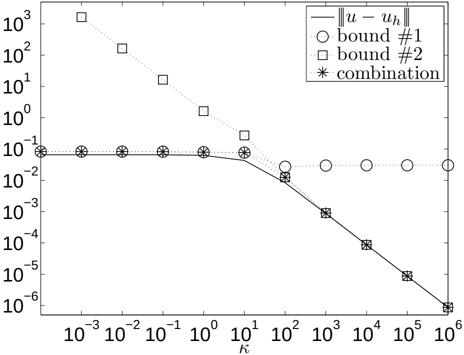


$N_{\text{DOF}} = 29791 \qquad h = 0.03125$

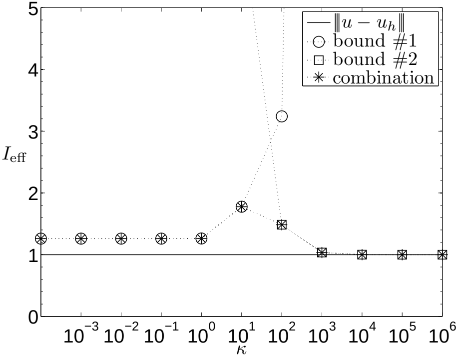
$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$

Example 2 (fixed mesh)

Error estimators



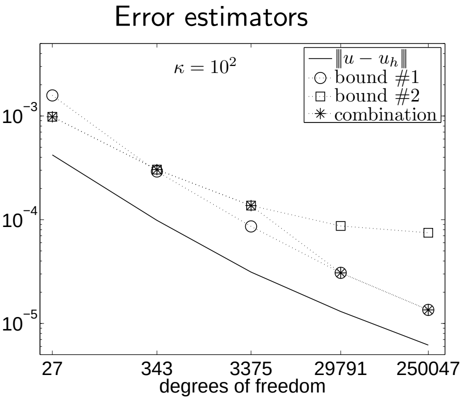
Effectivity indices

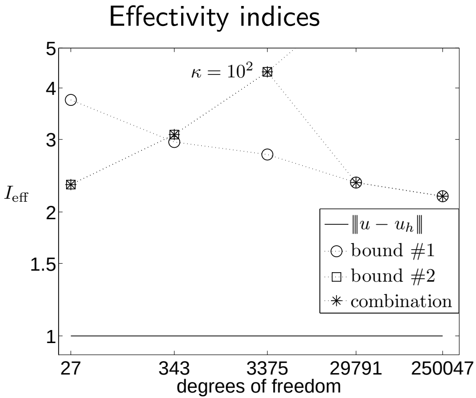


$N_{\mathrm{DOF}} = 751 \qquad h = 0.425096$

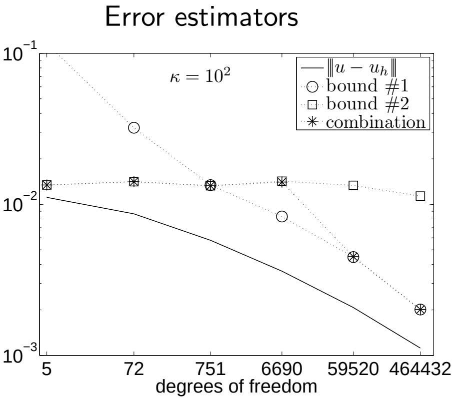
$$I_{\mathrm{eff}} = \frac{\eta}{\|u-u_h\|}$$

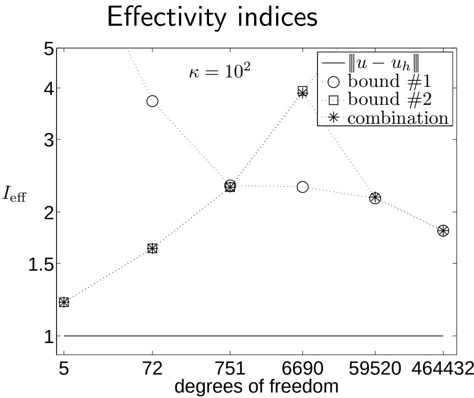
Example 1, uniform refinement, $\kappa = 100$





Example 2, uniform refinement, $\kappa = 100$





Conclusions

- Guaranteed upper bound
- Local efficiency
- Robustness
- No constants
- Explicit flux reconstruction
- Fast algorithm

Outlook

- Better reconstruction for small and intermediate κ
- Higher-order

Thank you for your attention

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