The influence of non-standard boundary conditions on the generation of spatial patterns

Milan Kučera, Filip Jaroš, Tomáš Vejchodský (Prague)

March 9, 2012

$$\frac{\partial u}{\partial t} = d_1 \Delta u + f(u,v) \text{ in } (0,\infty)\times\Omega,$$
$$\frac{\partial v}{\partial t} = d_2 \Delta u + g(u,v) \text{ in } (0,\infty)\times\Omega,$$
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N,\ \ u = v = 0 \text{ on } \Gamma_D,$$

where $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \partial\Omega$.

Assumptions:

There is a spatially homogeneous stationary solution $u \equiv \bar{u},\ \ v \equiv \bar{v}$, i.e. $\bar{u}, \bar{v} \in R^+,\ f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$.

Without loss of generality, we can shift this steady state to zero, and the new functions (denoted again by u and v) describe now the deviations from the original basic steady state $\bar{u},\ \bar{v}$.

$$\frac{\partial u}{\partial t} = d_1 \Delta u + f(u,v) \text{ in } (0,\infty) \times \Omega,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta u + g(u,v) \text{ in } (0,\infty) \times \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, \;\; u = v = 0 \text{ on } \Gamma_D,$$

where $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \partial\Omega$.

Assumptions:

There is a spatially homogeneous stationary solution $u \equiv \bar{u}, \;\; v \equiv \bar{v}$, i.e. $\bar{u}, \bar{v} \in R^+, \; f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$.

Without loss of generality, we can shift this steady state to zero, and the new functions (denoted again by u and v) describe now the deviations from the original basic steady state $\bar{u}, \;\bar{v}$.

$$\frac{\partial u}{\partial t} = d_1 \Delta u + f(u,v) \text{ in } (0,\infty) \times \Omega,$$
$$\frac{\partial v}{\partial t} = d_2 \Delta u + g(u,v) \text{ in } (0,\infty) \times \Omega,$$
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, \;\; u = v = 0 \text{ on } \Gamma_D,$$

where $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \partial\Omega$.

Assumptions:

There is a spatially homogeneous stationary solution $u \equiv \bar{u}, \;\; v \equiv \bar{v}$, i.e. $\bar{u}, \bar{v} \in R^+, \; f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$.

Without loss of generality, we can shift this steady state to zero, and the new functions (denoted again by u and v) describe now the deviations from the original basic steady state $\bar{u}, \;\; \bar{v}$.

After the shift of the steady state to zero, the new functions (denoted again by u and v) describe the deviations from the original basic steady state $\bar{u}$ and $\bar{v}$, and the system reads as

$$\frac{\partial u}{\partial t} = d_1 \Delta u + b_{11} u + b_{12} v + n_1(u, v), \text{ in } (0, \infty) \times \Omega,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta u + b_{21} u + b_{22} v + n_2(u, v), \text{ in } (0, \infty) \times \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, \ u = v = 0 \text{ on } \Gamma_D,$$

where $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \partial\Omega$.

The bifurcation of spatial patterns can occure only for $\frac{d_2}{d_1} > T$ – the slope of the joint tangent to the hyperbolas C_j.

After the shift of the steady state to zero, the new functions (denoted again by u and v) describe the deviations from the original basic steady state $\bar{u}$ and $\bar{v}$, and the system reads as

$$\frac{\partial u}{\partial t} = d_1 \Delta u + b_{11} u + b_{12} v + n_1(u,v), \text{ in } (0,\infty) \times \Omega,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta u + b_{21} u + b_{22} v + n_2(u,v), \text{ in } (0,\infty) \times \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N, \ u = v = 0 \text{ on } \Gamma_D,$$

where $\Gamma_N \cap \Gamma_D = \emptyset$, $\Gamma_N \cup \Gamma_D = \partial\Omega$.

The bifurcation of spatial patterns can occure only for $\frac{d_2}{d_1} > T$ – the slope of the joint tangent to the hyperbolas C_j.

Signorini boundary condition (source) for v in $\Gamma_U \subset \partial\Omega$

$$\frac{\partial u}{\partial t} = d_1 \Delta u + b_{11} u + b_{12} v + n_1(u,v) \text{ in } (0,\infty)\times\Omega,$$
$$\frac{\partial v}{\partial t} = d_2 \Delta u + b_{21} u + b_{22} v + n_2(u,v) \text{ in } (0,\infty)\times\Omega,$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma_D,\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma_U \cup \Gamma_D,$$
$$v \geq 0,\ \frac{\partial v}{\partial n} \geq 0,\ v\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_U.$$
$$u = v = 0 \text{ on } \Gamma_D.$$

Unilateral condition (source) for v in $\Omega_U \subset \Omega$

$$\frac{\partial u}{\partial t} = d_1 \Delta u + b_{11} u + b_{12} v + n_1(u,v) \text{ in } (0,\infty) \times \Omega,$$

$$\frac{\partial v}{\partial t} = d_2 \Delta u + b_{21} u + b_{22} v + n_2(u,v) \text{ in } (0,\infty) \times \Omega \setminus \Omega_U,$$

$$v \geq 0,\ \frac{\partial v}{\partial t} - d_2 \Delta u - b_{21} u - b_{22} v - n_2(u,v) \geq 0 \text{ in } (0,\infty) \times \Omega_U,$$

$$v\left(\frac{\partial v}{\partial t} - d_2 \Delta u - b_{21} u - b_{22} v - n_2(u,v)\right) = 0 \text{ in } (0,\infty) \times \Omega_U,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N,\ u = v = 0 \text{ on } \Gamma_D,\ \text{where } \Gamma_N \cup \Gamma_D = \partial\Omega.$$

P. Drábek, M. Kučera, M. Míková, Czechoslovak Math. J. **35** (1985), 639–660.

P. Quittner, J. Reine Angew. Math. **380** (1987), no. 2, 1–13.

M. Bosák, M. Kučera

J. Eisner, M. Kučera

J. Baltaev, M. Kučera, J. Math. Anal. Appl. (2008)

J. Eisner, M. Kučera, M. Väth, Z. Anal. Anwend.,

always the assumption $\Gamma_D \neq \emptyset$

Essential progress - Dirichlet conditions removed:

J. Eisner, M. Väth, Location of bifurcation points for a reaction-diffusion system with Neumann-Signorini conditions. Adv. Nonlin. Stud. 11 (2011), 809–836

M. Kučera, M. Väth, Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions. J. Diff. Eq. 252 (2012), 2951–2982

P. Drábek, M. Kučera, M. Míková, Czechoslovak Math. J. **35** (1985), 639–660.
P. Quittner, J. Reine Angew. Math. **380** (1987), no. 2, 1–13.
M. Bosák, M. Kučera
J. Eisner, M. Kučera
J. Baltaev, M. Kučera, J. Math. Anal. Appl. (2008)
J. Eisner, M. Kučera, M. Väth, Z. Anal. Anwend.,

always the assumption $\Gamma_D \neq \emptyset$

Essential progress - Dirichlet conditions removed:

J. Eisner, M. Väth, Location of bifurcation points for a reaction-diffusion system with Neumann-Signorini conditions. Adv. Nonlin. Stud. 11 (2011), 809–836

M. Kučera, M. Väth, Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions. J. Diff. Eq. 252 (2012), 2951–2982

P. Drábek, M. Kučera, M. Míková, Czechoslovak Math. J. **35** (1985), 639–660.

P. Quittner, J. Reine Angew. Math. **380** (1987), no. 2, 1–13.

M. Bosák, M. Kučera

J. Eisner, M. Kučera

J. Baltaev, M. Kučera, J. Math. Anal. Appl. (2008)

J. Eisner, M. Kučera, M. Väth, Z. Anal. Anwend.,

always the assumption $\Gamma_D \neq \emptyset$

Essential progress - Dirichlet conditions removed:

J. Eisner, M. Väth, Location of bifurcation points for a reaction-diffusion system with Neumann-Signorini conditions. Adv. Nonlin. Stud. 11 (2011), 809–836

M. Kučera, M. Väth, Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions. J. Diff. Eq. 252 (2012), 2951–2982

If $\Gamma_D \neq \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations also for $\frac{d_2}{d_1} < T$, but (d_1, d_2) to the left from the vertical asymptote of the first hyperbola.

If $\Gamma_D = \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations for arbitrarily small $\frac{d_2}{d_1}$.

Complications in a study of bifurcation and stability for unilateral problems, which can be formulated in terms of variational inequalities:
these problems cannot be linearized and therefore approaches based on linearization do not work
eigenvalues of very simple problems of thsi type need not be isolated (can cover an interval)

If $\Gamma_D \neq \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations also for $\frac{d_2}{d_1} < T$, but (d_1, d_2) to the left from the vertical asymptote of the first hyperbola.

If $\Gamma_D = \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations for arbitrarily small $\frac{d_2}{d_1}$.

Complications in a study of bifurcation and stability for unilateral problems, which can be formulated in terms of variational inequalities:

these problems cannot be linearized and therefore approaches based on linearization do not work

eigenvalues of very simple problems of thsi type need not be isolated (can cover an interval)

If $\Gamma_D \neq \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations also for $\frac{d_2}{d_1} < T$, but (d_1, d_2) to the left from the vertical asymptote of the first hyperbola.

If $\Gamma_D = \emptyset$ and $\Gamma_U \neq \emptyset$ or $\Omega_U \neq \emptyset$ then there are bifurcations for arbitrarily small $\frac{d_2}{d_1}$.

Complications in a study of bifurcation and stability for unilateral problems, which can be formulated in terms of variational inequalities:
these problems cannot be linearized and therefore approaches based on linearization do not work
eigenvalues of very simple problems of thsi type need not be isolated (can cover an interval)

QUESTIONS:

IS THERE A REASONABLE BIOLOGICAL MOTIVATION FOR UNILATERAL CONDITIONS?

DO UNILATERAL SOURCES OR SINKS PLAY A ROLE IN SPATIAL PATTERNING IN BIOLOGY?

IS IT POSSIBLE TO EXPLAIN SOME CONCRETE PHENOMENA IN THIS WAY?

WHAT ABOUT A DARK STRIPE AROUND THE SPINE OR A WHITE BELLY WHICH IS OBSERVED AMONG SOME FELIDS?

QUESTIONS:

IS THERE A REASONABLE BIOLOGICAL MOTIVATION FOR UNILATERAL CONDITIONS?

DO UNILATERAL SOURCES OR SINKS PLAY A ROLE IN SPATIAL PATTERNING IN BIOLOGY?

IS IT POSSIBLE TO EXPLAIN SOME CONCRETE PHENOMENA IN THIS WAY?

WHAT ABOUT A DARK STRIPE AROUND THE SPINE OR A WHITE BELLY WHICH IS OBSERVED AMONG SOME FELIDS?

QUESTIONS:

IS THERE A REASONABLE BIOLOGICAL MOTIVATION FOR UNILATERAL CONDITIONS?

DO UNILATERAL SOURCES OR SINKS PLAY A ROLE IN SPATIAL PATTERNING IN BIOLOGY?

IS IT POSSIBLE TO EXPLAIN SOME CONCRETE PHENOMENA IN THIS WAY?

WHAT ABOUT A DARK STRIPE AROUND THE SPINE OR A WHITE BELLY WHICH IS OBSERVED AMONG SOME FELIDS?

QUESTIONS:

IS THERE A REASONABLE BIOLOGICAL MOTIVATION FOR UNILATERAL CONDITIONS?

DO UNILATERAL SOURCES OR SINKS PLAY A ROLE IN SPATIAL PATTERNING IN BIOLOGY?

IS IT POSSIBLE TO EXPLAIN SOME CONCRETE PHENOMENA IN THIS WAY?

WHAT ABOUT A DARK STRIPE AROUND THE SPINE OR A WHITE BELLY WHICH IS OBSERVED AMONG SOME FELIDS?

OBRAZEK JAGUARA

Biological hypothesis concerning unilateral sources

(1) The morphogens diffuse and undergo a chemical reaction in the extracelular space or they interact via special cells with receptors which produce morphogens (ligands) according to their concentrations (autocrine signalling). The concentration of one of the morphogens determines a prepattern.

(2) In some tissues (e.g. a neural crest) there are cells which produce morphogens only if their concentration is below certain thresholds (something like homeostasis known in some processes, e.g. in metabolism). If the concentration u or v is above a given threshold u_0 or v_0 then the related source is inactive, if the concentration u or v is about to decrease below u_0 or v_0 then the corresponding morphogen is produced and supplemented into the extracellular space.

(3) The threshold u_0 and v_0 mentioned is either equal or slightly less than the value of the basic stationary spatially homogeneous stationary state $\bar{u}$ and $\bar{v}$, respectively.

Biological hypothesis concerning unilateral sources

(1) The morphogens diffuse and undergo a chemical reaction in the extracelular space or they interact via special cells with receptors which produce morphogens (ligands) according to their concentrations (autocrine signalling). The concentration of one of the morphogens determines a prepattern.

(2) In some tissues (e.g. a neural crest) there are cells which produce morphogens only if their concentration is below certain thresholds (something like homeostasis known in some processes, e.g. in metabolism). If the concentration u or v is above a given threshold u_0 or v_0 then the related source is inactive, if the concentration u or v is about to decrease below u_0 or v_0 then the corresponding morphogen is produced and supplemented into the extracellular space.

(3) The threshold u_0 and v_0 mentioned is either equal or slightly less than the value of the basic stationary spatially homogeneous stationary state $\bar{u}$ and $\bar{v}$, respectively.

Biological hypothesis concerning unilateral sources

(1) The morphogens diffuse and undergo a chemical reaction in the extracelular space or they interact via special cells with receptors which produce morphogens (ligands) according to their concentrations (autocrine signalling). The concentration of one of the morphogens determines a prepattern.

(2) In some tissues (e.g. a neural crest) there are cells which produce morphogens only if their concentration is below certain thresholds (something like homeostasis known in some processes, e.g. in metabolism). If the concentration u or v is above a given threshold u_0 or v_0 then the related source is inactive, if the concentration u or v is about to decrease below u_0 or v_0 then the corresponding morphogen is produced and supplemented into the extracellular space.

(3) The threshold u_0 and v_0 mentioned is either equal or slightly less than the value of the basic stationary spatially homogeneous stationary state $\bar{u}$ and $\bar{v}$, respectively.

A motivation of Hypotheses (2) and (3)

The stationary state $\bar{u}, \ \bar{v}$ is asymptotically stable for initial time period of the development (e.g. as far as the domain is sufficiently small, etc.)
During this period, the concentrations of morphogens converge to $\bar{u}, \ \bar{v}$ and are close to these values for considerable time.
Similarily to the principle of homeostasis, the organism accepts this steady state as natural and has a tendency to keep the concentration of morphogens close to $\bar{u}, \ \bar{v}$ even if the stability of this spatially homogeneous equilibrium is lost (i.e. if the parameters reach the domain of Turing instability, e.g. the domain becomes sufficiently large) and the concentrations tend to a new spatially non-homogeneous state (spatial pattern).

Similar regulations based on negative feedback are known, e.g. Hh ligand (hedgehog) is absorbed by the multipass membrain protein Patched if the concentration of Hh ligand exceeds a given threshold.
Perrimon, N. and McMahon, A. P. (1999)

A motivation of Hypotheses (2) and (3)

The stationary state $\bar{u}$, $\bar{v}$ is asymptotically stable for initial time period of the development (e.g. as far as the domain is sufficiently small, etc.) During this period, the concentrations of morphogens converge to $\bar{u}$, $\bar{v}$ and are close to these values for considerable time.
Similarily to the principle of homeostasis, the organism accepts this steady state as natural and has a tendency to keep the concentration of morphogens close to $\bar{u}$, $\bar{v}$ even if the stability of this spatially homogeneous equilibrium is lost (i.e. if the parameters reach the domain of Turing instability, e.g. the domain becomes sufficiently large) and the concentrations tend to a new spatially non-homogeneous state (spatial pattern).

Similar regulations based on negative feedback are known, e.g. Hh ligand (hedgehog) is absorbed by the multipass membrain protein Patched if the concentration of Hh ligand exceeds a given threshold.
Perrimon, N. and McMahon, A. P. (1999)

Biological hypothesis concerning unilateral sinks

There are regions (e.g. ventral side) where the efficiency of morphogens is reduced by the presence of proteinic agents. This can be modelled as a sink which is active only if the concentration exceeds a given threshold, which is close to the basic equilibrium as mentioned above.

The process might be similar to colour pattern formation in the case of mice (e.g. the mouse of genus Peromyscus). Agouti protein is expressed in ventral regions of the mice and delays the maturation of melanocytes, which are responsible for pigmentation. As a result, the ventral side of Peromyscus mice display white colouration.

M. Manceau, V. S. Domingues, R. Mallarino, H. E. Hoekstra 2011.

Biological hypothesis concerning unilateral sinks

There are regions (e.g. ventral side) where the efficiency of morphogens is reduced by the presence of proteinic agents. This can be modelled as a sink which is active only if the concentration exceeds a given threshold, which is close to the basic equilibrium as mentioned above.

The process might be similar to colour pattern formation in the case of mice (e.g. the mouse of genus Peromyscus). Agouti protein is expressed in ventral regions of the mice and delays the maturation of melanocytes, which are responsible for pigmentation. As a result, the ventral side of Peromyscus mice display white colouration.
M. Manceau, V. S. Domingues, R. Mallarino, H. E. Hoekstra 2011.

A concrete model - R. A. Barrio, C. Varea, J. L. Aragón, P. K. Maini 1999

R. T. Liu, S. S. Liaw, P. K. Maini 2006

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

Stationary spatially non-homogeneous solutions (spatial patterns) bifurcate from the trivial solutions only for D small (at least $D < 1$) and δ small in some sense (i.e. sufficiently large domain.

In the paper Liu, Liaw, Maini 2006, succesfull experiments were done for $D = 0.45$, $\delta = 6$ and in the second phase even $D = 0.15$, $\delta = 1.8$.

A concrete model - R. A. Barrio, C. Varea, J. L. Aragón, P. K. Maini 1999

R. T. Liu, S. S. Liaw, P. K. Maini 2006

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

Stationary spatially non-homogeneous solutions (spatial patterns) bifurcate from the trivial solutions only for D small (at least $D < 1$) and δ small in some sense (i.e. sufficiently large domain.

In the paper Liu, Liaw, Maini 2006, succesfull experiments were done for $D = 0.45$, $\delta = 6$ and in the second phase even $D = 0.15$, $\delta = 1.8$.

A concrete model - R. A. Barrio, C. Varea, J. L. Aragón, P. K. Maini 1999

R. T. Liu, S. S. Liaw, P. K. Maini 2006

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

Stationary spatially non-homogeneous solutions (spatial patterns) bifurcate from the trivial solutions only for D small (at least $D < 1$) and δ small in some sense (i.e. sufficiently large domain.

In the paper Liu, Liaw, Maini 2006, succesfull experiments were done for $D = 0.45$, $\delta = 6$ and in the second phase even $D = 0.15$, $\delta = 1.8$.

Unilateral boundary conditions - Signorini

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$
$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$v \geq 0, \ \frac{\partial v}{\partial n} \geq 0, \ v\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_U.$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

Unilateral boundary conditions - Signorini

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$
$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$
$$v \geq 0,\ \frac{\partial v}{\partial n} \geq 0,\ v\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_U.$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

Unilateral boundary conditions - Signorini

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$
$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$
$$v \geq 0,\ \frac{\partial v}{\partial n} \geq 0,\ v\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_U.$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

Unilateral source in the interior of the domain

$$\frac{\partial u}{\partial t} = D\delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 \text{ in } \Omega$$

$$\frac{\partial v}{\partial t} = \delta \Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{ in } \Omega \setminus \Omega_U$$

$$v \geq 0, \ \frac{\partial v}{\partial t} = \delta \Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \geq 0, \text{ in } \Omega \setminus \Omega_U,$$

$$v \cdot \Big(\frac{\partial v}{\partial t} = \delta \Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2\Big) \text{ in } \Omega \setminus \Omega_U.$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

Unilateral source in the interior of the domain

$$\frac{\partial u}{\partial t} = D\delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 \text{ in } \Omega$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{ in } \Omega \setminus \Omega_U$$

$$v \geq 0,\ \frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \geq 0,\ \text{ in } \Omega \setminus \Omega_U,$$

$$v \cdot \Big(\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2\Big) \text{ in } \Omega \setminus \Omega_U.$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 \text{in } \Omega$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{in } \Omega \setminus \Omega_U$$

$$v \geq -\varepsilon,\ v\Big(\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2,\Big) \text{in } \Omega_U$$

$$\frac{\partial v}{\partial t} \geq \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{in } \Omega_U$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 \text{in } \Omega$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{in } \Omega \setminus \Omega_U$$

$$v \geq -\varepsilon,\ \ v\Big(\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2,\Big)\text{in } \Omega_U$$

$$\frac{\partial v}{\partial t} \geq \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 \text{in } \Omega_U$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega,$$

There exist bifurcation points (and therefore also spatial patterns) for arbitrarily large D and for arbitrarily small domains.

Numerical experiments – Tomáš Vejchodský, Michal Kozák

A unilateral source of the inhibitor:

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)(v + \varepsilon^-)$$

A unilateral source of both - the activator and the inhibitor:

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 + a(y)(u + \varepsilon_1)^-$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)(v + \varepsilon_2)^-$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)v^-$$

A unilateral source of the inhibitor:

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)(v + \varepsilon^-)$$

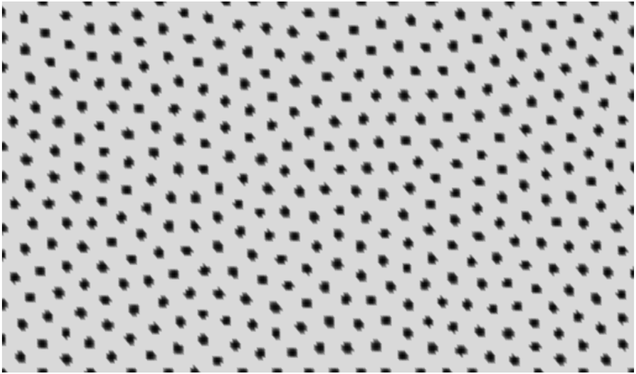
A unilateral source of both - the activator and the inhibitor:

$$\frac{\partial u}{\partial t} = D\delta\Delta u + \alpha u + v - r_2 uv - \alpha r_3 uv^2 + a(y)(u + \varepsilon_1)^-$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)(v + \varepsilon_2)^-$$

$$\frac{\partial v}{\partial t} = \delta\Delta v - \alpha u + \beta v + r_2 uv - \alpha r_3 uv^2 + a(y)v^-$$

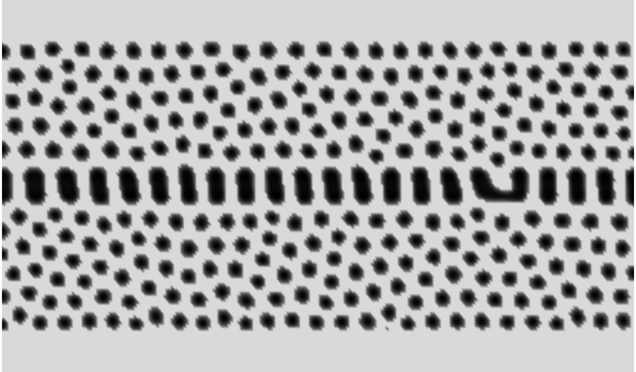
The classical case - no unilateral regulation



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 1: Initial condition noisy in $(-0.05, 0.05)$.

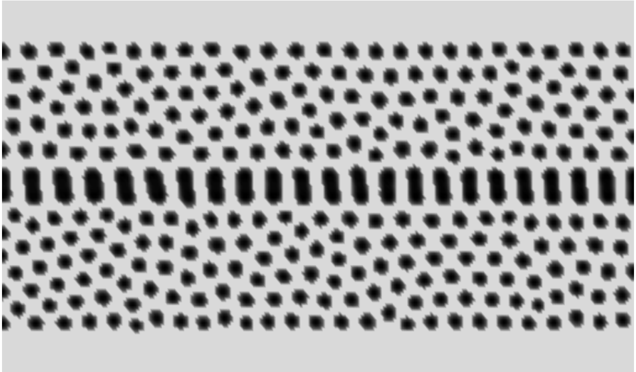
Unilateral regulation of v



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 2: Regulation of v:
$a_{v-} = 0.066$ (a very slight source on the neural crest)
$a^b_{v+} = 0.12$ (a slight sink on the belly)
$\theta_{v-} = \theta^b_{v+} = 0$ (thresholds at zero)
$w_1 = 28,\ w_2 = 30$ (width of the source and sink)

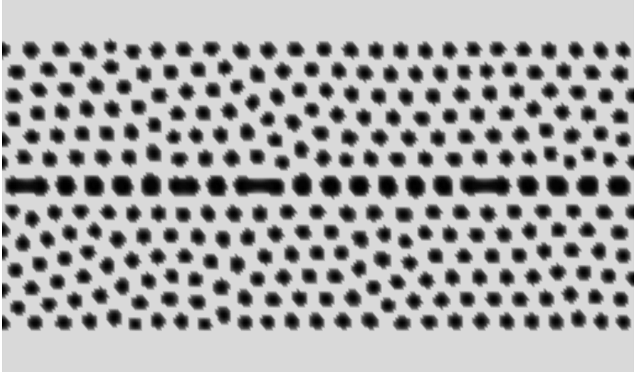
Unilateral regulation of v



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 3: Regulation for v:
$a_{v-} = 0.065$ (very slight source on the neural crest)
$a^b_{v+} = 0.12$ (slight sink on the belly)
$\theta_{v-} = 0.02$, $\theta_{v+} = 0$ (nonzero threshold)
$w_1 = 28$, $w_2 = 30$ (width of sources)

Unilateral regulation of v

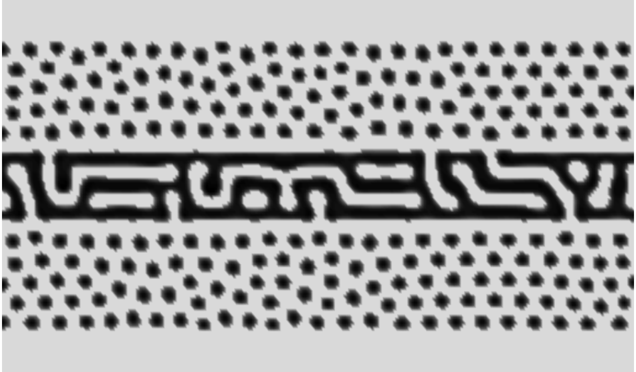


$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 4: Regulation for v:

$a_{v-} = 0.065$ (very slight source on the neural crest)
$a^b_{v+} = 0.12$ (slight sink on the belly)
$\theta_{v-} = 0.02$, $\theta_{v+} = 0$ (nonzero threshold)
$w_1 = 12$, $w_2 = 30$ (width of sources)

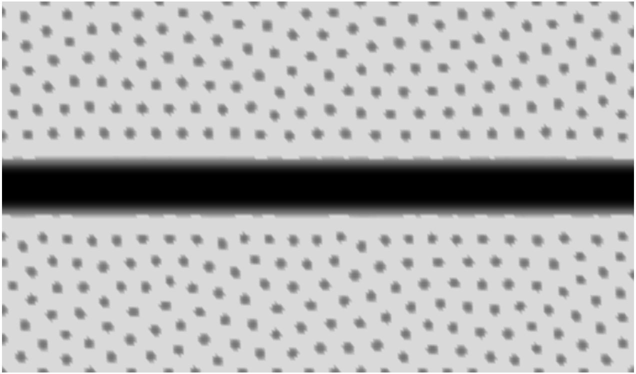
Unilateral regulation of v



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 5: Regulation for v:
$a_{v-} = 0.066$ (very slight source on the neural crest)
$a^b_{v+} = 0.12$ (slight sink on the belly)
$\theta_{v-} = -0.07$, $\theta_{v+} = 0$ (nonzero threshold)
$w_1 = 68$, $w_2 = 30$ (width of sources)

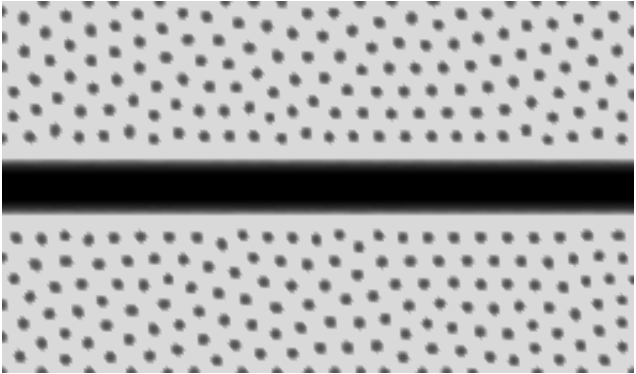
Unilateral regulation of both u and v



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 6: Regulation of both u and v:
$a_{u-} = a_{v-} = 5$ (strong source on the neural crest)
$a_{u+} = a_{v+} = 0.5$ (slight sink on the neural crest)
$\theta_{u-} = \theta_{v-} = \theta_{u+} = \theta_{v+} = 0.5$ (thresholds)
$w_1 = 40$ (width of the source and sink)

Unilateral regulation of both u and v



$D = 0.45$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 680 \times 400$

Fig. 7: Regulation of both u and v:
$a_{u-} = a_{v-} = 3.8$ (weaker source on the neural crest)
$a_{u+} = a_{v+} = 0.5$ (slight sink on the neural crest)
$\theta_{u-} = \theta_{v-} = \theta_{u+} = \theta_{v+} = 0.5$ (thresholds)
$w_1 = 40$ (width of the source and sink)

Higher ratio of diffusions $D = 0.8$

$D = 0.8$
$\delta = 6$
$\alpha = 0.899$
$\beta = 0.91$
$r_2 = 2$
$r_3 = 3.5$
$\Omega = 238 \times 140$

Fig. 8: Higher ratio of diffusions $D = 0.8$, regulation for v:
$a_{v-} = 0.08$ (small source on the neural crest)
$\theta_{v-} = 0$ (zero threshold)
$w_1 = 40$ (width of the source)