Guaranteed and robust error bounds for singularly perturbed problems in arbitrary dimension

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MAFELAP 2013, Brunel University, June 10–14, 2013

Outline

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = g_N \quad \text{on } \partial\Omega \qquad\qquad \kappa > 0$$

- Arbitrary dimension
- Neumann boundary conditions
- Guaranteed and robust upper bound on error
- Guaranteed bounds of trace constants

M. Ainsworth, T. Vejchodský: *Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems*, Numer. Math. 119 (2), 2011, 219–243.

Model problem

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Classical formulation:

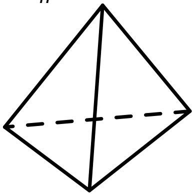
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = g_N \quad \text{on } \partial\Omega \qquad\qquad \kappa > 0$$

Weak formulation:

$$u \in V : \quad \mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V$$

Linear FEM on d-dimensional simplices:

$$u_h \in V_h : \quad \mathcal{B}(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h$$



Notation:

$V = H_0^1(\Omega)$

$\mathcal{B}(u, v) = (\nabla u, \nabla v)_\Omega + \kappa^2 (u, v)_\Omega$

$(f, v)_\Omega = \int_\Omega f v \,\mathrm{d}x$

$V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$

Main result

Upper bound:

$$\|\!|u-u_h\|\!|^2 \leq \sum_{K\in\mathcal{T}_h} \left[\eta_K(\mathbf{y}) + \mathrm{osc}_K(f) + \mathrm{osc}_{\partial\Omega\cap\partial K}(g_N)\right]^2$$

$$\forall \mathbf{y}\in\mathbf{H}(\mathrm{div},\Omega) : \mathbf{y}\cdot\mathbf{n} = \Pi_\gamma^K g_N \text{ on all } \gamma\subset\partial\Omega\cap\partial K$$

Local efficiency:

$$\eta_K(\mathbf{y}) \leq C\Big(\|\!|u-u_h\|\!|_{\widetilde{K}} + \min\{h_K,\kappa^{-1}\}\|f-\Pi f\|_{\widetilde{K}} + \min\{h_K,\kappa^{-1}\}^{1/2}\|g_N-\Pi_\gamma^K g_N\|_{\partial\Omega\cap\partial K}\Big) \text{ for a special } \mathbf{y}$$

- $\eta_K^2(\mathbf{y}) = \|\mathbf{y}-\nabla u_h\|_{0,K}^2 + \kappa^{-2}\|\Pi_K f - \kappa^2 u_h + \mathrm{div}\,\mathbf{y}\|_{0,K}^2$
- $\mathrm{osc}_K(f) = \min\left\{h_K/\pi, \kappa^{-1}\right\}\|f-\Pi_K f\|_{0,K}$
- $\mathrm{osc}_{\partial\Omega\cap\partial K}(g_N) = \min\{C_{\mathrm{T}}^K, C_{\mathrm{T}}^{K,\kappa}\}\|g_N-\Pi_\gamma^K g_N\|_{\partial\Omega\cap\partial K}$

- $\Pi_K f\in P^1(K) : \quad (f-\Pi_K f,\varphi)_K = 0 \quad \forall\varphi\in P^1(K)$
- $\Pi_\gamma^K g_N\in P^1(\gamma) : \quad (f-\Pi_\gamma^K g_N,\varphi)_\gamma = 0 \quad \forall\varphi\in P^1(\gamma)$

Flux reconstruction

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- Compute robust inter-element fluxes g_K
  - $g_K \approx \nabla u \cdot \mathbf{n}_K$ on ∂K

  [Ainsworth, Babuška, 1999], [Ainsworth, Vejchodský, 2011]
- For all elements K with $\kappa \rho_K \leq 1$ construct $\mathbf{y}_K^{(1)}$:
  - $\mathbf{y}_K^{(1)} \cdot \mathbf{n}_K = g_K \quad$ on ∂K
  - $\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K^{(1)} = 0$
- For all elements K with $\kappa \rho_K > 1$ construct $\mathbf{y}_K^{(2)}$:
  - $\mathbf{y}_K^{(2)} \cdot \mathbf{n}_K = g_K \quad$ on ∂K
  - Correct asymptotic behavior w.r.t. h and κ
- $\mathbf{y}|_K = \begin{cases} \mathbf{y}_K^{(1)} & \text{if } \kappa \rho_K \leq 1, \\ \mathbf{y}_K^{(2)} & \text{if } \kappa \rho_K > 1, \end{cases}$

Flux reconstruction #1

Definition:

$$\mathbf{y}_K^{(1)} = \nabla u_h + \mathbf{y}_K^L + \mathbf{y}_K^Q$$

$$\mathbf{y}_K^L = -\sum_{n=1}^{d+1} \lambda_n \sum_{\substack{m=1 \\ m\neq n}}^{d+1} R_{|\gamma_m}(\mathbf{x}_n)\, |\nabla \lambda_m|\, \mathbf{t}_{nm}$$

$$\mathbf{y}_K^Q = \frac{1}{d+1} \sum_{n=1}^{d+1} \sum_{\substack{m=2 \\ m>n}}^{d+1} \lambda_m \lambda_n \mathbf{t}_{mn} \mathbf{t}_{mn}^T \nabla r(\bar{\mathbf{x}}_K)$$

Notation:

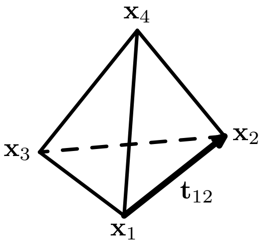
$R = g_K - \nabla u_h \cdot \mathbf{n}_K$

$r = \Pi_K f - \kappa^2 u_h$

λ_n ... barycentric coords in K

$\mathbf{t}_{mn} = \mathbf{x}_m - \mathbf{x}_n$ ... edge vector

$\bar{\mathbf{x}}_K$ ... barycentre of K



Flux reconstruction #2

Definition:

$$\mathbf{y}_K^{(2)} = \nabla u_h + \mathbf{y}_K^O$$

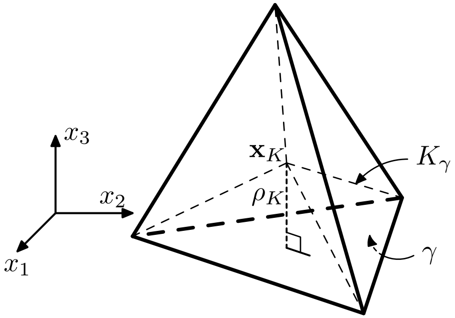
$$\mathbf{y}_K^O|_{K_\gamma} = \frac{1}{\rho_K} \mathrm{e}^{-\kappa x_d} (\mathbf{x} - \mathbf{x}_K) R(x_1, \ldots, x_{d-1}) \quad \text{for all } \gamma \subset \partial K$$

Notation:

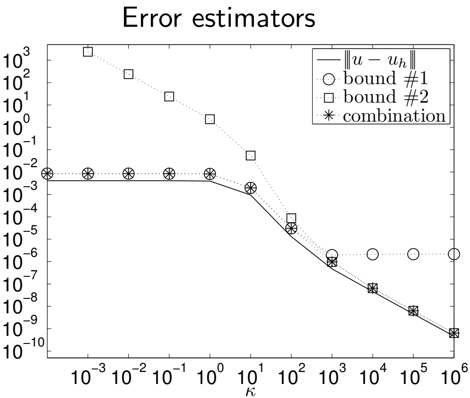
$R = g_K - \nabla u_h \cdot \mathbf{n}_K$

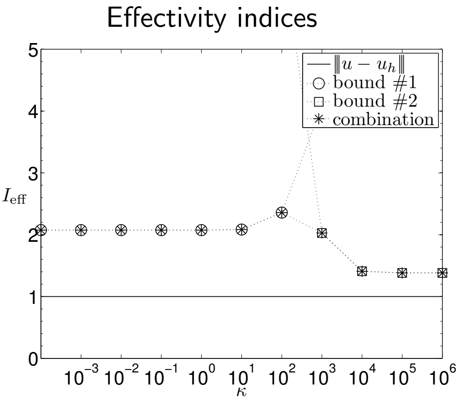
$\mathbf{x}_K$ ... incentre of K

ρ_K ... inradius of K



Example (cube)





$N_{\text{DOF}} = 29791 \qquad h = 0.03125$

$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$

Trace constants

Neumann oscillations:

$$\mathrm{osc}_{\partial\Omega\cap\partial K}(g_N) = \min\{C_{\mathrm{T}}^{K}, C_{\mathrm{T}}^{K,\kappa}\}\|g_N - \Pi_\gamma^K g_N\|_{\partial\Omega\cap\partial K}$$

Trace theorems:

(1) $\|v - \overline{v}\|_{0,\partial K} \leq C_{\mathrm{T}}^{K}\|\nabla v\|_{0,K} \quad \forall v \in H^1(K) \quad \overline{v} = \frac{1}{|\partial K|}\int_{\partial K} v\,\mathrm{d}x$

(2) $\|v\|_{0,\partial K} \leq C_{\mathrm{T}}^{K,\kappa}\left(\|\nabla v\|_{0,K}^2 + \kappa^2\|v\|_{0,K}^2\right)^{1/2} \quad \forall v \in H^1(K)$

Connection to eigenvalues:

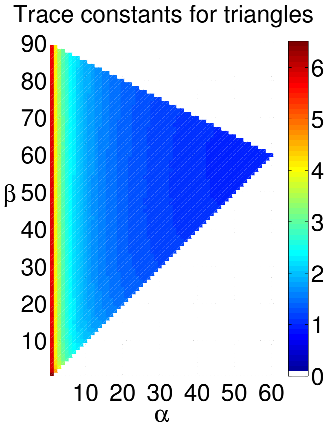
- $C_{\mathrm{T}}^{K} = \lambda_2^{-1/2}$, where λ_2 is the smallest positive eigenvalue:
  $u_i \in H^1(\Omega): \quad (\nabla u_i, \nabla v)_\Omega = \lambda_i (u_i, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$
- $C_{\mathrm{T}}^{K,\kappa} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:
  $u_i \in H^1(\Omega): \quad (\nabla u_i, \nabla v)_\Omega + \kappa^2 (u_i, v)_\Omega = \lambda_i (u_i, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$

Two-sided bounds: [Šebestová, Vejchodský, preprint 2013]

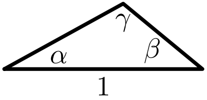
Trace constant (1) for triangles

$$\|v - \overline{v}\|_{0,\partial K} \leq C_{\mathrm{T}}^{K} \|\nabla v\|_{0,K} \quad \forall v \in H^1(K) \quad \overline{v} = \frac{1}{|\partial K|} \int_{\partial K} v \, \mathrm{d}x$$

Scaling: $C_{\mathrm{T}}^{K} = h_K^{1/2} C_{\mathrm{T},h=1}^{K}$



- Upper bound
- Error 1 %

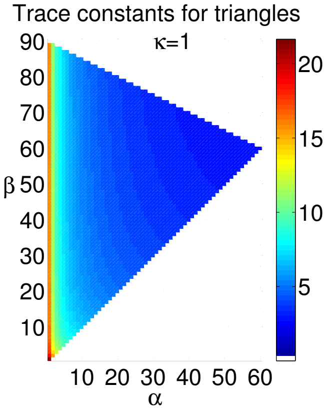


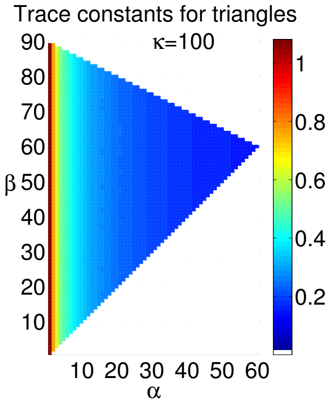
$\alpha \leq \beta \leq \gamma$

Trace constant (2) for triangles

$$\|v\|_{0,\partial K} \le C_{\mathrm{T}}^{K,\kappa}\left(\|\nabla v\|_{0,K}^2 + \kappa^2 \|v\|_{0,K}^2\right)^{1/2} \quad \forall v \in H^1(K)$$

Scaling: $C_{\mathrm{T}}^{K,\kappa} \le \max\{1,\kappa^{-1}\} h_K^{1/2} C_{\mathrm{T},h=1}^{K,\kappa=1}$





Trace constant (2) for triangles

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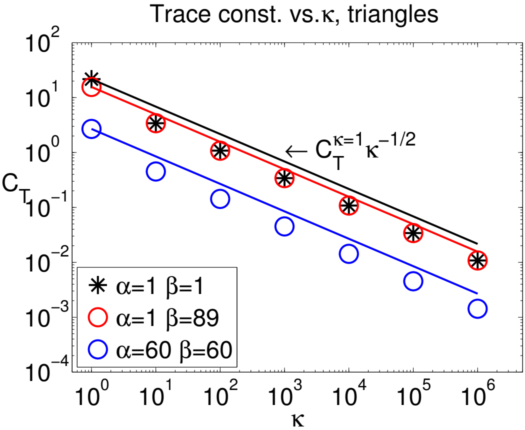
$$\|v\|_{0,\partial K} \leq C_{\mathrm{T}}^{K,\kappa} \left(\|\nabla v\|_{0,K}^2 + \kappa^2 \|v\|_{0,K}^2 \right)^{1/2} \quad \forall v \in H^1(K)$$

Numerical tests: $C_{\mathrm{T}}^{K,\kappa} \leq \kappa^{-1/2} h_K^{1/2} C_{\mathrm{T},h=1}^{K,\kappa=1}$ for $\kappa > 1$



Conclusions

- Fast flux reconstruction
- Guaranteed upper bound on error
- Robust for all values of κ
- Arbitrary dimension
- Neumann b.c. require bounds for trace constants

Thank you for your attention

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