Decentralized Control of Product (max+)-automata using Coinduction

Jan Komenda, Sébastien Lahaye, and Jean-Louis Boimond

Institute of Mathematics, Czech Academy of Sciences,
Brno, Czech Republic
and
LISA, ISTIA
Angers, France

2012 October 3,
WODES 2012
Guadalajara, Mexico

Outline

Outline

Control of (max,+) automata inspired by supervisory control

- **(Max,+) automata: weighted automata** with weights in $\overline{\mathbb{R}}_{\max} = (R \cup \{-\infty\}, \max, +)$.
- class of **Timed Discrete Event (dynamical) Systems (TDES)** with synchronization and resource sharing
- synchronous composition of (max,+)-automata: extended alphabet or non determinism
- Definitions by Coinduction of synchronous and supervised product
- Proofs by Coinduction of theorems modular synthesis equals global synthesis

Outline

(max,+)- automata

- **(max,+)-automata** generalize both logical automata and (max,+)-linear systems (e.g. timed event graphs)
- (max,+)-automata are quadruples $G = (Q, A, q_0, t)$, where Q set of states, q_0 initial state,
  A set of discrete events,
  $t : Q \times A \times Q \to \mathbb{R}_{max}$ transition function
  Meaning: output value $t(q, a, q') \in \mathbb{R}_{max}$ corresponds to the $a-$transition from q to q' and
  $t(q, a, q') = \varepsilon$ if there is no transition from q to q' labeled by a.
  Deterministic (max,+)-automata: t is deterministic, i.e.
  $t : Q \times A \to Q \times \mathbb{R}_{max}$

Deterministic (max,+)-automata as coalgebras

- Deterministic (max,+)-automata are coalgebras (S, t), where S is the set of states and the transition function is $t : S \to (1 + (\overline{\mathbb{R}}_{\max} \times S))^A$ with $1 = \{\emptyset\}$.
- Their behaviors are stream functions $f : A^\omega \to \overline{\mathbb{R}}_{\max}^\omega$.
  $f : A^\omega \to \overline{\mathbb{R}}_{\max}^\omega$ is *causal* if $\forall n \in \mathbb{N}, \sigma, \tau \in A^\infty$: $\forall i : i \leq n$: $\sigma(i) = \tau(i)$ then $f(\sigma)(n) = f(\tau)(n)$.
- Stream derivatives: $\omega = (\omega_0, \omega_1, \dots) \in K^\omega \to \omega' = (\omega_1, \omega_2, \dots)$.
- Stream functions form final coalgebra of (max,+)-automata with $t(f) = \langle f[a], f_a \rangle$ $f[a] = f(a : \sigma)(0)$ and $f_a(\sigma) = f(a : \sigma)'$
- $A^\infty = A^\omega \cup A^+$, where $A^+ = A^* \setminus \{\lambda\}$
  f is consistent if $\sigma \in A^\omega$: $f(\sigma)(k) = \emptyset \Rightarrow f(\sigma)(n) = \emptyset \; \forall n > k$.

Theorem. (Rutten 2006)

$\mathcal{F} = (\mathcal{F}, t_{\mathcal{F}})$ is the final coalgebra of (max,+)-automata:
$\mathcal{F} = \{f : A^\omega \to (1 + K)^\omega \mid f \text{ is causal and consistent}\}$.

$$t_{\mathcal{F}}(f)(a) = \begin{cases} \langle f[a], f_a \rangle & \text{if } f[a] \neq \emptyset \in 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Equivalent presentation of behaviors

- $s_0 \overset{\sigma(0)|k_0}{\longrightarrow} s_1 \overset{\sigma(1)|k_1}{\longrightarrow} s_2 \cdots \overset{\sigma(n)|k_n}{\longrightarrow} s_{n+1}$.
  We define $l(s_0)(\sigma)(n) = k_n$.
- $\mathcal{F}$ is isomorphic to functions between finite and infinite sequences!

$$\mathcal{F}_\infty = \{f : A^\infty \to \overline{\mathbb{R}}_{\max}^\infty | f \text{ preserve length, causal, } \& dom(f) \text{prefix-closed}\}$$

- $f[a] = f(a)(0)$ whenever f is defined for $a \in A$.
- $f_a : \; A^\infty \to (1 + \overline{\mathbb{R}}_{\max})^\infty$ given by $f_a(s) = f(a : s)'$

$$t_{\mathcal{F}_\infty}(f)(a) = \begin{cases} \langle f[a], f_a \rangle & \text{if } f[a] \text{ is defined} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

Residuation theory

Residuation theory generalizes inversion

An isotone $f : \mathcal{D} \rightarrow \mathcal{C}$, where $\mathcal{D}$ and $\mathcal{C}$ are dioids (naturally ordered $a \preceq b$ iff $a \oplus b = b$), is said to be **residuated** if there exists an isotone map $h : \mathcal{C} \rightarrow \mathcal{D}$ such that

$$f \circ h \preceq Id_{\mathcal{C}} \text{ and } h \circ f \succeq Id_{\mathcal{D}}.$$

h is unique residual of f, denoted by $f^\sharp$.

If f is residuated then $\forall y \in \mathcal{C}$, $\sup\{x \in \mathcal{D} | f(x) \preceq y\}$ exists and belongs to this subset and is equal to $f^\sharp(y)$.

Example: left and right multiplications are always residuated in complete dioids!

Notation.

$$\begin{aligned} a \backslash\!\!\circ\, y &= = max\{x \mid a \otimes x \leq y\} \text{ and} \\ y \,/\!\!\circ\, a &= \max\{x \mid x \otimes a \leq y\}. \end{aligned}$$

Supervisory control

Control framework: Given two deterministic (max,+) automata

$$G_c = (Q_c, q_{c,0}, Q_m^c, t_c), \ G = (Q_g, q_{g,0}, Q_m^g, t_g).$$

we consider their behaviors $y_c \in \mathcal{F}$ and $y \in \mathcal{F}$. Closed-loop system will be defined via ***supervised product***,
denoted $(y^c \otimes_{A_u} y$
Distinguish $A_c \subseteq A$ is the subset of *controllable events*, $A_u = A \backslash A_c$ is the subset of *uncontrollable events*.

Spec. $y^{ref} \in \mathcal{F}$ is **admissible** wrt $y \in \mathcal{F}$ if $L(y^{ref}) \subseteq L(y)$ and for all $w \in L(y^{ref})$ there is $y^{ref}(w) \geq y(w)$ (meant component-wise).

Controller $y^c \in \mathcal{F}$ is **admissible** wrt $y \in \mathcal{F}$ if $L(y^c) \subseteq L(y)$ and $\forall w \in L(y^{ref})$ there is $y^c(w) \geq 0$ (meant component-wise).

Supervisory control: coalgebraic framework

Notation. $y^{ref} : A^\infty \to (R_{max})^\infty$ is (an admissible) control specification

Natural order: for sequential functions $y, y' : A^\infty \to K^\infty$ we write $y \preceq y'$ iff $L(y) \subseteq L(y')$ and $\forall w \in L(y)$: $y(w) \leq y(w')$

Problem. Find a greatest admissible controller y_c such that $y^c \otimes_{A_u} y \preceq y^{ref}$.

Admissible controller: it does not disable nor delay uncontrollable events.

$L(y^{ref})$ is **controllable** wrt $L(y)$ and A_u if

$$\overline{L(y^{ref})} A_u \cap L(y) \subseteq \overline{L(y^{ref})}.$$

Outline

1. Introduction
2. Preliminaries from co/algebra and supervisory control
3. Centralized control using coalgebra
4. Decentralized control
5. CONCLUDING REMARKS

Supervised product by coinduction

Definition (Supervised product). Given System and Controller, resp. $y, y^c \in \mathcal{F}$, $\forall a \in A$:

$$(y^c \otimes_{A_u} y)_a = \begin{cases} y^c_a \otimes_{A_u} y_a & \text{if } y^c \xrightarrow{a} \text{ and } y \xrightarrow{a} \\ 0 \otimes_{A_u} y_a & \text{if } a \in A_{uc} \text{ and } y^c \not\xrightarrow{a} \text{ and } y \xrightarrow{a} \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$(y^c \otimes_{A_u} y)[a] = \begin{cases} y^c[a] \otimes y[a] & \text{if } a \in A_c \\ y[a] & \text{otherwise} \end{cases}$$

Example 1.

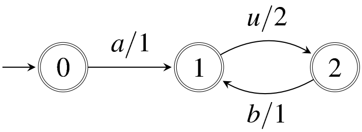

Figure: System automaton

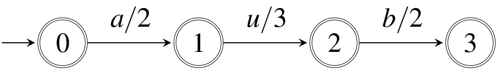

Figure: Controller automaton

Then $y(a(ub)^\omega) = (1(2,1)^\omega)$.

Example 1.

Controller y^c delays the first uncontrollable u, delays the first b, and tries to forbid the second u.

$$\begin{aligned}
(y^c \otimes_{A_u} y)[a] &= y^c[a] \otimes y[a] = 2 \otimes 1 = 3, \\
(y^c \otimes_{A_u} y)_a[u] &= y_a[u] = 2, \\
(y^c \otimes_{A_u} y)_{au}[b] &= y^c_{au}[b] \otimes y_{au}[b] = 2 \otimes 1 = 3, \\
(y^c \otimes_{A_u} y)_{aub}[u] &= y_{aub}[u] = 2.
\end{aligned}$$

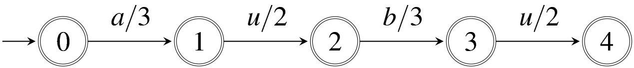

Figure: Closed-loop system automaton

Note that $(y^c \otimes_{A_u} y)_{aub} \xrightarrow{u}$, because $u \in A_u$ and $y_{aub} \xrightarrow{u}$ (even though $y^c_{aub} \not\xrightarrow{u}$).

Main result: least restrictive controller

Theorem 1. For any $y, y^{ref} \in \mathcal{F}$ with y^{ref} admissible with respect to y we have:

$$(y^{ref}/^{\sharp}_{A_u} y)_a = \begin{cases} (y^{ref})_a/^{\sharp}_{A_u} y_a & \text{if } \mathcal{C} \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$(y^{ref}/^{\sharp}_{A_u} y)[a] = \begin{cases} y^{ref}[a] \not\circ y[a] & \text{if } a \in A_c \text{ and } \mathcal{C} \\ \varepsilon & \text{if not } \mathcal{C} \\ T & \text{if } a \in A_u \text{ and } \mathcal{C} \end{cases},$$

where the auxiliary condition $\mathcal{C}$ is defined as
$y^{ref} \xrightarrow{a}$ **and** $y \xrightarrow{a}$ **and** $\forall u \in A_u^* : \ y_a \xrightarrow{u} \Rightarrow \ y_a^{ref} \xrightarrow{u}$.

Example 1 continued.

Let $y(a(ub)^{\omega}) = (1(2,1)^{\omega})$.

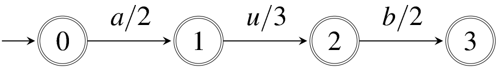

Figure: Specification automaton

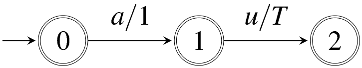

Figure: Controller $(y^{ref}/^{\sharp}_{A_u} y)$

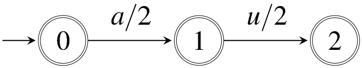

Figure: Closed-loop system $(y^{ref}/^{\sharp}_{A_u} y)) \otimes_{A_u} y$

Synchronous product defined by coinduction

Extended alphabet $\mathcal{A} = (A_1 \cap A_2) \cup (A_1 \setminus A_2)^* \times (A_2 \setminus A_1)^*$

For $l_i \in \mathcal{F}$ over A_i and $v_i = a_1 \dots a_k \in A_i^+$ we define for $i = 1, 2$:

$$l_i[v_i] = (l_i)[a_1] \otimes (l_i)_{a_1}[a_2] \otimes \cdots \otimes (l_i)_{a_1 \dots a_{k-1}}[a_k].$$

Definition. for $l_1, l_2 \in \mathcal{F}$ and $\forall v \in \mathcal{A}$:

$$\begin{aligned} (l_1 \| l_2)_v &= (l_1)_{P_1(v)} \| (l_2)_{P_2(v)} \text{ and} \\ (l_1 \| l_2)[v] &= l_1[P_1(v)] \otimes Bl_2[P_2(v)] \oplus Bl_1[P_1(v)] \otimes l_2[P_2(v)]. \end{aligned}$$

Synchronous product continued

$$(l_1 \| l_2)[v] = \begin{cases} max(l_1[P_1(v)], l_2[P_2(v)]) & \text{if } l_i[P_i(v)] \neq \varepsilon \text{ for } i = 1,2 \\ \varepsilon & \text{else, i.e. } \exists i = 1,2 : l_i[P_i(v)] = \varepsilon \end{cases}$$

Hint for understanding:
for partial languages $L_1 = (L_1^1, L_1^2)$, $L_2 = (L_2^1, L_2^2)$, and $w \in A^*$
we have in fact

$$(L_1 \| L_2)_w = (L_1)_{P_1(w)} \| (L_2)_{P_2(w)}.$$

Outline

Problem formulation

Definition. $(P_1(v), P_2(v))$ is controllable iff there exists a controllable event in either $P_1(v)$ or in $P_2(v)$.
Equiv.: $(P_1(v), P_2(v)) \in \mathcal{A}_u$ is uncontrollable iff both local strings $P_1(v) \in A_{u,1}^*$ and $P_2(v) \in A_{u,2}^*$.
Problem. When does global control synthesis equal decentralized one?

The global control synthesis amounts to computing

$$(y_1^{ref} \| y_2^{ref})/^{\sharp}_{A_u}(y_1 \| y_2) \otimes_{A_u} (y_1 \| y_2),$$

while local control synthesis amounts to computing

$$[y_1^{ref}/^{\sharp}_{A_u^1} y_1 \otimes_{A_u} y_1] \parallel [y_2^{ref}/^{\sharp}_{A_u^2} y_2 \otimes_{A_u} y_2].$$

Main result: decentralized vs. global control

We say that:

- local subsystems agree on the controllability status of shared events if $A_{u,1} \cap A_2 = A_{u,2} \cap A_1$.
- Local languages $L_i,\ i = 1, 2$ are mutually controllable if L_1 is controllable with respect to $P_1 P_2^{-1}(L_2)$ and $A_{u,1} \cap A_2$ and L_2 is controllable with respect to $P_2 P_1^{-1}(L_1)$ and $A_1 \cap A_{u,2}$.
- local specifications do not require to delay locally uncontrollable events if for all $u_i \in A_{u,i}$ we have $y_i[u_i] = y_i^{ref}[u_i]$.

Theorem. Let $y = y_1 \| y_2 : A^\infty \to (\mathbb{R}_{max})^\infty$ be the global behavior and $y^{ref} = y_1^{ref} \| y_2^{ref}$ the global specification. If the local languages $L(y_1)$ and $L(y_2)$ are mutually controllable, if the local subsystems agree on the controllability status of shared events and if local specifications do not require to delay locally uncontrollable events then

$$(y^{ref} /^{\sharp}_{A_u} y) \otimes_{A_u} y = ([y_1^{ref} /^{\sharp}_{A_{u,1}} y_1] \otimes_{A_u} y_1) \| ([y_2^{ref} /^{\sharp}_{A_{u,2}} y_2] \otimes_{A_u} y_2).$$

Main result: discussion

Remark.

- Problematic case: a concurrent event composed of a controllable component event in the first component and a uncontrollable component event in the second component $(b, c) \in \mathcal{A})$.
- From a timing viewpoint decentralized control yields the first output equal to $y_1^{ref}[b] \oplus y_2[c]$,
  while global control yields the first output equal to $y_1^{ref}[b] \oplus y_2^{ref}[c]$.
- Note that due to admissibility of the (local) specifications we always have $y_2[c] \leq y_2^{ref}[c]$ (the same as in the purely logical setting.).
- Conclusion: the same inequality holds in general as in the purely logical setting!

Outline

Conclusion

- synchronous and supervised composition of deterministic (max,+)-automata by coinduction
- Supervisory control: residuation theory
- Centralized supervision: coinductive formula for
- Application to decentralized supervisory control of (max,+)-automata: sufficient conditions for local control synthesis equals global control synthesis
- Controllability and Supremal controllable (max,+) series
- More work on control of (max,+)-automata is needed: control with partial observations, coordination control.