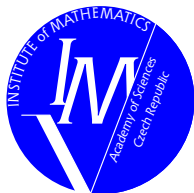


Computing upper bounds on Friedrichs' constant

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May 2–5, AM2012, Prague
in honor of the 60th birthday of Michal Křížek



Motivation – complementary error estimates

Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in V$$

Error bound: $u_h \in V$

$$\|u - u_h\| \leq C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Notation:

- ▶ $L^2(\Omega)$ scalar product: $(\varphi, \psi) = \int_{\Omega} \varphi \psi \, dx$
- ▶ $L^2(\Omega)$ norm: $\|v\|_0^2 = (v, v)$
- ▶ Energy norm: $\|e\|^2 = (\nabla e, \nabla e) = \|\nabla e\|_0^2$



Motivation – complementary error estimates

Friedrich's inequality: $\|v\|_0 \leq C_F \|\nabla v\|_0 \quad \forall v \in V$

Divergence thm: $(\operatorname{div} \mathbf{y}, v) + (\mathbf{y}, \nabla v) = 0 \quad \forall v \in V, \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

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Proof: $v = u - u_h \in V$

$$\begin{aligned} (\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) \\ &= (f + \operatorname{div} \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\ &\leq \|f + \operatorname{div} \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\ &\leq (C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0) \|v\| \end{aligned}$$

□



Relation with eigenvalues

Friedrichs' inequality:

$$\|v\|_0 \leq C_F \|\nabla v\|_0 \quad \forall v \in V \quad \Rightarrow \quad C_F = \sup_{v \in V} \frac{\|v\|_0}{\|\nabla v\|_0}$$

Laplace eigenvalue problem

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega, \quad u_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, \dots$$

Theorem: $C_F^2 = \frac{1}{\lambda_1}$ where $\lambda_1 = \min_i \lambda_i$.



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Proof:

Weak formulation: $u_i \in V : (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \quad \Leftrightarrow \quad \frac{1}{\lambda_1} = \sup_{v \in V} \frac{\|v\|_0^2}{\|\nabla v\|_0^2}$$



Rayleigh–Ritz approximation of λ_1

Weak formulation:

$$u_j \in V : \quad (\nabla u_j, \nabla v) = \lambda_j(u_j, v) \quad \forall v \in V$$

Rayleigh–Ritz method: $V^h \subset V$, $\dim V^h < \infty$

$$u_j^h \in V^h : \quad (\nabla u_j^h, \nabla v^h) = \lambda_j^h(u_j^h, v^h) \quad \forall v^h \in V^h$$

Theorem: $\lambda_1 \leq \lambda_1^h$



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Theorem: $\lambda_1 \leq \lambda_1^h$

Proof:

$$\lambda_1 = \inf_{v \in V} \frac{\|\nabla v\|_0^2}{\|v\|_0^2} \leq \inf_{v^h \in V^h} \frac{\|\nabla v^h\|_0^2}{\|v^h\|_0^2} = \lambda_1^h$$

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Corollary: $C_F^h \leq C_F$

Lower bound on λ_1

Method of *a priori-a posteriori inequalities*.

Theorem (Kuttler and Sigillito, 1978):

$$\Rightarrow \min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0},$$

where

- ▶ $\lambda_* \in \mathbb{R}$ and $u_* \in V$ are arbitrary
- ▶ $w \in V$: $(\nabla w, \nabla v) = (\nabla u_*, \nabla v) - \lambda_*(u_*, v) \quad \forall v \in V$

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$$\min_i \left| \frac{\lambda_i - \lambda_*}{\lambda_i} \right| \leq \frac{\|w\|_0}{\|u_*\|_0} \leq C_F \frac{\|\nabla w\|_0}{\|u_*\|_0}$$

Algorithm

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$$\|\nabla w\|_0 \leq \|\nabla u_* - \mathbf{q}\|_0 + C_F \|\lambda_* u_* + \operatorname{div} \mathbf{q}\|_0 \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}, \Omega).$$

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Algorithm



$$\min_i \left| \frac{\lambda_i - \lambda_1^h}{\lambda_i} \right| \leq \frac{1}{\sqrt{\lambda_1}} \left(\alpha + \frac{1}{\sqrt{\lambda_1}} \beta \right)$$

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- ▶ λ_1^h is closest to λ_1 (i.e. $i = 1$)



Algorithm

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}} \left(\alpha + \frac{1}{\sqrt{\lambda_1}} \beta \right)$$

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$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}} \left(\alpha + \frac{1}{\sqrt{\lambda_1}} \beta \right) \quad (*)$$

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- ▶ λ_1^h is closest to λ_1 (i.e. $i = 1$)
- ▶ (*) $\Leftrightarrow 0 \leq X^2 + \alpha X + \beta - \lambda_1^h$, where $X = \sqrt{\lambda_1}$

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- ▶ $\Rightarrow X_2^2 \leq \lambda_1$, where $X_2 = \left(\sqrt{\alpha^2 + 4(\lambda_1^h - \beta)} - \alpha \right) / 2$

$$\frac{\lambda_1^h - \lambda_1}{\lambda_1} \leq \frac{1}{\sqrt{\lambda_1}} \left(\alpha + \frac{1}{\sqrt{\lambda_1}} \beta \right) \quad (*)$$

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- ▶ $\Rightarrow C_F \leq 1/X_2$



Computing $\mathbf{q}_h \in \mathbf{H}(\text{div}, \Omega)$

$$\begin{aligned} & \left(\|\nabla u_* - \mathbf{q}\|_0 + C_F \|\lambda_* u_* + \text{div } \mathbf{q}\|_0 \right)^2 \\ & \approx \left(\|\nabla u_1^h - \mathbf{q}\|_0 + (\lambda_1^h)^{-1/2} \|\lambda_1^h u_1^h + \text{div } \mathbf{q}\|_0 \right)^2 \\ & \leq (1 + \varrho^{-1}) \|\nabla u_1^h - \mathbf{q}\|_0^2 + \frac{1 + \varrho}{\lambda_h} \|\lambda_1^h u_1^h - \text{div } \mathbf{q}\|_0^2, \quad \forall \varrho > 0 \end{aligned}$$

Minimize over $W_h \subset \mathbf{H}(\text{div}, \Omega)$:

Find $\mathbf{q}_h \in W_h$:

$$(\text{div } \mathbf{q}_h, \text{div } \boldsymbol{\psi}_h) + \frac{\lambda_1^h}{\varrho} (\mathbf{q}_h, \boldsymbol{\psi}_h) = \frac{\lambda_1^h}{\varrho} (\nabla u_1^h, \boldsymbol{\psi}_h) - (\lambda_1^h u_1^h, \text{div } \boldsymbol{\psi}_h) \quad \forall \boldsymbol{\psi}_h \in W_h$$

Solve by standard Raviart-Thomas finite elements.

Linear elliptic problem:

$$\begin{aligned} -\operatorname{div}(\mathcal{A}\nabla u) + cu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_D \\ n^T \mathcal{A}\nabla u &= 0 \text{ on } \Gamma_N \end{aligned}$$



Friedrich's inequality:

$$\|v\|_0 \leq C_F \|v\| \quad \forall v \in V$$

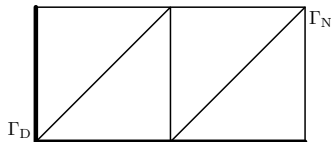
- ▶ $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$
- ▶ $\|v\|^2 = (\mathcal{A}\nabla u, \nabla u) + (cu, u)$

Example 1

$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$

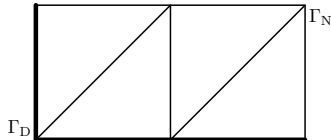


$$f = \frac{5\pi^2}{16} u$$

$$u = \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

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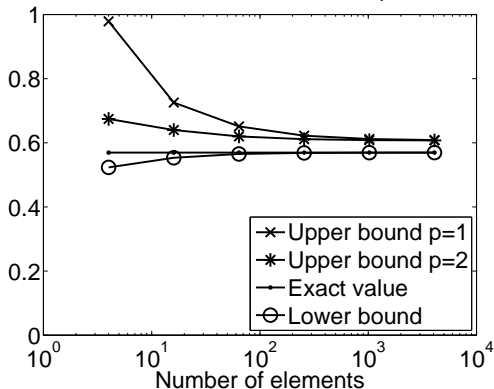
$$u = \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

$$C_F = \frac{4}{\sqrt{5}\pi} \doteq 0.5694$$

$$C_F^{\text{low}} = 0.5693$$

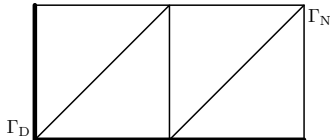
$$C_F^{\text{up}} = 0.6004$$

Friedrichs' constant – Example 1



Example 1

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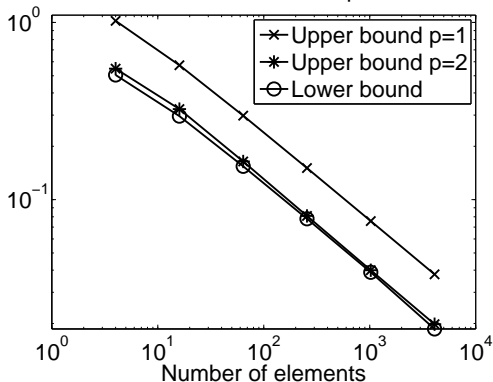
Lower bound:

$$\|u_h^{\text{ref}} - u_h\| \leq \|u - u_h\|$$

Upper bound:

$$\begin{aligned}
 &\|u - u_h\| \\
 &\leq C_F \|f + \text{div } \mathbf{y}\|_0 \\
 &\quad + \|\mathbf{y} - \nabla u_h\|_0 \\
 &\quad \forall \mathbf{y} \in \mathbf{H}(\text{div}, \Omega)
 \end{aligned}$$

Error bounds – Example 1

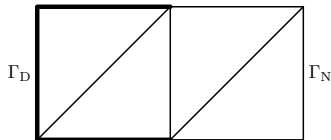


Example 2

$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$



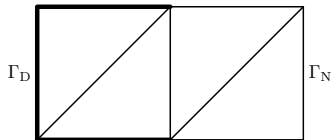
$$f = \frac{5\pi^2}{16} \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

Example 2

$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$



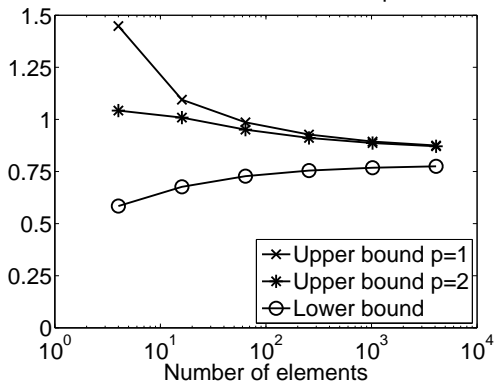
$$f = \frac{5\pi^2}{16} \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

$$C_F = ?$$

$$C_F^{\text{low}} = 0.7750$$

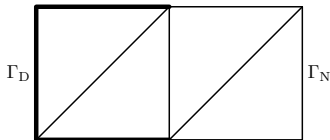
$$C_F^{\text{up}} = 0.8712$$

Friedrichs' constant – Example 2



Example 2

$$\begin{aligned}
 -\Delta u &= f \text{ in } (0, 2) \times (0, 1) \\
 u &= 0 \text{ on } \Gamma_D \\
 \mathbf{n}^\top \nabla u &= 0 \text{ on } \Gamma_N
 \end{aligned}$$



Error bounds – Example 2

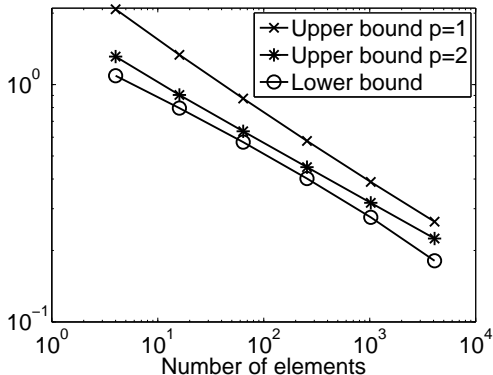
Lower bound:

$$\|u_h^{\text{ref}} - u_h\| \leq \|u - u_h\|$$

Upper bound:

$$\begin{aligned}
 &\|u - u_h\| \\
 &\leq C_F \|f + \text{div } \mathbf{y}\|_0 \\
 &\quad + \|\mathbf{y} - \nabla u_h\|_0
 \end{aligned}$$

$$\forall \mathbf{y} \in \mathbf{H}(\text{div}, \Omega)$$



Conclusions



- ▶ Practical method
- ▶ Guaranteed upper bound on Friedrichs' constant
- ▶ Easy to generalize to similar inequalities
- ▶ Computationally demanding
- ▶ Exact representation of the domain Ω

