Mathematical Properties of the Flows of Incompressible Fluids with Pressure and Shear Rate Dependent Viscosities†

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Contents

1. Introductory remarks

This thesis consists of a collection of eleven articles, ten of them published during the last five years, that have a relevance to the mathematical analysis of flows of

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incompressible fluids with the viscosity depending on the pressure and the shear-rate. The collection, completed by the introductory text that summarizes the main results, is formed by the following papers:

References

[D1] J. Hron, J. Málek, and K. R. Rajagopal. Simple flows of fluids with pressure dependent viscosities. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* IF†(1.326), 457:1603–1622, 2001.

[D2] J. Málek, J. Nečas, and K. R. Rajagopal. Global analysis of the flows of fluids with pressure and shear dependent viscosities. *Arch. Ration. Mech. Anal.* IF(1.769), 165:243–269, 2002.

[D3] J. Hron, J. Málek, J. Nečas, and K. R. Rajagopal. Numerical simulations and global existence of solutions of two dimensional flows of fluids with pressure and shear dependent viscosities. *Math. Comput. Simulation* IF(0.512), 61:297–315, 2003.

[D4] M. Franta, J. Málek, and K. R. Rajagopal. On steady flows of fluids with pressure and shear-dependent viscosities. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* IF(1.326), 461:651–670, 2005.

[D5] M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.* IF(0.784), 55:(to appear in the issue 6), 2006.

[D6] J. Frehse, J. Málek, and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.* IF(0.966), 34:1064–1083, 2003.

[D7] J. Málek and D. Pražák. Large time behavior via the method of ℓ-trajectories. *J. Differential Equations* IF(0.877), 181:243–279, 2002.

[D8] M. Bulíček, J. Málek, and D. Pražák. On the dimension of the global attractor for a class of fluids with pressure dependent viscosities. *Commun. Pure Appl. Anal.* IF(0.618), 4:805–822, 2005.

† IF stands for the impact factor.

[D9] J. Málek, J. Nečas and M. Růžička. On weak solutions to a class of non-Newtonian incompressible fluids in bounded threedimensional domains. *Adv. Differential Equations* (není zařazen mezi impaktované časopisy), 4:805–822, 2005.

[D10] P. Kaplický, J. Málek, and J. Stará. $C^{1,\alpha}$-regularity of weak solutions to a class of nonlinear fluids in two dimensions - stationary Dirichlet problem. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* (není zařazen mezi impaktované časopisy), 259:89–121, 1999. (také *J. Mat. Sci. (New York)*, 109:1867–1893 (2002))

[D11] P. Kaplický, J. Málek, and J. Stará. Global-in-time Hölder continuity of the velocity gradients for fluids with shear-dependent viscosities. *NoDEA Nonlinear Differential Equations Appl.* IF(0.396), 9:175–195, 2002.

The organization of the thesis is as follows. Firstly, we briefly describe the results and methods obtained and developed in [D1]–[D11].

At the first glance, incompressible fluids with pressure dependent viscosity can look as curiosities rather than physically well-sounded models. This point of view can even be supported by expositions on constraints in classical textbooks on continuum mechanics. For this reason, we discuss physical aspects of the models that are the object of our study in detail in Section 2. We introduce them taking two totally different points of view into account. Firstly, we derive the constitutive equation for the relevant form of the Cauchy stress using a suitable thermodynamic framework. Then we identify the same models within the framework of implicit constitutive theories. We also look at these models from historical perspective and provide a representative collection of experimental works supporting the viscosity-pressure relationships for incompressible fluids.

In Section 3 we shall provide a more detailed survey of the mathematical results concerning the analysis of the selected models; both the steady flows of fluids satisfying Dirichlet boundary conditions and the unsteady flows of these fluids under spatially periodic or Navier's slip boundary conditions will be considered. We also describe developed techniques.

Finally, in Section 4 we indicate why we think that the studies of the considered models are of importance and interest and we summarize the main novelties achieved in the above papers.

A brief survey of the results in this collection and some other relevant results

The article [D1] discusses the pressure-dependent fluid models from the point of view of their relevance to mechanics, and provides a survey of experimental works suggesting and supporting the idea to model behavior of fluid-like materials, particularly those subject to high pressures, as an incompressible fluid with the viscosity depending on the pressure. In addition, some interesting explicit solutions for steady flows of such fluids (for example those with an exponential or linear viscosity-pressure relationship) are computed†, and results of first numerical simulations are presented. These results show that the computed solutions have markedly different characteristic than the corresponding solutions to the classical Navier-Stokes fluid. The article [D1] has been a departure for our further research on analysis of relevant systems of partial differential equations (PDEs) that appear in this area.

To date there have been a few mathematically rigorous studies concerning fluids with pressure dependent viscosity. To our knowledge, to date there is no global existence theory that is in place for both steady and unsteady flows of fluids whose viscosity depends *purely* on the pressure. Previous studies by Renardy [55], Gazzola [22] and Gazzola and Secchi [23] either addressed existence of solutions that are both short-in-time and for small data or presume apriori (see [55]) that the solution ought to have certain properties while trying to establish the existence of solutions. We discuss this in more detail later (see Section 3).

The articles [D2], [D3] are the first ones where the long-time and large-data existence theory for incompressible fluids with pressure dependent viscosity were established. The paper [D2] deals with three-dimensional (unsteady) flows, in [D3] two-dimensional time-dependent flows are analyzed. The results are based on an observation that "adding" a suitable (sublinear) dependence of the viscosity on the shear rate may help, and on experiences achieved in analysis of time-dependent models with such sublinear dependence of the viscosity on the shear rate, see the monograph Málek et al. [38] and the papers Málek, Nečas, Růžička [37], Bloom, Bellout, Nečas [7], Frehse, Málek and Steinhauer [21], and the article [D9]. The fact that in [D2] and [D3] the governing equations are considered in the rather

† Vasudevaiah and Rajagopal [66] also considered the fully developed flow of a fluid that has a viscosity that depends on the pressure and shear rate in a pipe and were able to obtain explicit exact solutions for the problem, see also Renardy [56].

unrealistic spatially periodic setting can be considered as a "drawback" of these results.

More realistic *internal flows* are considered in the articles [D4] and [D5]. The paper [D4] deals with steady flows where the velocity satisfies the no-slip condition on the boundary. In [D5], long-time and large-data existence results for unsteady flows with the velocity fulfilling Navier's slip boundary conditions are established. The results as well as the difficulties to extend this theory to unsteady flows flufilling no-slip boundary consditions are addressed in Section 3.

The questions of further qualitative properties of solutions are more or less completely open in this area. In [D8], the authors observed that the assumptions on the viscosity required by the considered mathematical approach include some models where the Cauchy stress can grow linearly with the velocity gradient. Consequently, two-dimensional unsteady flows of such fluids are uniquely determined by their data and the solution operators form a semigroup. The long-time behavior of all solutions can then be investigated. Here, the authors incorporated the *method of trajectories*, a promising tool for studying long-time behavior of infinite-dimensional dynamical systems, and established the existence of a finite-dimensional global attractor, and the existence of an exponential attractor. The upper bound on the dimension of the attractor is also achieved. The method of trajectories is described in detail in [D7].

Higher differentiability and partial regularity of solutions are currently investigated by Málek, Mingione and Stará (see [35] for the announcement of the result). For fluids whose viscosity depends purely on the shear rate higher differentiability techniques were developed in the original papers [D9]–[D11]. More precisely, the article [D9] extends the results valid for spatially periodic setting (see the monograph Málek et al [38], and the papers [37] and [7]) to the homogeneous Dirichlet problem (no slip boundary condition). In [D10], $C^{1,\alpha}$-regularity for the velocity of two-dimensional steady flows is established (again for homogeneous Dirichlet problem). In both studies the results on regularity are established up to the boundary, i.e., they are global. In [D11], $C^{1,\alpha}$-regularity for the velocity of two-dimensional *unsteady* flows is established (here for simplicity the spatially periodic case is treated). The study of partial regularity for problems concerning fluids with pressure dependent viscosity initiated the analysis (existence, uniqueness, regularity) of a system of Stokes type where the Laplace operator is replaced by a general elliptic operator of second order with bounded measurable coefficients, and the gradient of the pressure is replaced by a general linear operator of first order. We refer to Huy and Stará [27] and the recent Ph.D. thesis by Huy [26] for details. The extension of the theory established in [D5] to models where the viscosity depends on the pressure, the shear rate and the *temperature* is performed in the recent Ph. D. thesis by Bulíček [11].

In the existence theory for non-linear partial differential equations, a key step represents the stability of nonlinear quantities with respect to weakly converging sequences. It is well known that weakly converging sequences do not commute with nonlinearities. For this purpose, several methods as for example monotone operator techniques, compact embedding theorems, regularity methods, div-curl lemma, parametrized (Young) and defect measures were developed. In these approaches it may happen that one is not allow to test by solution (or by the difference between the solution and its approximation) and suitably truncated (it means bounded) functions are used. In order to go beyond the limitations coming from their use, Lipschitz truncations can be incorporated. More details are given in Chapter 3. The paper [D6] addresses this issue in the analysis of power-law-like fluids. The approach has been recently simplified by Diening, Málek and Steinhauer in [14] by strengthening the properties of the Lipschitz truncations of Sobolev functions. The extension of this method to the system describing steady flows of incompressible fluids with the viscosity depending on the pressure and the shear rate is the topic of current research.

2. Models and their physical aspects

(*a*) *Kinematics and balance equations*

We shall keep our discussion of kinematics to a bare minimum. Let $\mathcal{B}$ denote the abstract body and let $\kappa : \mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is three dimensional Euclidean space, be a placer and $\kappa(\mathcal{B})$ the configuration (placement) of the body. We shall assume that the placer is one to one. By a motion we mean a one parameter family of placers. It follows that if $\kappa_R(\mathcal{B})$ is some reference configuration, and $\kappa_t(\mathcal{B})$ a configuration at time t, then we can identify the motion with a mapping $\chi_{\kappa_R} : \kappa_R(\mathcal{B}) \times \mathbb{R} \to \kappa_t(\mathcal{B})$ such that

$$x = \chi_{\kappa_R}(X, t)\,. \tag{2.1}$$

We shall suppose that χ_{κ_R} is sufficiently smooth to render the operations defined on it meaningful. Since χ_{κ_R} is one to one, we can define its inverse so that

$$X = \chi_{\kappa_R}^{-1}(x,t)\,. \tag{2.2}$$

Thus, any (scalar) property φ associated with an abstract body $\mathcal{B}$ can be expressed as (analogously we proceed for vectors or tensors)

$$\varphi = \varphi(P,t) = \hat{\varphi}(X,t) = \tilde{\varphi}(x,t)\,. \tag{2.3}$$

We define the following Lagrangean and Eulerian temporal and spatial derivatives:

$$\dot{\varphi} := \frac{\partial \hat{\varphi}}{\partial t}\,, \qquad \varphi_{,t} := \frac{\partial \tilde{\varphi}}{\partial t}\,, \qquad \nabla_X \varphi = \frac{\partial \hat{\varphi}}{\partial X}\,, \qquad \nabla_x \varphi := \frac{\partial \tilde{\varphi}}{\partial x}\,. \tag{2.4}$$

The Lagrangean and Eulerian divergence operators will be expressed as *Div* and *div*, respectively.

The velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are defined through

$$\mathbf{v} = \frac{\partial \chi_{\kappa_R}}{\partial t} \qquad \mathbf{a} = \frac{\partial^2 \chi_{\kappa_R}}{\partial t^2}\,, \tag{2.5}$$

and the deformation gradient $\mathbf{F}_{\kappa_R}$ is defined through

$$\mathbf{F}_{\kappa_R} = \frac{\partial \chi_{\kappa_R}}{\partial X}\,. \tag{2.6}$$

The velocity gradient $\mathbf{L}$ and its symmetric part $\mathbf{D}$ are defined through

$$\mathbf{L} = \nabla_x \mathbf{v}\,, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\,. \tag{2.7}$$

It immediately follows that

$$\mathbf{L} = \dot{\mathbf{F}}_{\kappa_R} \mathbf{F}_{\kappa_R}^{-1}\,. \tag{2.8}$$

It also follows from the notations and definitions given above, in particular from (2.4) and (2.5), that

$$\dot{\varphi} = \varphi_{,t} + \nabla_x \varphi \cdot \mathbf{v}\,. \tag{2.9}$$

Balance of Mass - Incompressibility - Inhomogeneity

A body is *incompressible* if

$$\int_{\mathcal{P}_R} dX = \int_{\mathcal{P}_t} dx \qquad \text{for all } \mathcal{P}_R \subset \kappa_R(\mathcal{B}) \;\text{ with } \mathcal{P}_t := \chi_{\kappa_R}(\mathcal{P}_R, t)\,.$$

Using the change of variables theorem, it leads to

$$\det \mathbf{F}_{\kappa_R}(X,t) = 1 \qquad \text{for all } X \in \kappa_R(\mathcal{B})\,. \tag{2.10}$$

If $\det \mathbf{F}_{\kappa_R}$ is continuously differentiable with respect to time, then by virtue of the identity

$$\frac{d}{dt}\det \mathbf{F}_{\kappa_R} = \operatorname{div}\mathbf{v}\det \mathbf{F}_{\kappa_R}\,,$$

we conclude, since $\det \mathbf{F}_{\kappa_R} \neq 0$, that

$$\operatorname{div}\mathbf{v}(x,t) = 0 \quad \text{ for all } t \in \mathbb{R} \text{ and } x \in \kappa_t(\mathcal{B})\,. \tag{2.11}$$

It is usually in the above form that the constraint of incompressibility is enforced in fluid mechanics.

The *balance of mass* in its Lagrangean form states that

$$\int_{\mathcal{P}_R} \varrho_R(X)dX = \int_{\mathcal{P}_t} \varrho(x,t)dx \;\; \text{for all } \mathcal{P}_R \subset \kappa_R(\mathcal{B})\,, \tag{2.12}$$

where ϱ_R and ϱ stand for the density at the reference and current configuration, respectively. Using again the change of variables theorem, (2.12) leads to

$$\varrho(x,t)\ \det \mathbf{F}_{\kappa_R}(X,t) = \varrho_R(X)\,. \tag{2.13}$$

From the Eulerian perspective, the balance of mass takes the form

$$\frac{d}{dt}\int_{\mathcal{P}_t} \varrho\,dx = 0 \qquad \text{for all } \mathcal{P}_t \subset \kappa_t(\mathcal{B})\,. \tag{2.14}$$

It immediately follows (for smooth functions) that

$$\varrho_{,t} + (\nabla_x \varrho)\cdot \mathbf{v} + \varrho \operatorname{div}\mathbf{v} = 0 \iff \varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) = 0\,. \tag{2.15}$$

If the fluid is incompressible, (2.15) simplifies to†

$$\varrho_{,t} + (\nabla_x \varrho)\cdot \mathbf{v} = 0 \iff \dot{\varrho} = 0 \iff \varrho(x,t) = \varrho(0,X) = \varrho_R(X)\,. \tag{2.16}$$

That is, for a fixed particle, the density is constant, as a function of time. However, the density of a particle may vary from one particle to another. The fact that the density varies over certain region of space, does not imply that the fluid is not incompressible. This variation is due to the fact that the fluid is inhomogeneous. We say that a fluid is *homogeneous* if $\varrho_R(X) = \varrho_R(Y)$ for all X, $Y \in \kappa_R(\mathcal{B})$. Thus,

† Note that the last equation in (2.16) can be obtained from (2.10) and (2.13).

if the fluid is homogeneous and incompressible, the equation (2.15) is automatically met.

Balance of Linear Momentum

The balance of linear momentum that originates from the second law of Newton in classical mechanics when applied to each subset $\mathcal{P}_t = \chi_{\kappa_R}(\mathcal{P}_R, t)$ of the current configuration takes the form

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\mathbf{v}\, dx = \int_{\mathcal{P}_t} \rho\mathbf{b}\, dx + \int_{\partial\mathcal{P}_t} \mathbf{T}^T\mathbf{n}\, dS\ , \tag{2.17}$$

where $\mathbf{T}$ denotes the Cauchy stress that is related to the surface traction $\mathbf{t}$ through $\mathbf{t} = \mathbf{T}^T\mathbf{n}$, and $\mathbf{b}$ denotes the specific body force. It then leads to the balance of linear momentum in its local Eulerian form:

$$\varrho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T}^T + \varrho\mathbf{b}\,. \tag{2.18}$$

Two comments are in order.

Firstly, considering the case when $\kappa_t(\mathcal{B}) = \kappa_R(\mathcal{B})$ for all $t \geq 0$ and setting $\Omega := \kappa_R(\mathcal{B})$, it is not difficult to conclude at least for incompressible fluids, that (2.17) and (2.14) imply that

$$\frac{d}{dt}\int_O \rho\mathbf{v}\, dx + \int_{\partial O}\Big[(\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n}) - \mathbf{T}^T\mathbf{n}\Big]\, dS = \int_O \rho\mathbf{b}\, dx\ , \tag{2.19}$$

$$\frac{d}{dt}\int_O \rho\, dx + \int_{\partial O} \rho(\mathbf{v}\cdot\mathbf{n})dS = 0, \tag{2.20}$$

valid for all (fixed) subsets O of Ω.

When compared to (2.17), this formulation is more suitable for further consideration in those problems where the velocity field $\mathbf{v}$ is taken as a primitive field defined on $\Omega \times \langle 0, \infty)$ (i.e., it is not defined through (2.5)).

To illustrate this convenience, we give a simple analogy from classical mechanics: consider a motion of a mass-spring system described by the second order ordinary differential equations for displacement of the mass from its equilibrium position and compare it with a free fall of the mass captured by the first order ordinary differential equations for the velocity.

Second, the derivation of (2.18) from (2.17) and similarly (2.15) from (2.14) requires certain smoothness of particular terms. In analysis, the classical formulations of the balance equations (2.18) and (2.15) are usually starting points for definition of various kinds of solutions. Following Oseen [41] (see also [17], [18]), we want to emphasize that the notion of a weak solution (or suitable weak solution) is very natural for the equations of continuum mechanics, since their weak formulation can be directly obtained from the original formulations of the balance laws (2.14) and (2.17) or better (2.19) and (2.20). This comment is equally applicable to the other balance equations of continuum physics as well.

Balance of Angular Momentum

In the absence of internal couples, the balance of angular momentum implies that the Cauchy stress is symmetric, i.e.,

$$\mathbf{T} = \mathbf{T}^T \,. \tag{2.21}$$

Balance of Energy

We shall merely record the local form of the balance of energy which is

$$\varrho \dot{\epsilon} = \mathbf{T} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{q} + \varrho r \,, \tag{2.22}$$

where ϵ denotes the internal energy, $\mathbf{q}$ denotes the heat flux vector and r the specific radiant heating.

Further Thermodynamic Considerations (The Second Law). Reduced dissipation equation

To know how a body is constituted and to distinguish one body from another, we need to know how bodies store energy. How, and how much of, this energy that is stored in a body can be recovered from the body? How much of the working on a body is converted to energy in thermal form (heat)? What is the nature of the latent energy that is associated with the changes in phase that the body undergoes? What is the nature of the latent energy (which is different in general from latent heat)? By what different means does a body produce the entropy? These are but few of the pieces of information that one needs to know in order to describe the response of the body. Merely knowing this information is insufficient to describe how the body will respond to external stimuli. A body's response has to meet the basic balance laws of mass, linear and angular momentum, energy and the second law of thermodynamics.

Various forms for the second law of thermodynamics have been proposed and are associated with the names of Kelvin, Plank, Claussius, Duhem, Carathéodory and others. Essentially, the second law states that the rate of entropy production has to be non-negative†. A special form of the second law, the Claussius-Duhem inequality, has been used, within the context of a continua, to obtain restrictions on allowable constitutive relations (see Coleman and Noll [12]). This is enforced by allowing the body to undergo arbitrary processes in which the second law is required to hold. The problem with such an approach is that the constitutive structure that we ascribe to a body is only meant to hold for a certain class of processes. The body might behave quite differently outside this class of processes. For instance, while rubber may behave like an elastic material in the sense that the stored energy depends only on the deformation gradient and this energy can be completely recovered in processes that are reasonably slow in some sense, the same rubber if deformed at exceedingly high strain rates crystallizes and not only does the energy that is stored not depend purely on the deformation gradient, all the energy that was supplied to the body cannot be recovered. Thus, the models for rubber depend on the process class one has in mind and this would not allow one to subject the body to arbitrary processes. We thus find it more reasonable to assume the constitutive structures for the rate of entropy production, based on physical grounds, that are automatically non-negative.

Let us first introduce the second law of thermodynamics in the form

$$\varrho\theta\dot{\eta} \geq -\operatorname{div}\mathbf{q} + \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} + \varrho r\,, \tag{2.23}$$

where η denotes the specific entropy.

On introducing the specific Helmholtz potential ψ through

$$\psi := \epsilon - \theta\eta\,,$$

and using the balance of energy (2.22), we can express (2.23) as

$$\mathbf{T}\cdot\mathbf{L} - \varrho\dot{\psi} - \varrho\dot{\theta}\eta - \frac{\mathbf{q}\cdot(\nabla_x\theta)}{\theta} \geq 0\,. \tag{2.24}$$

The above inequality is usually referred to as the dissipation inequality. This inequality is commonly used in continuum mechanics to obtain restrictions on the constitutive relations. A serious difficulty with regard to such an approach becomes immediately apparent. No restrictions whatsoever can be placed on the radiant

† There is a disagreement as to whether this inequality ought to be enforced locally at every point in the body, or only globally, even from the point of view of statistical thermodynamics.

heating. More important is that the radiant heating is treated as a quantity that adjusts itself in order to meet the balance of energy. But this is clearly unacceptable as the radiant heating has to be a constitutive specification. How a body responds to radiant heating is critical, especially in view of the fact that all the energy that our world receives is in the form of electromagnetic radiation which is converted to energy in its thermal form (see Rajagopal and Tao [52] for a discussion of these issues). As we shall be primarily interested in the mechanical response of fluids, we shall ignore the radiant heating altogether, but we should bear in mind the above observation when we consider more general processes.

We shall define the specific rate of entropy production ξ through

$$\xi := \mathbf{T} \cdot \mathbf{L} - \varrho\dot{\psi} - \varrho\dot{\theta}\eta - \frac{\mathbf{q} \cdot (\nabla_x \theta)}{\theta} \,. \tag{2.25}$$

We shall make constitutive assumptions for the rate of entropy production ξ and require that (2.25) holds in all admissible processes (see Green and Nagdhi [24]). Thus, the equation (2.25) will be used as a constraint that is to be met in all admissible processes. We shall choose ξ such that it is non-negative and thus the second law is automatically met.

We now come to a crucial step in our thermodynamic considerations. From amongst a class of admissible non-negative rate of entropy productions, we choose that which is maximal. This is asking a great deal more than the second law of thermodynamics. The rationale for the same is the following. Let us consider an isolated system. For such a system, it is well accepted that its entropy becomes a maximum and the system would reach equilibrium. The assumption that the rate of entropy production is a maximum ensures that the body attains its equilibrium as quickly as possible. Thus, this assumption can be viewed as an assumption of economy or an assumption of laziness, the system tries to get to the equilibrium state as quickly as possible, i.e., in the most economic manner. It is important to recognize that this is merely an assumption and not some deep principle of physics. The efficacy of the assumption has to be borne out by its predictions and to date the assumption has led to meaningful results in predicting the response of a wide variety of materials; see results pertinent to viscoelasticity, classical plasticity, twinning, solid to solid phase transition surveyed in the papers by Rajagopal and Srinivasa [48] and [49], crystallization in polymers [53] and [54], or single crystal supper alloys [44], etc.

Isothermal flows at uniform temperature

Here, we shall restrict ourselves to flows that take place at constant temperature for the whole period of interest at all points of the body. Consequently, the equations governing such flows for an incompressible homogeneous fluid are

$$\operatorname{div}\mathbf{v} = 0\,, \qquad \varrho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T} + \varrho\mathbf{b}\,. \tag{2.26}$$

Note also that (2.24) and (2.25) reduce to

$$\mathbf{T}\cdot\mathbf{D} - \varrho\dot{\psi} = \xi \quad \text{ and } \ \xi \geq 0\,, \tag{2.27}$$

where the symmetry of $\mathbf{T}$, see (2.21), is used.

In elasticity theory, it is common to use the stress-free configuration as the reference configuration $\kappa_R(\mathcal{B})$, while in classical fluids the current configuration $\kappa_t(\mathcal{B})$ is used as the reference configuration. Let $\kappa_{p(t)}(\mathcal{B})$ denote the configuration that the body would attain when all the external stimuli that act on the body in $\kappa_t(\mathcal{B})$ are removed. We shall refer to this configuration as the preferred natural configuration. The preferred configuration that the body attains depends on the process class that is permissible for the body under consideration. Thus, the body may attain a particular natural configuration if it is only allowed to undergo isothermal processes and another natural configuration if it is only subject to adiabatic processes, that is, the natural configuration attained depends on how the external stimuli are removed (for example instantaneously or slowly, etc.). We shall be interested in modeling the response of fluids whose current configuration is the natural configuration, i.e., removal of the external stimuli leaves the fluid in the configuration that it is in. The Navier-Stokes fluid is one such fluid.

(*b*) *Models within a consistent thermodynamic framework*

Following the work by Málek and Rajagopal [40] dealing with rate type fluids whose material moduli are pressure, shear rate and density dependent and where also the preferred natural configuration may differ from the current configuration, we derive an hierarchy of incompressible fluid models. For simplicity, we deal with homogeneous fluids where the density is uniformly constant. The hierarchy of incompressible fluid models generalizes the incompressible Navier-Stokes fluid in the following sense: the viscosity may not only be a constant, but it can be a function that may depend on the symmetric part of the velocity gradient $\mathbf{D}$ specifically through its second invariant $|\mathbf{D}|^2 := \mathbf{D}\cdot\mathbf{D}$ (we call this quantity the shear rate), or the mean normal stress, i.e., the pressure $p := -\frac{1}{3}\operatorname{tr}\mathbf{T}$, or it can depend on both of them. We shall consider the most general case within this setting by assuming that

$$\xi = \Xi(p,\mathbf{D}) = 2\nu(p,|\mathbf{D}|^2)\,|\mathbf{D}|^2\,. \tag{2.28}$$

Clearly, if $\nu \ge 0$ then automatically $\xi \ge 0$, ensuring that the second law is complied with.

We assume that the specific Helmholtz potential ψ is of the form $\psi = \Psi(\varrho)$, i.e., it is a constant for a fluid where the density is constant at any point. Consequently, $\dot{\psi}$ vanishes in (2.27) and we obtain

$$\mathbf{T}\cdot\mathbf{D} = \Xi\,. \tag{2.29}$$

Following the same procedure as in [40], we maximize Ξ with respect to $\mathbf{D}$ that is subject to the constraint (2.29) and the incompressibility constraint

$$\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = 0\,. \tag{2.30}$$

As a necessary condition for the extremum we obtain the equation

$$(1+\lambda_1)\Xi_{,\mathbf{D}} - \lambda_1\mathbf{T} - \lambda_0\mathbf{I} = 0\,, \tag{2.31}$$

where λ_0 and λ_1 are the Lagrange multipliers due to the constraints (2.30) and (2.29). We eliminate them as follows. Taking the scalar product of (2.31) with $\mathbf{D}$, and using (2.30) and (2.29) we obtain

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\Xi_{,\mathbf{D}}\cdot\mathbf{D}}\,. \tag{2.32}$$

Note that

$$\Xi_{,\mathbf{D}} = 4\Big(\nu(p,|\mathbf{D}|^2) + \nu_{,\mathbf{D}}(p,|\mathbf{D}|^2)|\mathbf{D}|^2\Big)\mathbf{D}\,. \tag{2.33}$$

Consequently, $\operatorname{tr}\Xi_{,\mathbf{D}} = 0$ by virtue of (2.30). Thus, taking the trace of (2.31) we have

$$-\frac{\lambda_0}{\lambda_1} = -p \quad \text{with } p = -\frac{1}{3}\operatorname{tr}\mathbf{T}\,. \tag{2.34}$$

Using (2.31)–(2.34), we finally find that (2.31) takes the form

$$\mathbf{T} = -p\mathbf{I} + 2\,\nu(p,|\mathbf{D}|^2)\mathbf{D}\,. \tag{2.35}$$

Mathematical issues related to the system (2.26) with the constitutive equation (2.35) will be discussed in the third part of this treatise.

Recall that taking constant ν in (2.35) we obtain the constitutive equation for an incompressible Navier-Stokes fluid.

Models in the context of non-Newtonian fluids

No fluid model has been scrutinized and studied by mathematicians, physicists and engineers as intensively as the Navier-Stokes fluid model. While this model describes adequately a large class of flows (primarily laminar flows) of the most ubiquitous fluids, air and water, it is inadequate in describing the laminar response of a variety of polymeric liquids, geological fluids, food products, and biological fluids, or for that matter the response of air and water undergoing turbulent flow. The departure from the behavior exhibited by the Navier-Stokes fluid is referred to as non-Newtonian behavior. Amongst the many points of departure, one that is encountered commonly is the dependence of viscosity on the shear rate (or to be more precise on the euclidean norm of the symmetric part of the velocity gradient), stress relaxation, non-linear creep, the development of normal stress differences in a simple shear flow and yield-like behavior (see Rajagopal [45] or Málek and Rajagopal [39]).

The fluid given by (2.35) has the ability to shear thin, shear thicken and pressure thicken. After adding the yield stress or activation criterion, the model could capture phenomena connected with the development of discontinuous stresses. Thus the model that we are interested in studying is a *non-Newtonian* fluid model. On the other hand, the model (2.26) together with (2.35) cannot stress relax or creep in a non-linear way, neither can it exhibit nonzero normal stress differences in a simple shear flow.

(*c*) *Models within the framework of implicit constitutive theories*

In the previous subsection we provided thermodynamic basis for incompressible models of the type (2.35). In this subsection, we show that (2.35) has a well-sounded physical basis using a completely different approach, namely the implicit constitutive theory, see [46].

The model (2.35) is markedly different from the models describing standard Navier-Stokes fluids that represent explicit relationships between the stress and $\mathbf{D}$; the equation (2.35) is an implicit relationship between $\mathbf{T}$ and $\mathbf{D}$ of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}\,. \tag{2.36}$$

To see this, recall that we are interested in the flows of incompressible fluids, i.e., the flows under consideration are isochoric and fulfil the constraint (2.30). It follows from (2.35) and (2.30) that

$$\mathbf{T} = \frac{1}{3}(\operatorname{tr}\mathbf{T})\mathbf{I} + \hat{\nu}(\operatorname{tr}\mathbf{T}, |\mathbf{D}|^2)\mathbf{D}\,, \tag{2.37}$$

which can be expressed in the form (2.36). We note that, in general, neither can $\mathbf{T}$ be expressed explicitly in terms of $\mathbf{D}$ nor vice-versa.

Let us start by considering fluids described by implicit constitutive relations of the form (2.36). If we require the function $\mathbf{f}$ to be isotropic, then $\mathbf{f}$ has to satisfy the restriction

$$\mathbf{f}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T, \mathbf{Q}\mathbf{D}\mathbf{Q}^T) = \mathbf{Q}\mathbf{f}(\mathbf{T},\mathbf{D})\mathbf{Q}^T \quad \forall \mathbf{Q} \in \mathcal{Q},$$

where $\mathcal{Q}$ denotes the orthogonal group. It immediately implies (see Spencer [61]) that

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{D}\mathbf{T}+\mathbf{T}\mathbf{D}) + \alpha_6(\mathbf{T}^2\mathbf{D}+\mathbf{D}\mathbf{T}^2)\\ +\alpha_7(\mathbf{T}\mathbf{D}^2+\mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{T}^2) = \mathbf{0}\,,\end{aligned}$$

where $\alpha_i, i = 0, \dots 8$ depend on the invariants

$$\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{D}^2, \operatorname{tr}\mathbf{T}^3, \operatorname{tr}\mathbf{D}^3, \operatorname{tr}(\mathbf{T}\mathbf{D}), \operatorname{tr}(\mathbf{T}^2\mathbf{D}), \operatorname{tr}(\mathbf{D}^2\mathbf{T}), \operatorname{tr}(\mathbf{T}^2\mathbf{D}^2).$$

We note that if

$$\begin{aligned}\alpha_0 &= -\frac{1}{3}\operatorname{tr}\mathbf{T},\\ \alpha_1 &= 1,\end{aligned}$$

and

$$\alpha_2 = -2\nu(-\frac{1}{3}\operatorname{tr}\mathbf{T}, |\mathbf{D}|^2) \qquad (\nu > 0),$$

we obtain the model

$$\mathbf{T} = (\frac{1}{3}\operatorname{tr}\mathbf{T})\mathbf{I} + 2\nu(-\frac{1}{3}\operatorname{tr}\mathbf{T}, |\mathbf{D}|^2)\mathbf{D}. \tag{2.38}$$

On defining

$$p := -\frac{1}{3}\operatorname{tr}\mathbf{T},$$

equation (2.38) reduces to

$$\mathbf{T} = -p\mathbf{I} + 2\nu(p, |\mathbf{D}|^2)\mathbf{D}, \tag{2.39}$$

the model whose mathematical properties we shall discuss in the next chapter.

We note that the model (2.39) automatically meets the constraint (2.30). That is, the model is only capable of undergoing isochoric motions; it describes an incompressible fluid. We obtained the model as a consequence of our constitutive choice for $\alpha_0 \ldots \alpha_8$, and not by imposing (2.30) as a constraint and thereby obtaining p as a Langrange multiplier that enforces the constraint.

It is important to note that the classical approach that is employed in most continuum mechanics textbooks to enforce constraints by requiring that they do no work, and splitting the stress into a constraint response stress $\mathbf{T}_C$ and an extra constitutively determinate stress $\mathbf{T}_E$ (see Truesdell [65]), i.e.,

$$\mathbf{T} = \mathbf{T}_E + \mathbf{T}_C$$

will not lead to the material moduli that appear in $\mathbf{T}_E$ to depend on the Lagrange multiplier, in our case the mean normal stress p (see Rajagopal and Srinivasa [50]). However, models such as (2.39) appear naturally within the context of the implicit relation of the form (2.36).

(*d*) *Works supporting the viscosity-pressure relationship*

Stokes, in his celebrated paper on the motion of fluids and the equilibrium of solids [62], clearly recognized that the viscosity of fluids such as water could depend on the mean normal stress, as evidenced by his comments: "...If we suppose ν to be independent of the pressure ..." and "Let us now consider in what cases it is allowable to suppose ν to be independent of the pressure". That is, according to Stokes, one cannot suppose ν to be independent of pressure in all processes that the fluid undergoes, and in this he was exactly on the mark over a century and half ago. Elastohydrodynamics is an example wherein one can sensibly approximate the fluid as being incompressible with the viscosity depending on the pressure. It is however worth emphasizing that there is a large class of flows, not those merely restricted to flows in pipes and channels wherein the viscosity can be assumed to be constant, so much so that this is the assumption that is usually made in fluid mechanics. The classical incompressible Navier-Stokes model bears testimony to the same.

A considerable amount of experimental work has been carried out concerning the pressure dependence of the material moduli, and the pertinent literature prior to 1930 can be found in the authoritative book by Bridgman [10]. Andrade [3] proposed the following relationship between the viscosity ν, the pressure p, the density ϱ and the temperature θ:

$$\nu(p,\varrho,\theta) = A\varrho^{\frac{1}{2}} \exp\left(\frac{B}{\theta}(p + D\varrho^2)\right) , \tag{2.40}$$

where A, B and D are constants. We note that Andrade [3] did not consider the possibility that the viscosity of the fluids which he experimented could depend upon the shear rate. We cannot be sure that the fluids that he tested did not posses a viscosity depending on the shear rate; the experiments that he carried out are inadequate to speak to this matter.

As early as 1893 Barus [6] suggested the following relationship for the viscosity for liquids:

$$\nu(p) = \nu_0 \exp(\alpha p) , \quad \alpha > 0 . \tag{2.41}$$

Such an expression has been used for several decades in elastohydrodynamics where the fluid is subject to a wide range of pressures and consequently a significant change in the viscosity occurs(see Szeri [63]). There are several recent experimental studies that indicate that the pressure gets to be so large that the fluid is very close to undergoing glass transition and at such high pressures the Barus equation (2.41) becomes inappropriate (see Bair and Kottke [4]). In fact the viscosity varies even more drastically than exponential dependence (see Figure in Bair and Kottke [4]). Other formulae for the variation of the viscosity with pressure, which better fit experimental results, can be found in the literature but they invariably involve an exponential relationship of sorts (see Cutler et al. [13], Griest et al. [25], Johnson and Cameron [29], Johnson and Greenwood [30], Johnson and Tevaarwerk [31], Bair and Winer [5], Roelands [57], Paluch et al. [42], Irving and Barlow [28], Bendler et al. [8]). The precise relationship between the viscosity and the pressure is not of consequence, what is important is the fact that the viscosity depends on the pressure.

The density changes in liquids such as water the pressure correlates well with the empirical expression (see Dowson and Higginson [16])

$$\varrho = \varrho_0 \left[1 + \frac{0.6p}{1 + 1.4p}\right] , \tag{2.42}$$

where ϱ_0 is the density in the liquid as the pressure tends to zero.

When liquids such as water and many organic fluids† are subject to a wide range of pressures, say from 2 GPa to 3 GPa, it is found that while the density of the fluid varies (say 3 to 10%) slightly, the viscosity of the fluid can change by as much as a factor of 10^8! (See recent experiments of Bair and Kottke [4].) This suggests that it would be reasonable to model such fluids as incompressible fluids with the viscosity depending upon the pressure.

We would be remiss if we did not emphasize that liquids are compressible and that the scatter amongst the compressibility of liquids can be quite large. As Bridgman points out, due to a certain pressure difference, while Glycerine can have a change of volume of approximately 13.5%, mercury changes by only 4%. The marginal compressibility, that is the change of density due to a change in pressure decreases as the pressures increase, as can be inferred from (2.42).

Despite recognizing the importance of the dependence of viscosity on pressure, the elastohydrodynamicists have failed to systematically incorporate the pressure dependence until recently. The classical Reynold's approximation for lubrication which forms one of the cornerstones of fluid mechanics is derived under the assumption that the viscosity is constant. This approximation has been subsequently generalized to the field of elastohydrodynamics. While the elastohydrodynamicist recognizes that the viscosity depends on the pressure, he merely substitutes this dependence after the approximation has been derived rather than incorporating the dependence of the viscosity, à priori, and subsequently deriving the approximations. Rajagopal and Szeri [51] have recently derived a consistent set of approximate equations that are the appropriate generalization of the celebrated Reynold's lubrication approximation.

3. Mathematical analysis of the models

There is a significant need to understand the mathematical properties of the solutions to the equations governing the flows of incompressible fluids with the viscosity depending on pressure and shear rate, both due to their use in various areas of engineering sciences, and due to difficulties that occur during numerical simulations of the relevant systems of partial differential equations (PDEs). This system of PDEs

† The variation of the viscosity of water is somewhat different from those of many organic liquids in that in certain range of pressure it exhibits anomalous response.

is of independent interest on its own in virtue of its structural simplicity on one hand and the very complicated relation between the velocity and the (nonlocal) pressure on the other hand that does not permit one to eliminate the pressure from the analysis of the problem by projecting the equations to the set of divergenceless functions, as it is frequently done in the analysis of NSEs and similar systems.

(*a*) *Systems of PDEs, boundary conditions and on a datum for the pressure*

We consider the case when flows take place inside a fixed container. It means that $\kappa_t(\mathcal{B})$ occupies for all $t \geq 0$ the same open bounded set $\Omega \subseteq \mathbb{R}^d$, i.e. $\kappa_t(\mathcal{B}) = \Omega$ for all $t \geq 0$. We consider for simplicity the case when the boundary $\partial\Omega$ is smooth as specified later.

On substituting (2.35) and $\varrho(x,t) = \bar{\varrho}$ (where $\bar{\varrho}$ is a positive number) into the balance of linear momentum $(2.26)_2$, we obtain the system of governing equations

$$-\nabla p + \operatorname{div}\left[\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\right] + \bar{\varrho}\mathbf{b} = \bar{\varrho}\frac{d\mathbf{v}}{dt}$$

that holds in $(0,T) \times \Omega$. To the above set of equations we add the conservation of mass that reduces to the divergenceless constraint $(2.26)_1$.

It is convenient to divide this form of the balance of linear momentum by the constant positive value of the density $\bar{\varrho}$. Then, relabeling $\frac{\nu}{\bar{\varrho}}$ and $\frac{p}{\bar{\varrho}}$ by ν and p, respectively, we can rewrite the above system as

$$\left.\begin{aligned} &\operatorname{div}\mathbf{v} = 0, \quad \mathbf{S} = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \\ &\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbf{S} = -\nabla p + \mathbf{b} \end{aligned}\right\} \text{ in } (0,T)\times\Omega, \tag{3.1}$$

If flows are steady the system reduces to the form

$$\left.\begin{aligned} &\operatorname{div}\mathbf{v} = 0, \quad \mathbf{S} = \nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \\ &\operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbf{S} = -\nabla p + \mathbf{b} \end{aligned}\right\} \text{ in } \Omega\,. \tag{3.2}$$

Naturally, if $\mathbf{v}^0$ is a given divergenceless initial velocity field, the relevant initial condition for the system (3.1) takes the form

$$\mathbf{v}(0,x) = \mathbf{v}^0(x) \qquad \text{at almost all } x \in \Omega\,. \tag{3.3}$$

Boundary conditions

Here we shall restrict ourselves to a discussion of internal flows which meet†

$$\mathbf{v} \cdot \mathbf{n} = 0. \tag{3.4}$$

This condition is considered on $\partial\Omega$ if we deal with (3.2), and on $[0, T] \times \partial\Omega$ if the evolutionary model (3.1) is studied. Regarding the tangential components of the velocity, we shall consider the no-slip boundary condition where

$$\mathbf{v}_\tau = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n} = \mathbf{0}, \tag{3.5}$$

or we take slip into account, according to what is usually referred to as Navier's slip, i.e.,

$$(\mathbf{Sn})_\tau + \alpha \mathbf{v}_\tau = \mathbf{0}. \tag{3.6}$$

We shall also present results corresponding to solutions of (3.1) that are spatially periodic and thus we shall assume that

$$\begin{aligned} &(\mathbf{v}, p) : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R} \text{ are } L\text{-periodic for each spatial variable} \\ &\qquad\qquad \text{and that} \int_\Omega \mathbf{v}(x, t)\, dx = \mathbf{0}, \quad \Omega := (0, L)^d. \end{aligned} \tag{3.7}$$

On a datum for the pressure

Unlike the classical Navier-Stokes equations wherein one only encounters the gradient of the pressure, in the problem under consideration the actual value of the pressure appears as the viscosity depends on the pressure. Within the context of the mathematical framework that we are using, it does not make sense to merely prescribe p at a specific point. Thus, in order to fix the pressure, we use for stationary problems the condition

$$\frac{1}{|\Omega|} \int_\Omega p\, dx = p_0, \tag{3.8}$$

where $p_0 \in \mathbb{R}$ is given and $|\Omega|$ denotes the volume of Ω, while for the evolutionary problems we incorporate the condition

$$\frac{1}{|\Omega|} \int_\Omega p(x, t)\, dx = Q(t) \quad \text{ for all } t \in (0, T), \tag{3.9}$$

where $Q : (0, T) \to \mathbb{R}$ is given.

† Here, $\mathbf{n} = \mathbf{n}(x)$ denotes the outer normal to the boundary $\partial\Omega$ at the point $x \in \partial\Omega$.

When the viscosity does not depend on the pressure, the constant that fixes the pressure is irrelevant, such is not the case in the current situation as the value of the viscosity depends on the value of the pressure. Thus, the flow characteristics corresponding to a flow in pipe due to a pressure of 100 psi at the inlet and 99 psi at the outlet would be significantly different than the flow due to an inlet pressure of $10^5 + 1$ psi and on outlet pressure 10^5 psi, although the pressure difference in the two cases is exactly the same. The effect of the pressure on the viscosity in the latter case is however significantly different than in the former. While, from physical considerations it might be best to fix the value of the pressure by knowing its value at one point, as we are interested in dealing with integrable functions fixing it on a set of measure zero is not meaningful. In view of this we fix the pressure by requiring it satisfies a certain mean value as defined through equation (3.8).

An alternative possibility of fixing the pressure is to prescribe the normal traction on a relevant portion of the boundary with non-zero area measure. The mathematical investigation of the systems of PDEs completed by boundary conditions of this type is a subject of current research.

(*b*) *On analysis of models when* $\nu = \nu(p)$

Up to now there have been few studies dealing with the mathematical analysis of incompressible fluid models wherein the viscosity is pressure-dependent. We first provide a survey of observations related to (3.2) or (3.1) where however ν is a function of the pressure only, i.e.,

$$\nu = \nu(p). \tag{3.10}$$

Renardy [55] seems to be the first who dealt with theoretical analysis of (3.2) and (3.1). Firstly, he asked the question whether a given velocity field uniquely determines the pressure and he showed that this happens if

$$\nu_\infty^* := \lim_{p\to+\infty} \nu'(p) < \infty, \tag{3.11}$$

i.e., the viscosity grows at most linearly with the pressure, and if

$$\text{eigenvalues of } \mathbf{D}(\mathbf{v}) \text{ are strictly less then } \frac{1}{\nu_\infty^*}\,. \tag{3.12}$$

The assumption (3.12) implies that the flow has to have sufficiently small velocity gradients. Let us remark that experimental results unequivocally indicate that

$$\nu'(p) > 0 \text{ for all } p \in \mathbb{R}. \tag{3.13}$$

In fact, if the pressure is sufficiently high glass transition takes place. The condition (3.11) is evidently not fulfilled if the viscosity depends on the pressure exponentially, as is the case in (2.40) or (2.41). One can however argue that the viscosity can be truncated at some higher value of $p_0 \gg 1$. Thus, for example, ν can be approximated by

$$\nu^*(p) = \begin{cases} \nu(p) & \text{if} \quad p \in (-\infty, p_0], \\ \nu(p_0) & \text{if} \quad p > p_0, \end{cases} \tag{3.14}$$

or it can be extended smoothly at p_0, but with sublinear growth so that (3.11) holds.

A much more severe restriction seems to be the condition (3.12) since not all velocity field are admissible. Working with higher order Sobolev spaces, and assuming the restriction (3.11) and (3.12), Renardy proves the existence and uniqueness of the solution to (3.1). Since the initial data also have to fulfill (3.12), this is a small-data result. Also, since the energy estimates ($\mathbf{b} \equiv \mathbf{0}$ for simplicity)

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t \int_\Omega \nu(p)|\mathbf{D}(\mathbf{v})|^2 \, dx\, ds = \frac{1}{2}\|\mathbf{v}_0\|_2^2 \qquad (\text{for all} t > 0) \tag{3.15}$$

are available here, it seems more natural to construct weak solution in the spaces determined by (3.15). Such a result is however not in place to our knowledge.

Gazzola [22] treating the evolutionary problem (3.1), and Gazzola and Secchi [23] when dealing with the stationary problem (3.2), established results without assuming (3.12). Nevertheless, their results are in some sense elementary even if the treatment of the pressure requires an approach totally different from that used usually in Navier-Stokes theory where one frequently eliminates the pressure and works with spaces of functions that are divergenceless. The results established in [22] and [23] are immediate consequences of the fact that only small initial conditions and almost potential external body forces ($\mathbf{b} \sim \nabla g$) are considered; one can immediately observe that $\mathbf{b} = \nabla g$ and $\mathbf{v}^0 \equiv \mathbf{0}$ leads to the energy identity ($\int \mathbf{b}\mathbf{v} = \int g \operatorname{div} \mathbf{v} = 0$)

$$\frac{1}{2}\|\mathbf{v}(t)\|_2^2 + \int_0^t \int_\Omega \nu(p)|\mathbf{D}(\mathbf{v})|^2 \, dx\, ds = 0,$$

which implies that $\mathbf{v} \equiv \mathbf{0}$, $p = g$ is the unique trivial solution of the problem. One obtains a similar conclusion for the stationary problem. Thus, if $\mathbf{b}$ is close to ∇g and ($\mathbf{v}^0$ is small for the time-dependent case) it is not "surprising" that in the case of (3.1), Gazzola obtains as an addition to all these restrictions, the existence of solution only on a certain interval $[0, T_0]$, T_0 determined by a suitable (small) norm of $\mathbf{v}_0$. Málek, Nečas and Rajagopal [D2] observed that the feature "to a given velocity field there is a uniquely defined pressure" can be achieved if one "allows" the viscosity to depend also on the shear rate in a suitable manner. The structure of the viscosities and the results established for such models till recently will be discussed next.

(c) On analysis of models when $\nu = \nu(p, |\mathbf{D}(\mathbf{v})|^2)$

It has been observed in Málek, Nečas and Rajagopal [D2] that the dependence of ν on $|\mathbf{D}(\mathbf{v})|^2$ may help, particularly if such a dependence is sublinear. To see this, let us assume that there are p^1, p^2 for a given $\mathbf{v}$ that fulfill (3.1). Then, by taking the divergence of (3.1) for $(\mathbf{v}, p^1)$ and $(\mathbf{v}, p^2)$ we come to the relation for the difference of the form

$$\begin{aligned} p^2 - p^1 &= (-\triangle)^{-1} \operatorname{div}\operatorname{div}\big([\nu(p^1, |\mathbf{D}(\mathbf{v})|^2) - \nu(p^2, |\mathbf{D}(\mathbf{v})|^2)]\mathbf{D}(\mathbf{v})\big) \\ &= (-\triangle)^{-1} \operatorname{div}\operatorname{div}\left(\frac{\partial \nu(p^1 + \delta(p^2 - p^1), |\mathbf{D}(\mathbf{v})|^2)}{\partial p}\mathbf{D}(\mathbf{v})(p^1 - p^2)\right), \end{aligned}$$

which on simplifying leads to

$$\|p^1 - p^2\|_q \le \max_{Q, \mathbf{D}(\mathbf{v})} \left|\frac{\partial \nu}{\partial p}(Q, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})\right| \|p^1 - p^2\|_q \,.$$

Thus, if

$$\max_{Q, \mathbf{D}} \left|\frac{\partial \nu}{\partial p}(Q, |\mathbf{D}|^2)\mathbf{D}\right| < 1,$$

we obtain $p^1 = p^2$.

This observation motivates the following assumptions. We assume that the viscosity ν is a $\mathcal{C}^1$-mapping of $\mathbb{R} \times \mathbb{R}_0^+$ into $\mathbb{R}_+$ satisfying for some fixed (but arbitrary) $r \in [1, 2]$ and all $\mathbf{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\mathbf{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $p \in \mathbb{R}$ the following inequalities

$$C_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2 \le \frac{\partial \nu(p, |\mathbf{D}|^2)\mathbf{D}_{ij}}{\partial \mathbf{D}_{kl}}\mathbf{B}_{ij}\mathbf{B}_{kl} \le C_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2, \tag{3.16}$$

$$\left|\frac{\partial \nu(p, |\mathbf{D}|^2)}{\partial p}\right| |\mathbf{D}| \le \gamma_0(1 + |\mathbf{D}|^2)^{\frac{r-2}{4}}, \tag{3.17}$$

where γ_0 is a positive constant whose value will be restricted in the formulations of the particular results.

Since $r = 2$ is included in the range of parameters, we see that (3.16) (and naturally also (3.17)) includes the Navier-Stokes model. Also, if ν is independent of p then (3.17) is irrelevant and (3.16) is fulfilled by generalized power-law-like fluids. We refer to [D6], [D9]–[D11], and Málek and Rajagopal [39] for a survey of mathematical results related to these special cases. In the next subsection we restrict ourselves to this class of fluids, it means we consider ν being purely a function of $|\mathbf{D}|^2$ satisfying (3.16), and we survey techniques developed to obtain compactness of the velocity gradients. These techniques, three of them developed in the last fifteen years, serve or will serve as important tools in the proofs of the results stated below for incompressible fluids with pressure and shear-rate dependent viscosities.

Note that the assumption (3.17) do not permit us to consider any model where the viscosity depends only on the pressure.

The following forms of the viscosities fulfil the assumptions (3.16) and (3.17).

Example 1. *Consider for $r \in (1,2]$*

$$\nu_i(p, |\mathbf{D}|^2) = \left(1 + \gamma_i(p) + |\mathbf{D}|^2\right)^{\frac{r-2}{2}}, \quad i = 1, 2, \tag{3.18}$$

where $\gamma_i(p)$ have the form ($s \geq 0$)

$$\left.\begin{aligned} &\gamma_1(p) = (1 + \alpha^2 p^2)^{-s/2}, \\ &\gamma_2(p) = \begin{cases} \exp(-\alpha s p) & \text{if } p > 0 \\ 1 & \text{if } p \leq 0 \end{cases} \end{aligned}\right\} \Rightarrow 0 \leq \gamma_i(p) \leq 1, \ (i = 1, 2). \tag{3.19}$$

Then (3.16) holds with $C_1 = 2^{\frac{r-2}{2}}(r-1)$ and $C_2 = A^{\frac{r-2}{2}}$ and (3.17) holds with $\gamma_0 = \alpha s \frac{2-r}{2}$ (see [D2] for details).

Example 2. *Within the class of the viscosities of the type*

$$\nu(p, |\mathbf{D}|^2) = \gamma_p(p)\nu_{\mathbf{D}}(|\mathbf{D}|^2) + \nu_\infty, \ \textit{with } \nu_\infty > 0$$

consider

$$\nu(p, |\mathbf{D}|^2) := \frac{\gamma_3(p)}{\sqrt{|\mathbf{D}|^2 + \varepsilon}} + \nu_\infty \tag{3.20}$$

such that for some γ_∞, $\gamma_0 > 0$

$$0 \leq \gamma_3(p) \leq \gamma_\infty \quad \textit{and} \quad |\gamma_3'(p)| \leq \gamma_0. \tag{3.21}$$

Then (3.20) satisfies the assumptions (3.16)-(3.17) with parameters γ_0 and $r = 2$. Setting $\varepsilon = \nu_\infty = 0$ and $\gamma_3(p) = \alpha(p)$ in (3.20), one obtains a model introduced by Schaeffer [58] in order to describe certain flows of granual materials.

In order to formulate clearly the results that have been recently established for fluids with pressure and shear rate dependent viscosity, we first introduce the notation for particular problems and for suitable function spaces. We set

$$\left.\begin{array}{l}(\mathcal{P}_{\text{steady}})_{\text{dir}}\\ (\mathcal{P}_{\text{steady}})_{\text{nav}}\\ (\mathcal{P}_{\text{evol}})_{\text{per}}\\ (\mathcal{P}_{\text{evol}})_{\text{dir}}\\ (\mathcal{P}_{\text{evol}})_{\text{nav}}\end{array}\right\} \text{ for the problem consisting of } \left\{\begin{array}{l}(3.2), (3.4), (3.8) \text{ and } (3.5),\\ (3.2), (3.4), (3.8) \text{ and } (3.6),\\ (3.1), (3.9) \text{ and } (3.3),\\ (3.1), (3.4), (3.9), (3.3) \text{ and } (3.5),\\ (3.1), (3.4), (3.9), (3.3) \text{ and } (3.6).\end{array}\right.$$

We write that $\Omega \in \mathcal{C}^{0,1}$ if $\Omega \subseteq \mathbb{R}^d, d \geq 2$ is a bounded open connected set with Lipschitz boundary $\partial\Omega$. If in addition the boundary $\partial\Omega$ is locally $\mathcal{C}^{1,1}$ we write $\Omega \in \mathcal{C}^{1,1}$.

Let $r \in [1, \infty]$. The Lebesgue spaces $L^r(\Omega)$ equipped with the norm $\|\cdot\|_r$ and the Sobolev spaces $W^{1,r}(\Omega)$ with the norm $\|\cdot\|_{1,r}$ are defined in the standard way. If X is a Banach space then $X^d = \underbrace{X \times X \times \ldots \times X}_{d-\text{times}}$. The trace of a Sobolev function u is denoted through $\operatorname{tr} u$, if $\mathbf{v} \in (W^{1,r}(\Omega))^d$ then $\operatorname{tr}\mathbf{v} := (\operatorname{tr} v_1, \ldots, \operatorname{tr} v_d)$. For our purpose we introduce the subspaces of vector-valued Sobolev functions which have zero normal part on the boundary. Let $1 \leq q \leq \infty$. We define

$$\begin{aligned}
W_0^{1,q} &:= \overline{\{\mathbf{v} \in (\mathcal{C}^\infty(\Omega))^d; \overline{\operatorname{supp} \mathbf{v}} \subset \Omega\}}^{\|\cdot\|_{1,q}},\\
W_{0,\operatorname{div}}^{1,q} &:= \left\{\mathbf{v} \in W_0^{1,q}; \operatorname{div} \mathbf{v} = 0\right\},\\
W_{\mathbf{n}}^{1,q} &:= \overline{\left\{\mathbf{v} \in (\mathcal{C}^\infty(\Omega))^d \cap (\mathcal{C}(\overline{\Omega}))^d; \operatorname{tr} \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}}^{\|\cdot\|_{1,q}},\\
W_{\mathbf{n},\operatorname{div}}^{1,q} &:= \left\{\mathbf{v} \in W_{\mathbf{n}}^{1,q}; \operatorname{div} \mathbf{v} = 0\right\},\\
L_{\mathbf{n}}^q &:= \overline{\left\{\mathbf{v} \in W_{\mathbf{n},\operatorname{div}}^{1,q}\right\}}^{\|\cdot\|_q}.
\end{aligned}$$

We also introduce the notation for the dual spaces:

$$\begin{aligned}
&W^{-1,q'} := \left(W_0^{1,q}\right)^*, W_{\operatorname{div}}^{-1,q'} := \left(W_{0,\operatorname{div}}^{1,q}\right)^*,\\
&W_{\mathbf{n}}^{-1,q'} := \left(W_{\mathbf{n}}^{1,q}\right)^* \text{ and } W_{\mathbf{n},\operatorname{div}}^{-1,q'} := \left(W_{\mathbf{n},\operatorname{div}}^{1,q}\right)^*.
\end{aligned}$$

All the spaces introduced above are Banach spaces. Moreover, if $1 < q < \infty$ they are also reflexive and separable.

The first theorem discusses the results dealing with stationary problems $(\mathcal{P}_{\text{steady}})_{\text{dir}}$ and $(\mathcal{P}_{\text{steady}})_{\text{nav}}$.

Theorem 1. *Let $\Omega \in \mathcal{C}^{0,1}$ and $p_0 \in \mathbb{R}$ be given. Let $\mathbf{b} \in W_{\mathbf{n}}^{-1,r'}$ for the problem $(\mathcal{P}_{steady})_{nav}$ or $\mathbf{b} \in W_0^{-1,r'}$ for the problem $(\mathcal{P}_{steady})_{dir}$. Assume that ν satisfies* (3.16)-(3.17) *with the parameters r and γ_0 such that*

$$\frac{2d}{d+1} < r \leq 2 \text{ and } 0 \leq \gamma_0 < \frac{C_1}{2C_*(\Omega,2)(C_1+C_2)}, \tag{3.22}$$

where $C_*(\Omega,2)$ is specified below. Then there exists a weak solution to the problem $(\mathcal{P}_{steady})_{dir}$ and $(\mathcal{P}_{steady})_{nav}$ such that

$$\mathbf{v} \in \begin{cases} W^{1,r}_{0,\mathrm{div}} & \text{for } (\mathcal{P}_{steady})_{dir} \\ W^{1,r}_{\mathbf{n},\mathrm{div}} & \text{for } (\mathcal{P}_{steady})_{nav} \end{cases}$$

$$p \in \begin{cases} L^{r'}(\Omega) & \text{for } r \in \left(\frac{3d}{d+2}, 2\right], \\ L^{\frac{dr}{2(d-r)}}(\Omega) & \text{for } r \in \left(\frac{2d}{d+1}, \frac{3d}{d+2}\right], \end{cases}$$

and the weak formulation

$$-(\mathbf{v}\otimes\mathbf{v}, \nabla\boldsymbol{\varphi}) + \big(\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi})\big) + \alpha\int_{\partial\Omega} \mathbf{v}\cdot\boldsymbol{\varphi}\, dS = (p, \operatorname{div}\boldsymbol{\varphi}) + \langle \mathbf{b}, \boldsymbol{\varphi}\rangle \tag{3.23}$$

is valid for all $\boldsymbol{\varphi}$ having the property†

$$\boldsymbol{\varphi} \in \begin{cases} W^{1,q}_{0,\mathrm{div}} & \text{for } (\mathcal{P}_{steady})_{dir} \\ W^{1,q}_{\mathbf{n},\mathrm{div}} & \text{for } (\mathcal{P}_{steady})_{nav} \end{cases}$$

where

$$q = \max\left\{r, \frac{dr}{(d+2)r-2d}\right\} = \begin{cases} r & \text{if } r \in \left(\frac{3d}{d+2}, 2\right], \\ \frac{dr}{(d+2)r-2d} & \text{if } r \in \left(\frac{2d}{d+1}, \frac{3d}{d+2}\right]. \end{cases}$$

The constant $C_*(\Omega, q)$ has relevance to the following problem: for a given $g \in L^q(\Omega)$ with zero mean value, find $\mathbf{z}$ by solving

$$\operatorname{div}\mathbf{z} = g \quad \text{in } \Omega, \qquad \mathbf{z} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.24}$$

It is known (see Bogovskii [9] or Amrouche and Girault [2]) that there is a bounded linear operator $\mathcal{B}$ that maps $L^q(\Omega)$ into $W_0^{1,q}(\Omega)$, for every $q \in (1,\infty)$, such that $\mathbf{z} := \mathcal{B}(g)$ solves (3.24). Particularly, we have

$$\|\mathbf{z}\|_{1,q} = \|\mathcal{B}(g)\|_{1,q} \leq C_*(\Omega,q)\|g\|_q.$$

† Note that the boundary integral $\alpha\int_{\partial\Omega}\mathbf{v}\cdot\boldsymbol{\varphi}\, dS = 0$ for $\boldsymbol{\varphi} \in W_0^{1,q}$.

Comments concerning the proof: For $r \in \left(\frac{3d}{d+2}, 2\right]$ the result for $(\mathcal{P}_{\text{steady}})_{\text{dir}}$ is established in Franta et al. [D4]. The results for $r \in \left(\frac{2d}{d+1}, \frac{3d}{d+2}\right]$ and for $(\mathcal{P}_{\text{steady}})_{\text{nav}}$ can be deduced in a straightforward way from Bulíček et al. [D5] where the authors treat a more complicated evolutionary model $(\mathcal{P}_{\text{evol}})_{\text{nav}}$. While it is not at all clear how to extend the result to $(\mathcal{P}_{\text{evol}})_{\text{dir}}$, for steady flows the extension from $(\mathcal{P}_{\text{steady}})_{\text{nav}}$ to $(\mathcal{P}_{\text{steady}})_{\text{dir}}$ requires only modifications in definitions of relevant function spaces (see also Comments concerning the proof of Theorem 3). Partial regularity of weak solution to $(\mathcal{P}_{\text{steady}})_{\text{dir}}$ for $d = 2, 3$ is established in [36], see the annoucement in [35]. This study initiated the analysis of a generalized Stokes system whereas the generalization consists in replacing the Laplace operator by a general elliptic operator of second order and by replacing the gradient of the pressure by a general first order operator. Such generalized Stokes system are studied by Huy and Stará in [27]. See also the recent thesis [26] by Huy.

The second theorem deals with the problem $(\mathcal{P}_{\text{evol}})_{\text{per}}$. For this purpose we denote L^r_{per} and $W^{1,r}_{\text{per}}$ the standard Lebesgue and Sobolev spaces of L-periodic functions. The Sobolev spaces contain only functions with zero mean value over the periodic cell $\Omega := (0, L)^d$.

Theorem 2. *Let $d = 2, 3$. Let $\Omega \in \mathcal{C}^{0,1}$, $\mathbf{b} \in L^{r'}(0, T; W^{-1,r'}_{per})$, $\mathbf{v}_0 \in L^2_{per,\text{div}}$ and $Q \in L^2(0, T)$ be given. Assume that ν satisfies* (3.16)-(3.17) *with the parameters r and γ_0 such that*

$$r \in \begin{cases} \left(\dfrac{4}{3}, 2\right] & \text{if } d = 2 \\ \left(\dfrac{9}{5}, 2\right] & \text{if } d = 3 \end{cases} \quad \text{and } \gamma_0 = \min\left\{\frac{1}{2}, \frac{C_1}{4C_2}\right\}.$$

Then there exists a weak solution to the problem $(\mathcal{P}_{evol})_{per}$ such that

$$\begin{aligned} &\mathbf{v} \in \mathcal{C}(0, T; L^2_{weak}) \cap L^r(0, T; W^{1,r}_{per,\text{div}}), \\ &\mathbf{v}_{,t} \in \begin{cases} L^r(0, T; W^{-1,r}_{per}) & \text{if } d = 2, \\ L^{\frac{5r}{6}}(0, T; W^{-1,\frac{5r}{6}}_{per}) & \text{if } d = 3, \end{cases} \\ &p \in \begin{cases} L^r(0, T; L^r(\Omega)) & \text{if } d = 2, \\ L^{\frac{5r}{6}}(0, T; L^{\frac{5r}{6}}(\Omega)) & \text{if } d = 3, \end{cases} \end{aligned}$$

and the weak formulation

$$\langle \mathbf{v}_{,t}, \boldsymbol{\varphi} \rangle - (\mathbf{v} \otimes \mathbf{v}, \nabla \boldsymbol{\varphi}) + \big(\nu(p, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi})\big) = (p, \operatorname{div} \boldsymbol{\varphi}) + \langle \mathbf{b}, \boldsymbol{\varphi} \rangle \quad (3.25)$$

is valid for almost all $t \in (0,T)$ *and all* $\boldsymbol{\varphi}$ *having the property*

$$\boldsymbol{\varphi} \in \begin{cases} L^{r'}(0,T;W^{1,r'}_{per}) & \text{if } d=2,\\ L^{\frac{5r}{5r-6}}(0,T;W^{1,\frac{5r}{5r-6}}_{per}) & \text{if } d=3. \end{cases}$$

Comments concerning the proof: The result in three dimensions is established in [D2], and the two-dimensional case is treated in [D3].

The final theorem deals with the problem $(\mathcal{P}_{\text{evol}})_{\text{nav}}$.

Theorem 3. *Let* $d=2,3$*. Let* $\Omega \in \mathcal{C}^{1,1}$*,* $\mathbf{b} \in L^{r'}(0,T;W^{-1,r'}_{\mathbf{n}})$*,* $\mathbf{v}_0 \in L^2_{\mathbf{n},\text{div}}$ *and* $Q \in L^2(0,T)$ *be given. Assume that* ν *satisfies* (3.16)-(3.17) *with the parameters* r *and* γ_0 *such that*

$$r \in \begin{cases} \left(\frac{3}{2},2\right) & \text{if } d=2\\ \left(\frac{8}{5},2\right) & \text{if } d=3 \end{cases} \qquad \text{and} \qquad \gamma_0 < \frac{1}{C_{\#}(\Omega,2)}\frac{C_1}{C_1+C_2}.$$

where $C_{\#}(\Omega,2)$ *appears in* (3.27) *below. Then there exists a weak solution to the problem* $(\mathcal{P}_{evol})_{nav}$ *such that*

$$\begin{aligned} \mathbf{v} &\in \mathcal{C}(0,T;L^2_{weak}) \cap L^r(0,T;W^{1,r}_{\mathbf{n},\text{div}}),\\ \mathbf{v}_{,t} &\in \left(L^{\frac{r(d+2)}{r(d+2)-2d}}(0,T;W^{1,\frac{r(d+2)}{r(d+2)-2d}}_{\mathbf{n}})\right)^*,\\ p &\in \begin{cases} L^r(0,T;L^r(\Omega)) & \text{if } d=2,\\ L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)) & \text{if } d=3, \end{cases} \end{aligned}$$

and the weak formulation

$$\begin{aligned} \langle \mathbf{v}_{,t},\boldsymbol{\varphi}\rangle - (\mathbf{v}\otimes\mathbf{v},\nabla\boldsymbol{\varphi}) + \big(\nu(p,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}),\mathbf{D}(\boldsymbol{\varphi})\big)&\\ + \alpha\int_{\partial\Omega}\mathbf{v}\cdot\boldsymbol{\varphi}\,dS = (p,\operatorname{div}\boldsymbol{\varphi}) + \langle \mathbf{b},\boldsymbol{\varphi}\rangle& \end{aligned} \tag{3.26}$$

is valid for almost all $t \in (0,T)$ *and all* $\boldsymbol{\varphi}$ *having the property*

$$\boldsymbol{\varphi} \in \begin{cases} L^{r'}(0,T;W^{1,r'}_{\mathbf{n}}) & \text{if } d=2,\\ L^{\frac{5r}{5r-6}}(0,T;W^{1,\frac{5r}{5r-6}}_{\mathbf{n}}) & \text{if } d=3. \end{cases}$$

Moreover, if instead of (3.17) *one assumes the condition (8) from* [D5] *then the results holds also for the case* $r=2$*. If in addition* $d=2$ *then the weak solution is unique.*

The constant $C_{\#}(\Omega, q)$ occurs in the solvability of the following problem. For $q \in (1, \infty)$ and $\Omega \in \mathcal{C}^{1,1}$ and an arbitrary $\varphi \in L^q(\Omega)$ with zero mean value to find $g \in W^{2,q}(\Omega)$ solving

$$\triangle g = \varphi \quad \text{in } \Omega, \quad \nabla g \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega g(x)\, dx = 0$$

satisfying

$$\|g\|_{2,q} \le C_{\#}(\Omega, q)\|\varphi\|_q. \tag{3.27}$$

Comments: Even for steady flows, there is a remarkable difference in how the pressure is introduced. While for problems where the vicosity is independent of pressure, the pressure can be easily identified using for example de Rham's theorem, the same method cannot be used for problems with pressure dependent viscosity since one needs to have knowledge of the nature of the pressure à priori. Our approach uses the quasi-compressible approximations.

There is also another crucial difference in introducing the pressure for the evolutionary NSEs and time-dependent models with non-constant viscosity. For the NSEs, we can identify the model with an evolutionary Stokes system, where the convective term is included in the right-hand side, and apply results concerning the L^p-estimates available for such systems (see [60]). For the models where ν is not constant (and may depend on p or $|\mathbf{D}(\mathbf{v})|^2$) an analogous theory for generalized Stokes system is not available thus far. In the article [D5] (see Theorem 3), Bulíček et al. show that the problem with Navier's slip boundary conditions, contrary to that with no-slip boundary conditions, does not suffer such a deficiency and it is possible to introduce the pressure globally.

Theorem 3 establishes the first result concerning long-time and large-data existence of weak solutions to any incompressible fluid model where the viscosity depends on the pressure and where flows take place in a bounded container. Theorem 3 covers several interesting results even for fluids whose viscosity is independent of the pressure (as are for example Navier-Stokes or power-law fluids). We refer to [D5] for details.

(*d*) *Compactness criteria for the velocity gradient*

In the analysis of problems $(\mathcal{P}_{\text{steady}})_{\text{nav}}$, $(\mathcal{P}_{\text{steady}})_{\text{dir}}$, $(\mathcal{P}_{\text{evol}})_{\text{per}}$, $(\mathcal{P}_{\text{evol}})_{\text{dir}}$ or $(\mathcal{P}_{\text{evol}})_{\text{nav}}$, an important role played the knowledge of various methods to obtain almost everywhere convergence for the velocity gradient. These methods were developed when analyzing the systems of PDEs describing flows of incompressible fluids with the viscosity depending purely on the shear rate. The aim of this subsection is to provide a survey of these techniques. For this purpose we restrict ourselves to the case when

$$\mathbf{T} = -p\mathbf{I} + \nu(|\mathbf{D}|^2)\mathbf{D}\,, \tag{3.28}$$

assuming the validity of the assumption (3.16). In fact, just for brevity we consider a typical example of (3.28), namely, the power-law fluids given by

$$\mathbf{T} = -p\mathbf{I} + \nu_0|\mathbf{D}|^{r-2}\mathbf{D}\,, \qquad \text{with } r > 1\,. \tag{3.29}$$

A key issue in the proof of the existence of a weak solution to any (system of) nonlinear PDEs is the stability of weak solutions with respect to weak convergence. This property, called *weak stability* of the system of PDEs, written symbolically as $PDE(u) = b$, can be made more precise in the following way. Assume that a sequence $\{u^\varepsilon\}$ enjoys two properties: *(i)* $\{u^\varepsilon\}$ satisfies uniformly apriori estimates in a (reflexive Banach) function space Y that are available for a given problem, i.e. $\sup_\varepsilon \|u^\varepsilon\|_Y \le K$, and *(ii)* $\{u^\varepsilon\}$ solves $PDE(u^\varepsilon) = b^\varepsilon$ with the right-hand side b^ε converging strongly to b in a suitable (dual) norm. It follows from *(i)* that modulo a subsequence u^ε converges weakly to u in Y. It is well known that weakly converging sequences do not commute with nonlinearities in general. Taking the limit $\varepsilon \to 0$ in $PDE(u^\varepsilon) = b^\varepsilon$, we say that the problem is weakly stable if one can show that u solves the original problem $PDE(u) = b$.

In order to investigate the weak stability of our problems $(\mathcal{P}_{\text{steady}})_{\text{dir}}$, $(\mathcal{P}_{\text{evol}})_{\text{per}}$, etc., we first summarize the basic energy estimates and their consequences. For simplicity, we restrict ourselves to the most interesting three-dimensional flows, i.e., $d = 3$.

The balance of mechanical energy implies the following estimates for suitable approximations $(\mathbf{v}^\varepsilon, p^\varepsilon)$ of (3.1)

$$\|\mathbf{v}^\varepsilon(t)\|_2^2 + \int_0^T \|\nabla\mathbf{v}^\varepsilon\|_r^r\,ds + \int_0^T \|\mathbf{v}^\varepsilon\|_{\frac{5r}{3}}^{\frac{5r}{3}}\,ds \le K\,. \tag{3.30}$$

From here we have the following bound on $\partial_t\mathbf{v}^\varepsilon$ (see (5.2.25) in Málek et al. [38], p.207)

$$\|\partial_t\mathbf{v}^\varepsilon\|_{L^{r'}(0,T;(W^{s,2}_{div})^*)} \le K\,, \qquad s > \frac{5}{2}\,,\; r' = \frac{r}{r-1}\,. \tag{3.31}$$

This immediately implies (at least for a subsequence) that

$$\begin{aligned} \nabla \mathbf{v}^\varepsilon \to \nabla \mathbf{v} \quad & \text{weakly in} \;\; L^r(0,T;L^r(\Omega))\,, \\ \mathbf{v}^\varepsilon_t \to \mathbf{v}_t \quad & \text{weakly in} \;\; L^{r'}(0,T;(W^{s,2}_{div})^*)\,, \end{aligned} \tag{3.32}$$

and if $r > \frac{6}{5}$ the Aubin-Lions compactness lemma gives

$$\mathbf{v}^\varepsilon \to \mathbf{v} \quad \text{strongly in } L^s(0,T;L^2(\Omega)) \qquad (\forall s \in \langle 1,\infty))\,. \tag{3.33}$$

It then easily follows from (3.32) and (3.33) that if $r > \frac{6}{5}$ one can pass to the limit both in the convective term and in the time derivative term.

The key point is the passage to the limit in the nonlinear elliptic term involving $\mathbf{S}(\mathbf{D}(\mathbf{v})) = 2\nu_0|\mathbf{D}(\mathbf{v})|^{r-2}\mathbf{D}(\mathbf{v})$. Since $|\mathbf{S}(\mathbf{D})| \le C\,|\mathbf{D}|^{r-1}$, this will follow from

$$\text{almost everywhere convergence of } \nabla \mathbf{v}^k \text{ to } \nabla \mathbf{v} \text{ in } Q_T\,. \;\big(\{\mathbf{v}^k\} \subset \{\mathbf{v}^\varepsilon\}\big) \tag{3.34}$$

We present four different ways how to obtain (3.34).

1. Higher regularity method. It is based on standard difference quotient technique, which leads for spatially periodic or Cauchy problem (see Málek et al. [38] and Pokorný [43]) to the inequality

$$\begin{aligned} \frac{d}{dt}\|\nabla \mathbf{v}^\varepsilon\|_2^2 + J(\mathbf{v}^\varepsilon) \le C\|\nabla \mathbf{v}^\varepsilon\|_2^{2\lambda}\|\nabla \mathbf{v}^\varepsilon\|_r^r \qquad & \text{with } \lambda = \frac{2(3-r)}{3r-5} \\ \text{and } \; J(\mathbf{v}^\varepsilon) = \int_\Omega |\mathbf{D}(\mathbf{v}^\varepsilon)|^{r-2}\,|\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2\,dx\,dt\,. & \end{aligned} \tag{3.35}$$

If $\lambda \le 1$, which happens if $r \ge \frac{11}{5}$, then we obtain higher regularity from which (3.34) follows using the compact imbedding. See Málek et al. [38] for more details; note however that $J(\mathbf{v}^\varepsilon)$ is not degenerate therein.

If $\lambda > 1$ (it means that $r < \frac{11}{5}$), the inequality (3.35) does not, in general, improve the smoothness of the solution. However, it follows from (3.35) and (3.30) that

$$\int_0^T \frac{J(\mathbf{v}^\varepsilon)}{(1+\|\nabla \mathbf{v}^\varepsilon\|_2^2)^\lambda}\,ds \le K\,. \tag{3.36}$$

The further arguments differ in dependence whether $r > 2$ or $r < 2$. For simplicity, let us consider the latter. Then (3.36) together with (3.30) implies that if $(2 \ge) r > \frac{9}{5}$ we have

$$\int_0^T \|\nabla^{(2)}\mathbf{v}^\varepsilon\|_r^{2\beta}\,dt \le K \qquad \text{with } \beta \in (0, \tfrac{1}{3}]\,. \tag{3.37}$$

All these estimates together with an interpolation technique for certain $\sigma > 0$ and $r_0 \in (1, r)$ lead then to the following

$$\left.\begin{aligned} \int_0^T \|\mathbf{v}^\varepsilon\|_{1,r}^r \, dt \le K \\ \int_0^T \|\mathbf{v}^\varepsilon\|_{2,r}^{2\beta} \, dt \le K \end{aligned}\right\} \implies \int_0^T \|\mathbf{v}^\varepsilon\|_{1+\sigma,r}^{r_0} \, dt \le K \,. \tag{3.38}$$

Having (3.38) and (3.31) at hand, we apply the Aubin-Lions compactness lemma again and conclude the strong convergence of $\nabla \mathbf{v}^\varepsilon$ to $\nabla \mathbf{v}$ in $L^{r_0}(0, T; W^{1,r})$, which implies (3.34).

This method has been extended to the no-slip boundary condition (see Málek, Nečas and Růžička [D9] for details), which seems to be a nontrivial task as we have to apply the regularity techniques up to the boundary and in addition to overcome the presence of the nonlocal quantity, the pressure. In this case, the presence of the boundary leads to results that are worse than in the spatially-periodic case. On the other hand, it follows from Málek, Nečas and Růžička [D9] that there is a weak solution to 3d NSEs with the no-slip boundary satisfying

$$\int_0^T \|\nabla^2 \mathbf{v}\|_2^{2/3} \, dt \le C \,,$$

the result that has been known before only for the spatially periodic case, see Foias, Guillopé and Temam [19].

The following three methods have a common goal: to show that for some $\theta \in (0, 1)$, say $\theta = \frac{1}{2}$, and for an arbitrarily chosen $\eta > 0$ there is $\{\mathbf{v}^n\} \subset \{\mathbf{v}^\varepsilon\}$ such that

$$\limsup_{n\to\infty} \int_0^T \int_\Omega \Big[\{\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\} : \mathbf{D}(\mathbf{v}^n - \mathbf{v}) \Big]^\theta \, dx \, dt \le \eta \,. \tag{3.39}$$

Since the corresponding nonlinear operator is strictly monotone, once we obtain (3.39), the almost everywhere convergence (3.34) (at least for a suitable chosen subsequence) follows.

2. Standard monotone operator method. To achieve (3.39) it seems natural to multiply the approximate problem by $\mathbf{v}^n - \mathbf{v}$. This is possible for all type of considered boundary conditions, as shown by O. A. Ladyzhenskaya for the first time† (see [32] for example), but it restricts ourselves to the range of r,

$$r \ge \frac{11}{5} \,, \tag{3.40}$$

† See also Lions [34].

due to the required finiteness of the integral

$$\int_{Q_T} v_k^n \partial_{x_k} \mathbf{v}^n \cdot (\mathbf{v}^n - \mathbf{v})\, dx\, dt\,. \tag{3.41}$$

Once the integrability is assured, the term in (3.41) vanishes as $n \to \infty$ thanks to (3.33).

3. Method of truncated Sobolev functions. In order to obtain (3.39) for $r < \frac{11}{5}$, one can attempt to replace the multiplier $\mathbf{v}^n - \mathbf{v}$ by its bounded truncation

$$\boldsymbol{\psi}^n := (\mathbf{v}^n - \mathbf{v})(1 - \min\{1, \delta^{-1}|\mathbf{v}^n - \mathbf{v}|\})\,. \qquad (\delta > 0) \tag{3.42}$$

That it is in principle possible the reader can verify in Frehse, Málek and Steinhauer [21], where the existence of weak solution to the spatially periodic problem and to the problem with slip boundary conditions (set $\alpha = 0$ in (3.6)) are established for

$$r > \frac{8}{5}\,. \tag{3.43}$$

Note however that δ has to be chosen in a suitable way so that the difference between the integral in (3.39) and the term

$$\int_0^T \int_\Omega \{\mathbf{S}(\mathbf{D}(\mathbf{v}^n)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\} : \mathbf{D}(\boldsymbol{\psi}^n)\, dx\, dt$$

can be made as small as needed (at least for a subsequence). This requirement leads to show that for all $\eta > 0$ there is $\{\mathbf{v}^k\} \subset \{\mathbf{v}^n\}$ and δ independent of k such that

$$\int_{Q_\delta^k} \{\mathbf{S}(\mathbf{D}(\mathbf{v}^k)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\} : \mathbf{D}((\mathbf{v}^k - \mathbf{v}) \min\{1, \delta^{-1}|\mathbf{v}^k - \mathbf{v}|\})\, dx\, dt \le \eta\,,$$

$$\text{where } Q_\delta^k := \{(x,t) \in Q_T; |\mathbf{v}^k - \mathbf{v}| < \delta\}\,. \tag{3.44}$$

Note that functions behind the integral sign (restricted to Q_δ^k) are uniformly bounded in $L^1(Q_T)$. A proof of (3.44) is based on the following assertion:
Let $\{g^n\}$ be such that $\|g^n\|_{L^1(Q_T)} \le M$ for all n. Then for all $\eta > 0$ there are $\{\mathbf{v}^k\} \subset \{\mathbf{v}^n\}$, $\{g^k\} \subset \{g^n\}$ and $\delta < \frac{1}{M}$ independent of k such that

$$\int_{A_\delta^k} g^k\, dx\, dt \le \eta\,, \qquad \textit{where } A_\delta^k := \{(x,t) \in Q_T; \delta^2 \le |\mathbf{v}^k - \mathbf{v}| < \delta\}\,.$$

The bound (3.43) is due to the convective term: it follows from (3.30) that $\mathbf{v}_k^n \partial_{x_k} \mathbf{v}^n \in L^1(Q_T)$ uniformly if (3.43) holds.

The reason why the problem completed by the no-slip boundary conditions is not covered in Frehse et al. [21] has its origin in the fact that $\operatorname{div} \boldsymbol{\psi}^n \neq 0$. As indicated in Sect. 2.1, it is in general difficult to introduce the pressure to (3.1). Thus we can modify (3.42) by subtracting a suitable vector field so that the result is divergenceless. Subtracting $\mathbf{h}^n$ satisfying $\operatorname{div} \mathbf{h}^n = \operatorname{div} \boldsymbol{\psi}^n$ with no-slip boundary conditions we obtain test functions with correct boundary conditions but not enough regularity, while subtracting ∇z^n, where $-\Delta z^n = \operatorname{div} \boldsymbol{\psi}^n$ with homogeneous Neumann boundary conditions, we obtain suitable test functions only for problems with Navier's boundary or without boundary.

Despite these difficulties, we have conjectured (see Frehse and Málek [20]) that even in the case of no-slip boundary conditions the existence of weak solution for $r > \frac{8}{5}$ can be proved via this method. This conjecture is proved in the recent work of Wolf [67].

4. Method of Lipschitz truncations of Sobolev functions. This method gives the existence of a weak solution (see Frehse, Málek and Steinhauer [D6] and Diening, Málek and Steinhauer [14] for stationary problems, and the recent study by Diening, Růžička and Wolf [15] for evolutionary ones) for

$$r > \frac{6}{5}\,. \tag{3.45}$$

In stationary problems, this method stems from the following assertion (see Acerbi and Fusco [1] and Landes [33] for example):
There exists a $C > 0$ such that for all $u^n \to 0$ weakly in $W_0^{1,r}(\Omega)$ and all $\lambda > 0$ there are $u_\lambda^n \in W_0^{1,\infty}(\Omega)$ such that

$$\|\nabla u_\lambda^n\|_{L^\infty(\Omega)} \leq C(d)\,\lambda\,, \tag{3.46}$$

$$u_\lambda^n \to 0 \textit{ strongly in } L^\infty(\Omega)\,, \tag{3.47}$$

$$u_\lambda^n \to 0 \textit{ weakly in } W_0^{1,s}(\Omega)\,, \qquad (\forall s \in [1,\infty)) \tag{3.48}$$

$$|\{x \in \Omega;\, u^n(x) \neq u_\lambda^n(x)\}| \leq C\lambda^{-r}\|u^n\|^r_{W^{1,r}(\Omega)}\,. \tag{3.49}$$

We remark that it follows from the proof that

$$|\{x \in \Omega;\; u^n(x) \neq u_\lambda^n(x)\}| = |\{x \in \Omega;\; M(|\nabla u^n(x)|) + M(u^n(x)) > \lambda\}|\,.$$

where $M(f)$ denotes the Hardy-Littlewood maximal function to f.

Note that in general for any $\mu > 0$ one has

$$|\{x \in \Omega;\; M(|\nabla u^n(x)|) + M(u^n(x)) > \mu\}| \leq C\mu^{-r}\|u^n\|^r_{W^{1,r}(\Omega)}\,. \tag{3.50}$$

.

Using these observations, we can replace the test function $\mathbf{v}^n - \mathbf{v}$ by its Lipschitz truncations $(\mathbf{v}^n - \mathbf{v})_\lambda$. The bound (3.45) is due to the convective term, surprisingly. We have

$$\int_\Omega \mathbf{v}^n \otimes \mathbf{v}^n : \nabla(\mathbf{v}^n - \mathbf{v})_\lambda \, dx = \int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla(\mathbf{v}^n - \mathbf{v})_\lambda \, dx + \int_\Omega [(\mathbf{v}^n - \mathbf{v}) \otimes \mathbf{v}^n + \mathbf{v} \otimes (\mathbf{v}^n - \mathbf{v})] : \nabla(\mathbf{v}^n - \mathbf{v})_\lambda \, dx \, .$$

Taking $\lambda > 0$ and $r > \frac{6}{5}$ fixed and letting $n \to \infty$ we observe that the first term at the right hand vanishes thanks to (3.48) applied to $\mathbf{u}^n = (\mathbf{v}^n - \mathbf{v})$, while the second term tends to 0 thanks to (3.46) and the strong convergence of $\mathbf{v}^n \to \mathbf{v}$ in $L^2(\Omega)$.

Note again that λ has to be chosen in an appropriate way so that the integral

$$\int_{\Omega^k_\lambda} \{\mathbf{S}(\mathbf{D}(\mathbf{v}^k)) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\} : \mathbf{D}((\mathbf{v}^k - \mathbf{v})_\lambda) \, dx \qquad \text{with } \Omega^k_\lambda := \{x \in \Omega; (\mathbf{v}^k - \mathbf{v})_\lambda(x) \neq (\mathbf{v}^k - \mathbf{v})(x)\} \tag{3.51}$$

is small. Here, the following assertion is used (see [D6] for details):
Let $\{g^n\}$ be such that $\|g^n\|_{L^1(\Omega)} \leq K$ for all n. Then for all $\eta > 0$ there are $\{g^k\} \subset \{g^n\}$, $\{\mathbf{v}^k\} \subset \{\mathbf{v}^n\}$ and $\lambda \geq \frac{1}{\eta}$ independent of k such that $\int_{A^k_\lambda} g^k \, dx \leq \eta$, where $A^k_\lambda := \{x \in \Omega; \lambda < M(|\nabla(\mathbf{v}^k - \mathbf{v})(x)|) + M(\mathbf{v}^k(x) - \mathbf{v}(x)) \leq \lambda^2\}$.

4. Novelties of the studies in this collection

The following list aims to clarify why we think that the mathematical analysis of models studied in this thesis is of interest and what are the main new observations achieved in the theoretical analysis of the relevant problems.

- The studies [D1]–[D5] opened a new topic of investigations in the area of mathematical analysis of incompressible Navier-Stokes equations and its generalizations. At the first glance, the governing systems have very similar structure, and the same unknowns as the Navier-Stokes equations. On the other hand, even investigation of simple flows in special geometrical setting shows that flows for fluids with varying viscosity may differ tremendously from flows of the Navier-Stokes fluid. The studies [D1]–[D5] switch the attention in the analysis from the velocity field to the pressure. Note that the pressure is frequently elimininated from the analysis of the Navier-Stokes equations and similar systems by projecting the equations to the set of divergenceless functions. On the other hand, there are recent regularity criteria for the evolutionary Navier-Stokes system in three dimensions (see Seregin and Šverák [59] or Titi [64]) calling for understanding of the properties of the pressure. Naturally, the pressure cannot be eliminated from analysis in problems where the material coefficients are pressure dependent.

- To our knowledge, to date there is no global existence theory that is in place for both steady and unsteady flows of fluids whose viscosity depends *purely* on the pressure. Previous studies by Renardy [55], Gazzola [22] and Gazzola and Secchi [23] provide rather restrictive results not supporting the fact that fluids with pressure dependent viscosities are popular in many areas of engineering science. The studies [D1]-[D5] show that one can build a consistent mathematical theory for such fluids. A key step is based on the observation that adding the sublinear dependence of the viscosity on the shear rate may help significantly. Need to say that in most experimental studies of the viscosity-pressure dependence the dependence of the viscosities on the shear rate is not measured. The exponential dependence of the viscosity on the pressure, which is quite popular in applications, does not fulfil the assumptions need in the presented consistent theory. However, the exponential viscosity-pressure relationship can be approximated by a suitable sequence of viscosities, depending on the shear rate and the pressure, for which the theory developed in [D1]–[D5] is applicable.

- The models for incompressible fluids with the vicosities depending on the mean normal stress (the pressure) serve as an example of implicit constitutive models. Implicit constitutive theory is very recent and modern approach in continuum physics used to capture in elegant way a complicated response of complex materials, see Rajagopal [46] and [47]. In our opinion, it is of interest that the mathematical results to some of these implicit models are already in place.

- In the existence theory the notion of weak (or suitable weak) solution seems to be very natural. This solution belongs to function spaces that come from the apriori estimates that are in our case giving the weak convergence of the velocity gradient and the pressure. Methods how one can obtain the compactness (almost everywhere convergence) for the velocity gradient and the pressure represent in our opinion a main discovery in the theory.

- The analysis of flows of incompressible fluids with pressure (and shear-rate) dependent viscosity requires to have the pressure in hands from the beginning. In order to be able to establish the results for the evolutionary model in a bounded domain we observe that Navier's slip boundary conditions are more suitable for introducing the pressure globally. It is an open question whether one can introduce the pressure globally also for no-slip boundary conditions. Interestingly, the results obtained for the Navier's slip boundary conditions seem to be new even for the classical Navier-Stokes or power-law fluids.

- The method of trajectories developed in [D7] represents a new tool to study long-time dynamics of infinite-dimensional dynamical system. While standard approaches investigate the properties of all equilibria described by stationary problems, the method of trajectories deals with solution trajectories of a fixed finite length and looks for their long-time properties in Bochner spaces. The behavior of these trajectories is still described by evolutionary PDEs, which reveals to be a significant advantage.

- The method how one can use Lipschitz truncations of Sobolev functions to obtain compactness of the velocity gradient seems to be an original approach developed in [D6] in order to extend the available existence theory for large range of model parameters, interesting from the point of view of engineering applications.

- In the regularity theory of weak solutions to systems of nonlinear PDEs, the extensions of the results up to the boundary is frequently missing. In addition, the results may differ in dependence on the prescribed boundary conditions. The articles [D9] and [D10] not only present the results valid up to the boundary for homogeneous Dirichlet problem, but more importantly they provide techniques how the regularity near the boundary of a domain should be investigated.

- The results, methods and tools developed in the analysis of models considered in this dissertation are used or extended in the analysis of more complicated models for incompressible fluids: inhomogeneous fluids, rate type fluids, fluids with the viscosity dependent on the temperature, electric or magnetic field, concentration, power-law like fluids with a variable power-law exponent, etc.

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