# Inferring coupling structure from dynamics in complex systems: consequences for graph-theoretical analysis

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## Introduction I: Motivation

Characterisation of complex systems commonly involves the study of their structure using graph theory [1]. For both practical and theoretical reasons, the links are commonly quantified by the dependence of the observed time series (functional connectivity, FC) rather than the underlying physical or coupling network of connections (structural connectivity, SC).

It is increasingly recognised that various confounding factors may affect graph theoretical analysis of complex systems, such as the system spatial structure and its sampling [4]. In the presented study, we show that there are tendencies towards specific graph structures in functional connectivity matrices. A prime example of such an effect is the increased levels of clustering in Pearson correlation coefficient-based graphs, further affecting the estimates of small-world indices.

## Introduction II: Theory

Network described by an unweighted graph $G = (V, E)$: $V$ is the set of nodes of $G$, $n = \#V$ is the number of nodes, $E \subset V^2$ is the set of the edges of $G$. A graph $G$ can be characterised by global properties, including the average path length

$$L = \frac{1}{n \cdot (n-1)} \cdot \sum_{i,j} d_{i,j}; \quad d_{i,j} \in D, \tag{1}$$

where $D$ is the distance matrix; and the clustering coefficient

$$C = \frac{1}{n} \sum_{i \in V} c_i; \quad c_i = \frac{\sum_{j,\ell} a_{i,j} a_{j,\ell} a_{\ell,i}}{k_i(k_i - 1)}. \tag{2}$$

A network si considered to be "small world" if it has similar average path length, but increased clustering coefficient compared to a corresponding random graph ($\lambda = \frac{L}{L_{rand}} \sim 1$, $\gamma = \frac{C}{C_{rand}} \gg 1$). These properties are summarized by $\sigma = \frac{\gamma}{\lambda} \gg 1$.

## Methods

Linear model: Multivariate AR(p) process of dimension $n$:

$$y_t = c + A_1 y_{t-1} + A_2 y_{p-2} + \ldots + A_p y_{t-p} + e_t, \tag{3}$$

where $c$ is a $n \times 1$ vector of constants, $A_i$ is a $n \times n$ matrix (for every $i = 1, \ldots, p$) and $e_t$ is a $n \times 1$ vector of error terms. For simplicity we choose $c = \mathbf{0}_{n,1}$ and $e_t \sim \mathcal{N}(0, 1)$, $p = 1, A \equiv A_1$.

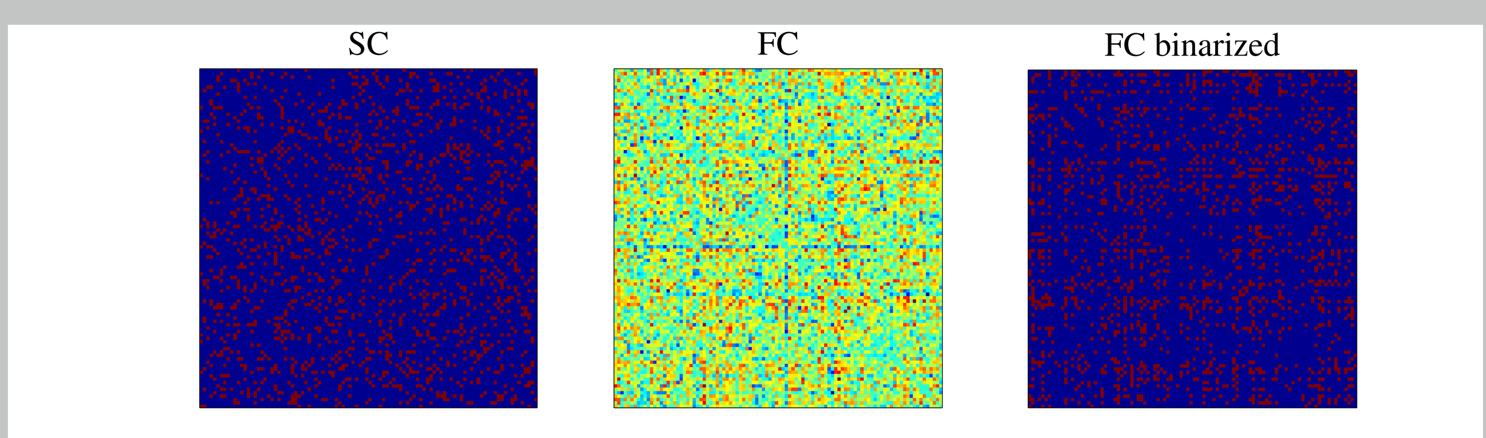
The matrix A is generated randomly by the following procedure:

- A binary structural connectivity matrix $S(n, p)$ is generated as a realization of the Erdős-Rényi model $G(n, p)$.
- Autocorrelation of the time series and further controllable parameters are introduced by setting $A = \frac{a}{|\lambda_{max}|}(S(n, p) + \alpha\mathbb{I})$ where $\lambda_{max}$ is the largest (in absolute value) eigenvalue of the matrix $S(n, p) + \alpha\mathbb{I}$.
- The choice of $a \in (0, 1)$ assures that the AR model is stationary.
- The value of $s = \frac{\alpha}{\sqrt{p(n-1)}}$ controls the ratio of variance due to autocorrelation and cross-interactions, we set $s = 1$.

Functional connectivity graph construction: FC is computed by thresholding the $n \times n$ correlation matrix of the generated time series $y_t$ (diagonal is set to 0). Two thresholding methods were used:

1) Threshold is chosen such that $\#E_{FC} = \#E_{SC}$, FC compared to SC
2) A range of thresholds; for each the FC graph is compared to a corresponding random graph that conserves the average degree (Erdős-Rényi model $G(n, p)$), or the degree distribution.

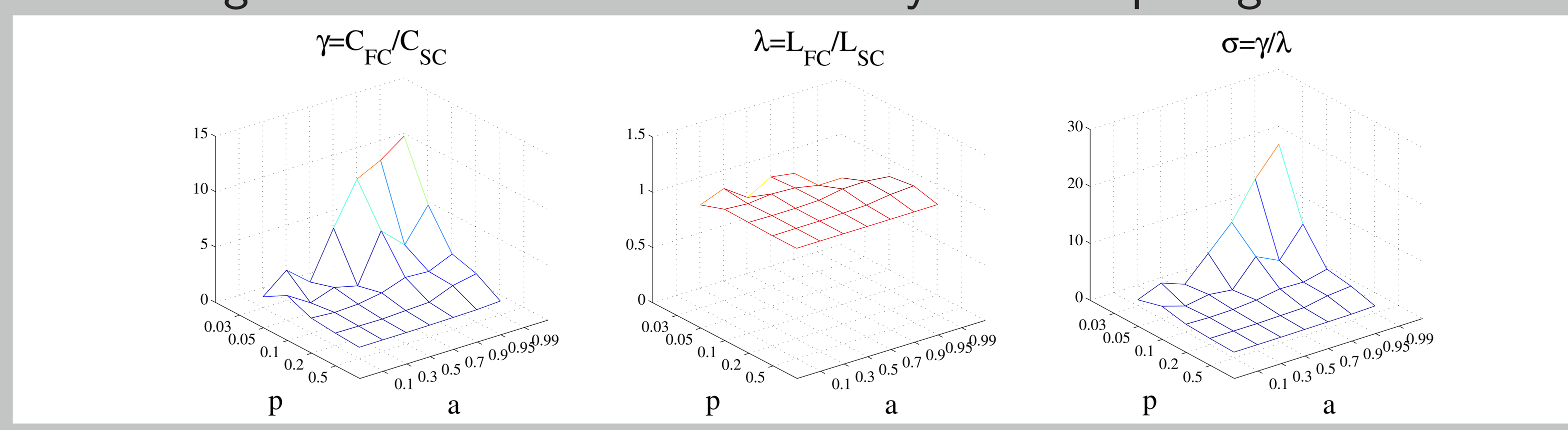
- Graph measures $L$, $C$ are computed and compared by means of $\lambda, \gamma, \sigma$
- Non-random structure of functional connectivity (FC) visible by naked eye:



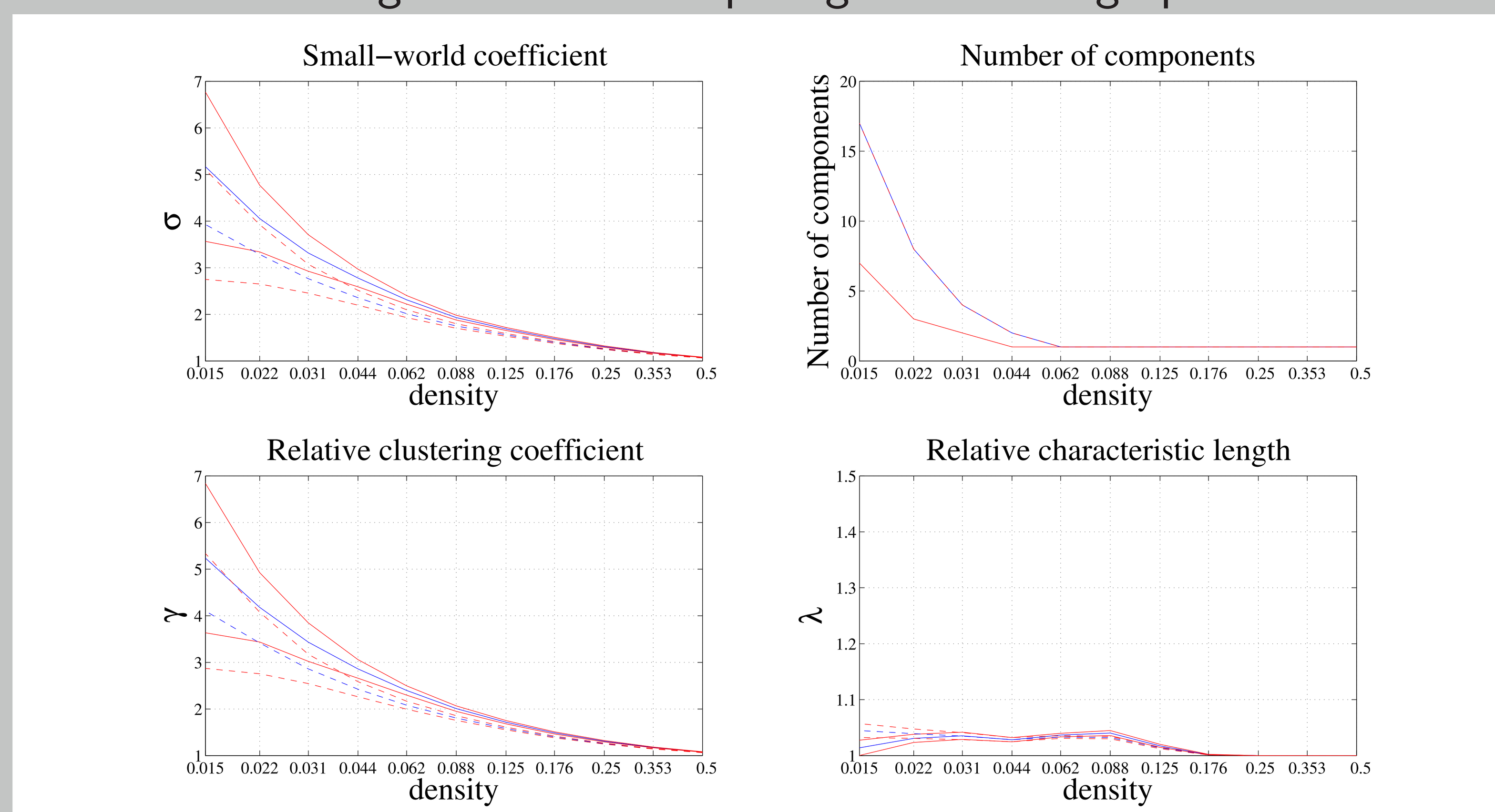

## Results

Even with a random coupling matrix, the functional connectivity matrix shows signs of apparent small-world structure.

- Thresholding FC at structural matrix density and comparing to SC

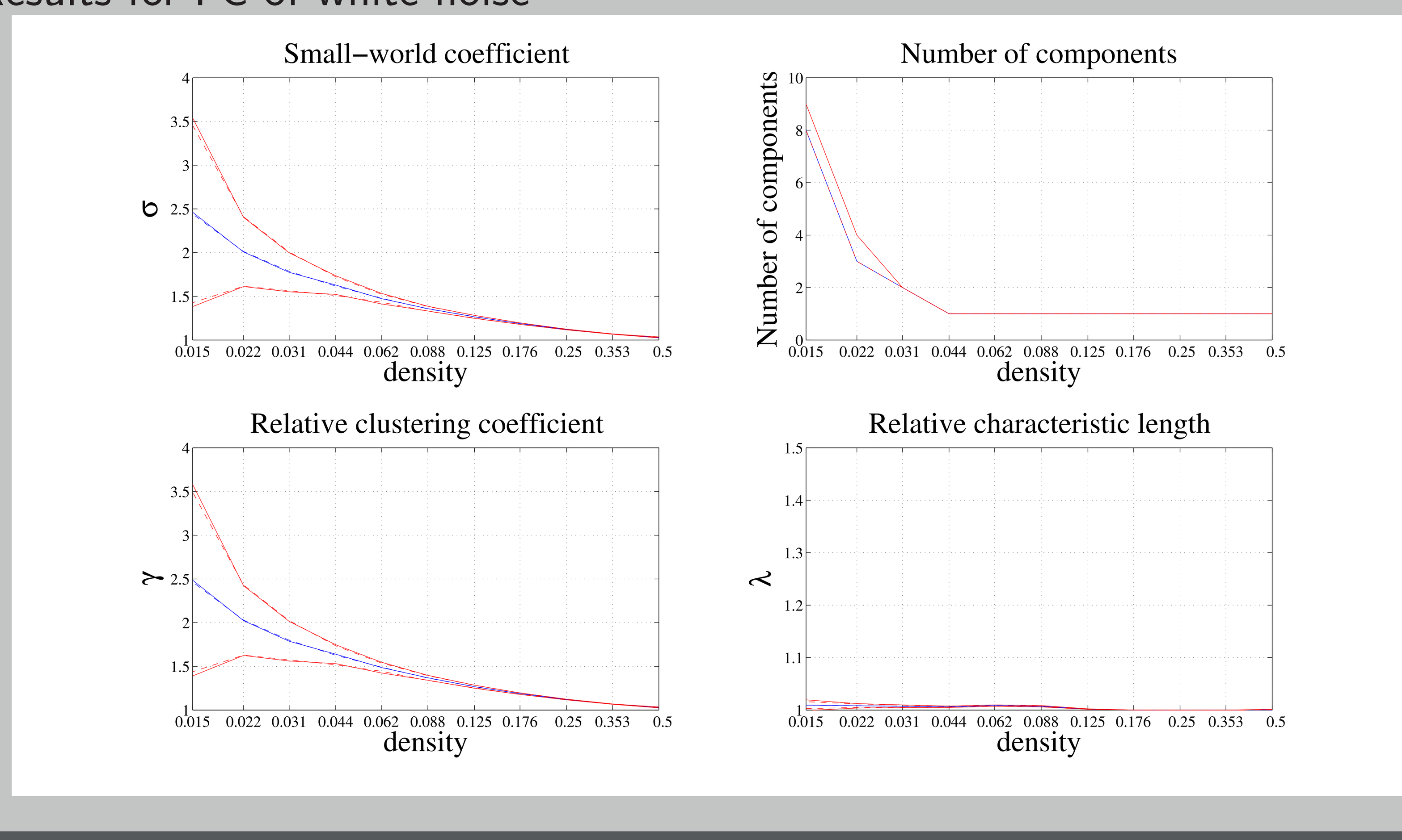



- Variable thresholding of FC and comparing to random graph




Statistics for $N = 100$ realisations.Maximum number of components shown (blue: FC graph, red: random graph). For $\sigma, \gamma, \lambda$ we show the mean(blue) $\pm$ the standard deviation (red). Reference is (Erdős-Rényi model $G(n, p)$, full lines), or degree distribution conserving model (dashed lines). $n = 300$, $a = 0.9$, $T = 301$, $s = 10$

- Results for FC of white noise




## Discussion and conclusions

- Network graphs computed from functional connectivity, in particular by correlation of time series, show small-world properties even if the underlying system connectivity is random (or completely missing).
- Potential compensation effect of partial correlation strongly depends on time series length, ranging from over- to under-compensation (not shown)
- The described phenomena have direct relevance for the study of a wide range of complex dynamical systems, including but not limited to climatic [3] and brain [4] networks.
- While linear correlation might sufficiently capture the dependence in time series of weakly non-linear real-world systems [5], non-linear connectivity measures are suitable for some systems. Theoretical considerations and preliminary results (not shown) suggest possible generalisation of the described effect to range of nonlinear data and coupling measures.

## References

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