A LINEAR ALGORITHM TO RECOGNIZE MAXIMAL GENERALIZED OUTERPLANAR GRAPHS

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Abstract. In this work, we get a combinatorial characterization for maximal generalized outerplanar graphs (mgo graphs). This result yields a recursive algorithm testing whether a graph is a mgo graph or not.

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1. Introduction

The main concept of this paper was introduced by Sedláček in [6]. He defined *generalized outerplanar graphs* as graphs with a planar representation such that, at least, one end-vertex of each edge lies on the outer face. Also, he gave a characterization in terms of forbidden subgraphs (see Figure 1).

Clearly, this is a way to generalize the well-known concept of outerplanar graph. These two kinds of graphs have been used in the design of printed boards where it is required that all terminals (or one end-terminal of each wire) be placed on the periphery of the chip of the board [3].

Of course, it would be very useful to get linear algorithms for recognizing outerplanar and generalized outerplanar graphs. The former was obtained by Mitchell in [4], and, in this paper, we present a linear algorithm for the recognition of maximal generalized outerplanar graphs since a test whether a graph is generalized outerplanar or not, easily follows from our algorithm.

A *maximal generalized outerplanar* (mgo) graph is a generalized outerplanar graph such that no edge can be added without violating this property.

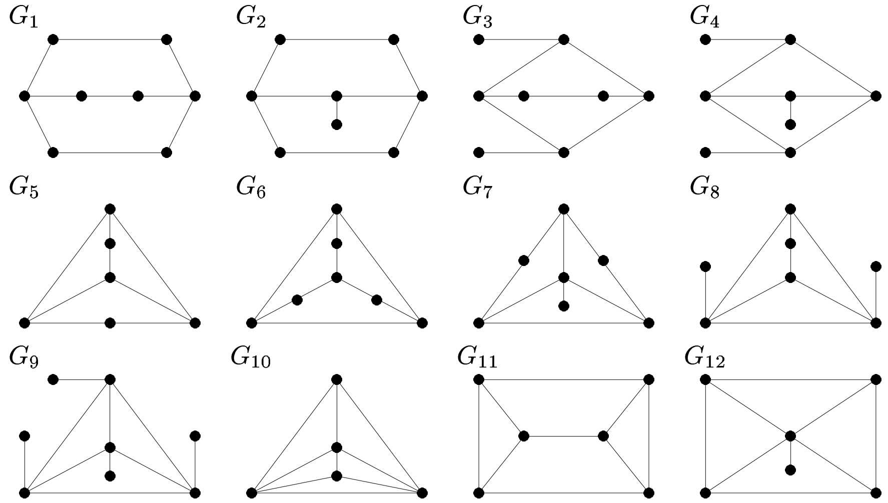

Figure 1. The forbidden subgraphs of Sedláček

We will use [1] for the common graph notation, except for using the terms vertex instead of point, and edge instead of line. Nonetheless, let us recall some helpful concepts.

A graph G is *2-connected* when at least two vertices of G must be removed to disconnected it; if there are two vertices, u and v, of G such that their removal disconnects G, we call them a *separation pair.* The graph induced by u, v and the vertices of the connected component of $G - \{u, v\}$ is called a *split graph.* A 2-connected graph with no separation pair is said to be *3-connected* and a maximal 3-connected subgraph is called a *3-component.*

Also, we mention a special kind of graphs. For $n \geqslant 4$, the *wheel* W_n is defined to be the graph $K_1 + C_{n-1}$. In [7], Tutte showed that every 3-connected graph either is a wheel or can be built by a sequence of the following two operations over W_n:

1. Add a new edge.
2. Replace a vertex w having a degree at least 4 by two adjacent vertices w' and w'' such that each vertex formerly joined to w is joined to exactly one of w' and w'' so that in the resulting graph, w' and w'' have a degree at least 3.

2. The results

The previous result yields

Lemma 1. *A 3-connected maximal generalized outerplanar graph is a wheel.*

P r o o f . If we try to construct a mgo 3-connected graph from a wheel by applying the two Tutte's operations then we realize that we are just allowed to add an edge between two non-consecutive vertices of the periphery of the wheel but this new graph is not generalized outerplanar because it contains a subgraph homeomorphic to the forbidden subgraph G_{10} (see Figure 1).

On the other hand, the only vertex with a degree at least 4 is the center of the wheel, so we can apply operation 2 just to this vertex. But, again, we obtain a non-planar graph or a graph which contains a subgraph homeomorphic to the Sedláček subgraph G_{11} (see Figure 1). □

After we have characterized 3-connected mgo graphs, it is straightforward to check the characterization in the 2-connected case.

Lemma 2. *The only 2-connected maximal generalized outerplanar graph without 3-components is K_3.*

P r o o f . There exists only one 2-connected graph with 3 vertices: K_3, it has no 3-components and it is mgo. So, consider a graph G with at least four vertices. In an outerplane embedding of G, if all vertices lie in the exterior face then the graph is outerplanar, and it is easy to check that an outerplanar graph with at least four vertices cannot be mgo.

Thus, there exists a vertex which does not lie in the exterior face but this vertex must be joined only with the vertices $v_1, \dots v_n$ of the exterior cycle (every edge has an end-vertex on the exterior face). Without loss of generality, we can suppose that the vertices $v_1, \dots v_n$ are consecutive in the cycle.

Now, if the edge $v_1 v_n$ exists, then the graph has a 3-component and if the edge $v_1 v_n$ does not exist, then G is not maximal because $G + v_1 v_n$ is generalized outerplanar. So, there are no graphs with at least four vertices under the conditions of the lemma, and we have the proof.

□

Now we are ready to solve the general case.

Theorem 3. *Let $\{u, v\}$ be a separation pair of a 2-connected graph G that splits the graph in $G_1, \dots, G_p$. G is a mgo graph if and only if the following four conditions are satisfied:*

1. *uv is an edge of $G_1, \dots, G_p$.*
2. *$G_1, \dots, G_p$ are mgo graphs.*
3. *At most two of the components $G_1, \dots, G_p$ are not isomorphic to K_3.*
4. *Each $G_1, \dots, G_p$ has a generalized outerplane embedding such that the edge uv belongs to the exterior face.*

Proof. Assume that G is a 2-connected graph and $\{u, v\}$ is a separation pair. Let Z be the exterior cycle of a generalized outerplane embedding of G. Since this cycle connects the graph, the vertices u and v belong to Z. If u and v are consecutive in Z then the edge uv exists, otherwise, since the endvertices of every edge are in the same split graph, the maximality of G implies that the edge uv belongs to G and so, it belongs to $G_1, \ldots, G_p$ (condition 1).

Condition 2 follows from the fact that $G_1, \ldots, G_p$ inherit the mgo property of G.

By virtue of Lemma 2, all but at most two of the components $G_1, \ldots, G_p$ are isomorphic to K_3. On the other hand, if G_1, G_2 and G_3 are different from K_3 then G_3 must be embedded in an internal face of either G_1 or G_2 and so, G would not be generalized outerplanar (condition 3).

Clearly, u and v lie on the exterior face of the embedding of G_i $(1 \leqslant i \leqslant p)$ induced by the generalized outerplane embedding of G. If u and v are not consecutive in the exterior cycle then we can split G_i into a new 2-connected component and a K_3. Thus, uv belongs to the exterior cycle and condition 4 holds.

Conversely, we build the generalized outerplane embedding of G in the following way. Consider two components, G' and G'', such that they are not simultaneously K_3 (see condition 3), and its generalized outerplane embedding such that uv belongs to the exterior cycle (see condition 4). Merging their planar representations, we obtain a generalized outerplane embedding of $G' \cup G''$ and also, if G' and G'' are maximal then this new graph is maximal as well. Now, we can join components isomorphic to K_3 losing neither the maximality property nor the generalized outerplanarity property. □

3. The linear algorithm

Theorem 3 is the result we need to design a recursive algorithm for testing whether a graph is mgo or not. Roughly speaking, the algorithm works in the following way: it splits the input graph into two 2-connected components, checks conditions 1 and 3 of Theorem 3 and recursively uses these components as input. During the backtrack of the algorithm, condition 4 is tested and the recursion ends when the situation of Lemma 1 or Lemma 2 occurs.

One important step of the algorithm is to find the 3-component of the input graph. The linear algorithm of Hopcroft and Tarjan (see [2]) can be used to do this. Also, in [5], the authors choose to explore the graph by using *depth-first search* (DFS) and this is the method that our algorithm uses.

MGO-TEST Algorithm. Let G be a 2-connected graph with M vertices having a list of vertices V and edges E. Let us suppose that all vertices are labelled with 'interior'.

Step 1: If $|E| > 3M - 6$ then G is not mgo and stop. If $G = K_3$ then G is mgo and stop. Otherwise:

Step 2: Using DFS, check whether G is 3-connected.

1. G is 3-connected. Check whether G is a wheel (the degree of $M - 1$ vertices is 3). If it is not then G is not mgo and stop. Else:
   (a) $G = K_4$. G is mgo if and only if there exists a vertex labelled 'interior'. Stop.
   (b) $G \neq K_4$. G is mgo if and only if the vertex with a greater degree is labelled 'interior'. Stop.
2. G is not 3-connected. Look for a separation pair $\{u, v\}$ of G. If $uv \notin E$ then G is not mgo and stop. Else, build split graphs $G_1, \ldots, G_p$ labelling u and v as 'exterior' in each G_i. Except one or two, all the graphs $G_1, \ldots, G_p$ must be K_3, or G is not mgo and stop. If G' and G'' are such graphs then G is mgo if and only if both G' and G'' are mgo.

Step 1 ensures that we are not in a trivial case. In 2.2, we define a recursion loop that splits the input graph, checks whether there are the correct number of K_3 and deletes them. The condition for ending this recursion is in 2.1 where the algorithm checks whether we have a wheel and whether its center can be placed not in the exterior face.

Theorem 4. *To test whether a graph is mgo or not, needs time $O(n)$ with the MGO-TEST algorithm, and this is optimal.*

Proof. Clearly, the step that dominates the calculation is to check by DFS whether the graph is 3-connected. The cost of this test is $O(n)$, which completes the proof. □

References

[1] *F. Harary*: Graph Theory. Addison Wesley, Reading Mass., 1969.

[2] *J. E. Hopcroft and R. E. Tarjan*: Dividing a graph into triconnected components. SIAM J. Comput. *2* (1973), 135–158.

[3] *M. C. van Lier and R. H. J. M. Otten*: C.A.D. of masks and wiring. T. H. Rept. 74-E-44, Dept. Elect. Engrg. Eindhoven University of Technology.

[4] *S. Mitchell*: Linear algorithms to recognize outerplanar and maximal outerplanar graphs. Inform. Process. Lett. *9* (1979), 229–232.

[5] *T. Nishizeki, N. Chiba*: Planar Graphs: Theory and Algorithms. North-Holland, Amsterdam, 1969.

[6] *J. Sedláček*: On a generalization of outerplanar graphs. Časopis Pěst. Mat. *113* (1988), 213–218.
[7] *W. T. Tutte*: A theory of 3-connected graphs. Indag. Math. *23* (1961), 441–455.

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