GEOMETRY OF SECOND-ORDER CONNECTIONS AND ORDINARY DIFFERENTIAL EQUATIONS

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Summary. The geometry of second-order systems of ordinary differential equations represented by 2-connections on the trivial bundle $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$ is studied. The formalism used, being completely utilizable within the framework of more general situations (partial equations), turns out to be of interest in confrontation with a traditional approach (semisprays), moreover, it amounts to certain new ideas and results. The paper is aimed at discussion on the interrelations between all types of connections having to do with integral sections (geodesics), integrals and symmetries of the equations studied.

Keywords: connection, semispray, differential equation, integral, symmetry

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1. Introduction

The goal of the paper is to direct the attention to some less traditional aspects and points of view applicable to the geometry of first and second-order systems of ordinary differential equations represented by connections on the tangent bundle $\tau_M : TM \to M$ in the autonomous situation, and mainly on the trivial bundle $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$ in the time-dependent case. The equations for integral sections of connections (i.e. the equations solved with respect to the highest derivatives or the Pfaffian systems of 'horizontal form') do not represent the most general situation, nevertheless, their role e.g. in both the autonomous and the nonautonomous mechanics is well-known. Consequently, the investigation of generators of such equations, called *semisprays* (or second-order differential equation fields) is very extensive (cf. [6] and references therein). The methods and ideas used are closely related to particular properties of tangent bundles and the underlying canonical structures and morphisms. It appears

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that the absence of such tools in the general situation on fibered manifolds calls for the application of a more general approach, the most characteristic feature of which is the change

$$\text{semispray} \quad \rightsquigarrow \quad \text{2-connection} = \text{semispray distribution}$$

[25], [27]. This change causes the assistance of deep theories acting on fibered manifolds – the theory of *differential equations* (e.g. [22], [26] and references therein) and the theory of *connections* (e.g. [14], [19], [20], [25] and references therein). Both theories integrated additionally by the theory of *natural operations* (e.g. [14], [15] and references therein) admit a transparent description of the geometry of the systems studied, even for the case of partial differential equations or in higher-order theories; we refer to [8], [9], [17], [28], [30], [31] for details.

The paper represents an attempt at the illustration of this approach in the particular situation of the trivial bundle $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$, where the developed formalism meets canonical tangent and related structures, hence it can stand for certain different points of view to the traditional results. Moreover, some new results and motivations appear, as well, whose possible areas of application could be for example the theory of *variational equations* (e.g. [11], [16], [18]) and the formalism of *differential forms along a map* [24]. In addition, the role of connections on $\mathrm{pr}_2 \circ \pi_{1,0} : \mathbb{R} \times TM \to M$, studied e.g. in [1], appears to be of interest and a discussion of this topic will be presented in a forthcoming paper.

All notions and results of the paper are presented in the form with the ambition for the work to be more or less self-contained; by $\mathcal{F}(X)$ we mean the module of smooth functions on X, all manifolds and maps are smooth and the standard summation convention is used.

2. Connections on $\tau_M : TM \to M$

Let M be an n-dimensional manifold, $\tau_M : TM \to M$ its *tangent bundle*. Denote by (x^i) or $(x^i, \dot{x}^i)$ local coordinates on M or the induced fibered coordinates on TM, respectively. The corresponding fibered coordinates on the *first jet prolongation* $J^1\tau_M$ of τ_M which is the set of 1-jets of local vector fields on M will be $(x^i, \dot{x}^i, \dot{x}^i_j)$, i.e. for $v = v^i \partial/\partial x^i$, $v^i = \dot{x}^i \circ v$, we have $\dot{x}^i_j \circ j^1 v = \partial v^i / \partial x^j$. Recall that $(\tau_M)_1 : J^1\tau_M \to M$ is again a vector bundle while according to the general theory of fibered manifolds, $(\tau_M)_{1,0} : J^1\tau_M \to TM$ is an affine bundle modelled on the vector bundle

$$V_{\tau_M} TM \otimes T^*M \to TM, \tag{2.1}$$

where $V_{\tau_M}TM = \mathrm{span}\{\partial/\partial \dot{x}^i\}$ is a subbundle of τ_M-vertical vectors from TTM (note that T^*M should be considered pulled-back by τ_M). The sections of (2.1) are called *soldering forms* on τ_M; the local expression of any such vertical tangent valued 1-form (or equivalently a $(1,1)$-tensor field on M or an endomorphism on TM) is

$$\varphi = \varphi^i_j \frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}x^j. \tag{2.2}$$

There is a canonical *basic* soldering form

$$J = \delta^i_j \frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}x^j$$

on τ_M, called an *almost tangent structure* on TM, which finds wide application in the tangent bundle geometry realizing e.g. the *vertical lift* of vectors from TM to TTM.

A (generally *nonlinear*) *connection* on τ_M is a section $\Lambda\colon TM \to J^1\tau_M$ of $(\tau_M)_{1,0}$. Local equations of Λ are $\dot{x}^i_j \circ \Lambda = \Lambda^i_j(x^k, \dot{x}^k)$, where Λ^i_j are the *components* of Λ transformed like the coordinates $\dot{x}^i_j$:

$$\bar{\Lambda}^i_j = \Lambda^p_q \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} - \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \dot{\bar{x}}^k.$$

It is easy to see that Λ can be identified in particular with the tangent valued 1-form $h_\Lambda\colon TM \to TTM \otimes T^*M$, called a *horizontal form* of Λ. Locally, $h_\Lambda = D_{\Lambda i} \otimes \mathrm{d}x^i$, where

$$D_{\Lambda i} = \frac{\partial}{\partial x^i} + \Lambda^j_i \frac{\partial}{\partial \dot{x}^j},$$

for $i = 1, \dots, n$, is the *i-th absolute derivative* with respect to Λ. As h_Λ creates a splitting of the canonical exact sequence

$$0 \to V_{\tau_M}TM \to TTM \to TM \times_M TM \to 0$$

and thus it realizes a *horizontal lift* of vectors, a connection Λ on τ_M is identified with the decomposition $TTM = V_{\tau_M}TM \oplus H_\Lambda$ with the *horizontal subbundle* $H_\Lambda = \mathrm{span}\{D_{\Lambda i}\}$. The complementary projection $v_\Lambda = I - h_\Lambda$ on TTM is called the *vertical form* of Λ.

The structure of $(\tau_M)_{1,0}$ implies the meaning of soldering forms on τ_M as *deformations* of connections on τ_M; namely, for each pair Λ_1, Λ_2 of connections, $\varphi = h_{\Lambda_1} - h_{\Lambda_2}$ is a soldering form, and conversely, for Λ being a connection and φ a soldering form, $h_\Lambda + \varphi$ is a horizontal form of a connection.

A connection Λ on τ_M represents a (generally nonlinear) first-order system of partial differential equations. A global (or coordinate-free) expression is given by the submanifold $\operatorname{Im}(\Lambda) \subset J^1\tau_M$, while the coordinate expression is

$$\frac{\partial v^i}{\partial x^j} = \Lambda^i_j(x^k, v^k(x^\ell))$$

for the unknown family of functions v^k. The *integral sections* of Λ are thus (local) vector fields v on M satisfying $j^1 v = \Lambda \circ v$ on its domain. Evidently, an equivalent condition for v to be an integral section of Λ is to be an integral mapping of the corresponding horizontal distribution H_Λ, which can be expressed by the condition $Tv \in H_\Lambda$. By

$$M \xrightarrow{j^1 v} J^1\tau_M \to J^1\tau_M \times_M TM \xrightarrow{\mathrm{id} \times \Lambda} J^1\tau_M \times_{TM} J^1\tau_M \to V_{\tau_M}TM \otimes T^*M,$$

the *covariant derivative* $\nabla_\Lambda v$ of a vector field v with respect to Λ is defined. In view of

$$\nabla_\Lambda v = \left(\frac{\partial v^i}{\partial x^j} - \Lambda^i_j \circ v\right) \frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}x^j,$$

the integral sections of Λ are vector fields *parallel* with respect to Λ, which means $\nabla_\Lambda v = 0$. In this arrangement, the *integrability* of Λ means that there is a uniquely determined maximal vector field v on M passing through each point of TM being an integral section of Λ. Equivalent conditions are e.g. both the integrability of H_Λ and the vanishing of the tangent valued 2-form

$$R_\Lambda = \frac{1}{2}[h_\Lambda, h_\Lambda] \colon TM \to V_{\tau_M}TM \otimes \Lambda^2 T^*M$$

called the *curvature* of Λ, which locally means

$$\frac{\partial \Lambda^k_i}{\partial x^j} + \frac{\partial \Lambda^k_i}{\partial \dot{x}^\ell}\Lambda^\ell_j = \frac{\partial \Lambda^k_j}{\partial x^i} + \frac{\partial \Lambda^k_j}{\partial \dot{x}^\ell}\Lambda^\ell_i \qquad \text{for any } i, j, k = 1, \ldots, n.$$

Let φ be a soldering form (2.2) on τ_M. The *φ-torsion* $\mathcal{T}_\varphi$ is another important notion closely related to a connection Λ on τ_M, defined again to be a tangent valued 2-form

$$\mathcal{T}_\varphi = [h_\Lambda, \varphi] \colon TM \to V_{\tau_M}TM \otimes \Lambda^2 T^*M.$$

Using a natural choice $\varphi = J$, the corresponding J-torsion will be briefly called a *torsion* $\mathcal{T}$ of Λ. Locally,

$$\mathcal{T} = \frac{\partial \Lambda^i_k}{\partial \dot{x}^j} \frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}x^j \wedge \mathrm{d}x^k.$$

There is a bijective correspondence between the connections on τ_M and the endomorphisms on TM satisfying $JG = J$, $GJ = -J$, called *Grifone connections*. The identification is expressed by $G_\Lambda = 2h_\Lambda - I$ and a distinguished soldering form $\mathcal{H}$, called the *tension* of Λ, is defined by

$$\mathcal{H} = \frac{1}{2}\mathcal{L}_C G_\Lambda,$$

where $C = \dot{x}^i \partial/\partial\dot{x}^i$ is the *Liouville vector field* on TM and $\mathcal{L}$ is the *Lie derivative*. Locally,

$$\mathcal{H} = \left(\frac{\partial\Lambda^i_j}{\partial\dot{x}^k}\dot{x}^k - \Lambda^i_j\right)\frac{\partial}{\partial\dot{x}^i}\otimes \mathrm{d}x^j. \tag{2.3}$$

It appears that the above situation represents a natural generalization of the classical one concerning linear connections on M. An additional requirement for Λ to realize a vector bundle morphism between τ_M and $(\tau_M)_1$ over M implies the linearity of Λ^i_j in $\dot{x}^k$, i.e. $\Lambda^i_j(x^\ell, \dot{x}^k) = -\Lambda^i_{jk}(x^\ell)\dot{x}^k$, where the functions Λ^i_{jk} are the classical *Christoffels*. By (2.3), this condition is equivalent to $\mathcal{H} = 0$. More generally, if $\mathcal{H}$ is basic, then evidently Λ is *affine*.

The *geodesics* of a connection Λ on τ_M are the curves $c\colon \mathbb{R} \supset J \to M$ whose tangent vector field $\dot{c}\colon J \to TM$ along c is parallel with respect to Λ, which means $\ddot{c} \in H_\Lambda$. Locally, this condition is represented by a (generally again nonlinear) system of second-order differential equations (ODE) for $c^i = x^i \circ c$:

$$\frac{\mathrm{d}^2 c^i}{\mathrm{d}t^2} = \Lambda^i_j\left(c^k, \frac{\mathrm{d}c^k}{\mathrm{d}t}\right)\frac{\mathrm{d}c^j}{\mathrm{d}t}. \tag{2.4}$$

Clearly, if v is an integral section of Λ then each integral curve c of v is a geodesic of Λ. Accordingly, for Λ being integrable (recall that under the condition $R_\Lambda = 0$, Λ is more traditionally called *flat*), the second-order system (2.4) is reduced to the first-order system $\mathrm{d}c^i/\mathrm{d}t = v^i(c^k)$ on a certain neighbourhood of each point from TM.

For a more detailed discussion on the above concepts and for the relations to the classical theory of linear connections we refer to [14], [20], [25] and [6].

When speaking of the *second-order tangent bundle*, we bear an (embedded) submanifold $T^2M \subset TTM$ of those tangent vectors from TTM for which the bundle projections $T\tau_M$ and τ_{TM} coincide, in mind. Local fibered coordinates on T^2M are $(x^i, \dot{x}^i, \ddot{x}^i)$, where briefly $\ddot{x}^i = \mathrm{d}\dot{x}^i/\,\mathrm{d}t = \mathrm{d}^2x^i/\,\mathrm{d}t^2$. A *semispray* on TM is a (global) section w of $\tau_M^{2,1} = \tau_{TM}|_{T^2M} \colon T^2M \to TM$, i.e. a vector field on TM of the particular type

$$w = \dot{x}^i \frac{\partial}{\partial x^i} + w^i(x^j, \dot{x}^j) \frac{\partial}{\partial \dot{x}^i}. \tag{3.1}$$

An essential property of a semispray is that of its integral curves $c \colon \mathbb{R} \supset J \to TM$,

$$c = T\tau_M \circ \dot{c}. \tag{3.2}$$

Consistently, a curve $c \colon \mathbb{R} \supset J \to M$ is called a *geodesic* of a semispray w if $\dot{c}\colon J \to TM$ is an integral curve of w. Relative to (3.2), the geodesics satisfy $T\tau_M \circ \ddot{c} = \dot{c}$ and the equations for them are

$$\frac{\mathrm{d}^2 c^i}{\mathrm{d}t^2} = w^i \left(c^k, \frac{\mathrm{d}c^k}{\mathrm{d}t} \right),$$

which corresponds to an alternative name for semisprays: *second-order differential equation fields* (SODE).

A semispray w on TM is called a *spray* if $w = [C, w]$, which means that the components w^i of w are functions homogeneous of order two in $\dot{x}^i$, i.e.

$$\frac{\partial w^i}{\partial \dot{x}^j} \dot{x}^j = 2w^i.$$

There is a uniquely determined (global) semispray w_Λ on TM *associated* to a connection Λ on τ_M, which is a spray in the case of Λ linear. Globally it can be defined as a generator of the one-dimensional distribution $H_w = H_\Lambda \cap T^2M$, which locally gives

$$w_\Lambda = \dot{x}^i \frac{\partial}{\partial x^i} + \Lambda^i_j \dot{x}^j \frac{\partial}{\partial \dot{x}^i}, \tag{3.3}$$

and consequently Λ and w_Λ have the same geodesics.

Moreover, the integral sections of Λ can be thus equivalently defined as vector fields on TM such that $w_\Lambda \circ v = Tv \circ v$.

Conversely, if w is a semispray on TM, one might ask for connections on τ_M associated to w. It turns out (e.g. [2], [3], [5], [10]) that

$$h_\Lambda = \frac{1}{2}\left(I_{TTM} - \mathcal{L}_w J\right) \tag{3.4}$$

is a horizontal form of a canonically determined connection Λ with the components

$$\Lambda^i_j = \frac{1}{2}\frac{\partial w^i}{\partial \dot{x}^j}.$$

Obviously, Λ is torsion free but generally it is not associated to w except for w being a spray. If this is the case, Λ is linear with a zero *strong torsion* $\boldsymbol{\mathcal{T}}$, which is a soldering form

$$\boldsymbol{\mathcal{T}} = i_w\mathcal{T} - \mathcal{H},$$

locally expressed by

$$\boldsymbol{\mathcal{T}} = \left(\Lambda^i_j - \frac{\partial \Lambda^i_k}{\partial \dot{x}^j}\dot{x}^k\right)\frac{\partial}{\partial \dot{x}^i} \otimes \mathrm{d}x^j,$$

where $\mathcal{H}$ and $\mathcal{T}$ are the tension and the torsion of Λ, and w is an *arbitrary* semispray on TM. It is worth mentioning here that all first-order natural operators transforming semisprays on TM into connections on τ_M form a one-parameter family expressed by $h_\Lambda + kJ$, where $k \in \mathbb{R}$ and h_Λ is given by (3.4) (see [4] or [14]), which corresponds to the fact that J is the only natural soldering form on τ_M.

Both the necessary and the sufficient form of deformations of (3.1) for the obtained connection to be associated to the given w, and the role of the strong torsion in the previous considerations are described in the so-called *Decomposition Theorem* [6]: for any semispray w on TM and a soldering form φ on τ_M such that

$$2i_w\varphi = w - [C, w], \tag{3.5}$$

there exists a unique connection Λ on τ_M whose associated semispray is w and its strong torsion $\boldsymbol{\mathcal{T}} = 2\varphi$; this connection is given by

$$h_\Lambda = \frac{1}{2}\left(I_{TTM} - \mathcal{L}_w J\right) + \varphi.$$

For a related discussion of the material we refer to Sec. 6; it should be noticed here that the distinguished role of sprays in the above considerations can be seen e.g. from the local expression of (3.5):

$$\varphi^i_j \dot{x}^j = w^i - \frac{1}{2}\frac{\partial w^i}{\partial \dot{x}^j}\dot{x}^j. \tag{3.6}$$

Notice the role played by the relationships between semisprays on TM and connections on τ_M in the theory of derivations of forms (even vector valued) along τ_M, (see [21] and references therein). In this situation, a semispray w is combined with the prolonged objects studied and the connection (3.4) coming from w is used essentially, as well.

4. First and second-order connections on $\mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$

Let now M be an m-dimensional manifold with local coordinates redenoted (for some technical reasons) to (q^σ) on M and $(q^\sigma, q^\sigma_{(1)})$ on TM. Fibered coordinates on the total space $Y = \mathbb{R} \times M$ of the trivial bundle $\pi := \mathrm{pr}_1 : \mathbb{R} \times M \to \mathbb{R}$ are consequently (t, q^σ) with (t) being a *global* canonical coordinate on $\mathbb{R}$. Local sections of π are evidently of the form $\gamma = (\mathrm{id}_\mathbb{R}, c)$, where c is a curve in M. By $j^1_0\gamma \mapsto \dot{c}(0)$, where γ is an arbitrary section of π on a neighbourhood of zero, a canonical isomorphism $J^1_0\pi \cong TM$ is realized. Analogously, by $j^1_t\gamma \mapsto (t, \dot{c}(t))$ one gets the well-known isomorphism $J^1\pi \cong \mathbb{R} \times TM$. This implies that in contradistinction to the general situation where $\pi_{1,0} \colon J^1\pi \to Y$ is an affine bundle,

$$\pi_{1,0} \cong \mathrm{id}_\mathbb{R} \times \tau_M \colon \mathbb{R} \times TM \to \mathbb{R} \times M$$

is now a *vector* bundle with fibres T_xM over (t, x). Another useful identification

$$J^1\pi \cong V_\pi Y \tag{4.1}$$

of $\mathbb{R} \times TM$ with the submanifold $V_\pi Y \subset T(\mathbb{R} \times M)$ of π-vertical vector fields on $\mathbb{R} \times M$ is given by $j^1_t\gamma \mapsto \dot{\gamma}(t)$, and locally by $\dot{t} = 1$.

In this arrangement, a (first-order) *connection* on π, i.e. a section $\Gamma \colon Y \to J^1\pi$, can be identified with a π-vertical vector field

$$v = \Gamma^\sigma(t, q^\lambda)\frac{\partial}{\partial q^\sigma} \tag{4.2}$$

on $\mathbb{R} \times M$, which is equivalently a vector field along $\mathrm{pr}_2 : \mathbb{R} \times M \to M$ called a *time-dependent* vector field on M. The one-dimensional π-horizontal distribution $H_\Gamma \subset TY$ defining the decomposition $TY = H_\Gamma \oplus V_\pi Y$, is thus generated by the *absolute derivative* D_Γ with respect to Γ, $D_\Gamma = \partial/\partial t + v$, where $D_\Gamma = D \circ \Gamma$ with $D = \partial/\partial t + q^\sigma_{(1)}\partial/\partial q^\sigma$ is the *formal derivative* (a vector field along $\pi_{1,0}$). The *horizontal form* of Γ is a tangent valued 1-form $h_\Gamma = D_\Gamma \otimes \mathrm{d}t$ while $v_\Gamma = I_{TY} - h_\Gamma$ is the *vertical form* of Γ.

The *integral sections* of Γ are the sections γ of π satisfying $j^1\gamma = \Gamma \circ \gamma$ on its domain. In terms of the above identifications, integral sections of Γ are the 'graphs' of integral curves of v (4.2), i.e. $\gamma = (\mathrm{id}_{\mathbb{R}}, c)$ such that $v \circ \gamma = \dot{c}$. The corresponding first-order system of ODE reads

$$\frac{\mathrm{d}c^\sigma}{\mathrm{d}t} = \Gamma^\sigma(t, c^\lambda). \tag{4.3}$$

In general, the equations (4.3) are globally represented by a submanifold $\Gamma(\mathbb{R}\times M) \subset \mathbb{R} \times TM$; in particular, in case that v does not depend on t, this submanifold is $\mathbb{R} \times \mathrm{Im}\, v \subset \mathbb{R} \times TM$.

If we are interested in the structure of some other prolongations of fibrations under consideration we find that for $\pi_1 \colon \mathbb{R} \times TM \to \mathbb{R}$ we have

$$J^1\pi_1 \cong \mathbb{R} \times TTM \quad \text{and} \quad J^2\pi \cong \mathbb{R} \times T^2M,$$

where the additional induced coordinates on the first prolongation $J^1\pi_1$ of π_1 or on the second prolongation $J^2\pi$ of π are denoted by $\dot{q}^\sigma, \dot{q}^\sigma_{(1)}$ or $q^\sigma_{(2)}$, respectively. Moreover, evidently $T^2M \cong J^2_0\pi$ and also for π_1-vertical vectors on $\mathbb{R}\times TM$ we have $V_{\pi_1}J^1\pi \cong \mathbb{R}\times TTM \subset T(\mathbb{R}\times TM)$; it is important to note that $\mathbb{R}\times TTM$ is a total space of $(\mathrm{id}_{\mathbb{R}} \times \tau_{TM})$ in the above identifications. On the other hand, by (4.1) we have $J^1(\pi \circ \tau_Y|_{V_\pi Y}) \cong J^1\pi_1 \cong \mathbb{R}\times TTM$, however, now the presented fibration over $\mathbb{R}\times TM$ is $(\mathrm{id}_{\mathbb{R}} \times T_{\tau_M})$ and accordingly the realization of the isomorphism

$$J^1(\pi \circ \tau_Y|_{V_\pi Y}) \cong V_{\pi_1}J^1\pi \tag{4.4}$$

calls for the canonical involution $\kappa_M \colon TTM \to TTM$ (see [6], [14] etc.) by $(\mathrm{id}_{\mathbb{R}}, \kappa_M)$.

A second-order connection (briefly a 2-*connection*) on π is in accordance with [19], [25] a section $\Gamma^{(2)}$ of $\pi_{2,1} \colon J^2\pi \to J^1\pi$, which means

$$\Gamma^{(2)} \colon \mathbb{R}\times TM \to \mathbb{R}\times T^2M$$

in the situation studied. Any 2-connection $\Gamma^{(2)}$ on π is equivalently characterized by its horizontal form $h_{\Gamma^{(2)}} = D_{\Gamma^{(2)}} \otimes \mathrm{d}t$, where the absolute derivative $D_{\Gamma^{(2)}}$ with respect to $\Gamma^{(2)}$

$$D_{\Gamma^{(2)}} = \frac{\partial}{\partial t} + q^\sigma_{(1)}\frac{\partial}{\partial q^\sigma} + \Gamma^\sigma_{(2)}\frac{\partial}{\partial q^\sigma_{(1)}}$$

is a *semispray* on $\mathbb{R}\times TM$ where its alternative name *semispray connection* comes from. Again, $D_{\Gamma^{(2)}} = D \circ \Gamma^{(2)}$, where the formal derivative $D = \partial/\partial t + q^\sigma_{(1)}\partial/\partial q^\sigma + q^\sigma_{(2)}\partial/q^\sigma_{(1)}$ is now a vector field along $\pi_{2,1}$. Thus, $\Gamma^{(2)}$ can be identified with a one-dimensional π_1-horizontal distribution $H_{\Gamma^{(2)}} = \mathrm{Im}\, h_{\Gamma^{(2)}}$, realizing a decomposition $TJ^1\pi = H_{\Gamma^{(2)}} \oplus V_{\pi_1}J^1\pi$; recall that evidently $H_{\Gamma^{(2)}} \subset C_{\pi_{1,0}}$, where $C_{\pi_{1,0}}$ is a canonical *Cartan distribution* on $J^1\pi$, and that the soldering forms on π_1 realizing the *deformations* of 2-connections on π are the sections of the vector bundle $V_{\pi_{1,0}}J^1\pi\otimes\pi_1^*(T^*\mathbb{R})\to J^1\pi$, associated to $\pi_{2,1}$, locally expressed by

$$\varphi^{(2)}=\varphi^{\sigma}_{(2)}\frac{\partial}{\partial q^{\sigma}_{(1)}}\otimes\mathrm{d}t.$$

Due to the previous considerations, $D_{\Gamma^{(2)}}$ and thus $\Gamma^{(2)}$ itself can be represented by the (generally *time-dependent*) semispray

$$w=q^{\sigma}_{(1)}\frac{\partial}{\partial q^{\sigma}}+\Gamma^{\sigma}_{(2)}(t,g^{\lambda},q^{\lambda}_{(1)})\frac{\partial}{\partial q^{\sigma}_{(1)}}\tag{4.5}$$

on TM, which can be viewed as a vector field along $\mathrm{pr}_2:\mathbb{R}\times TM\to TM$. Clearly, the autonomous situation (denoted here by $w=w(q^{\lambda},q^{\lambda}_{(1)})$) means that (4.5) is an ordinary semispray on TM in the sense of Sec. 3.

The *integral sections* of $\Gamma^{(2)}$ are the sections γ of π such that $j^2\gamma=\Gamma^{(2)}\circ j^1\gamma$, hence they are the 'graphs' of geodesics of w from (4.5) which means $\gamma=(\mathrm{id}_{\mathbb{R}},c)$ such that $w\circ j^1\gamma=\ddot{c}$. Consequently, the corresponding second-order system of ODE, represented by $\Gamma^{(2)}(\mathbb{R}\times TM)\subset\mathbb{R}\times T^2M$, is

$$\frac{\mathrm{d}^2c^{\sigma}}{\mathrm{d}t^2}=\Gamma^{\sigma}_{(2)}\left(t,c^{\lambda},\frac{\mathrm{d}c^{\lambda}}{\mathrm{d}t}\right).$$

Recall that both for Γ and for $\Gamma^{(2)}$, the integral sections coincide with the maximal integral mappings of the corresponding horizontal distributions.

The following ideas will appear to be profitable in Sec. 7.

Let $\Gamma\colon Y\to J^1\pi$ be a connection on π. Using the vertical functor V one gets the mapping $V\Gamma\colon V_\pi Y\to V_{\pi_1}J^1\pi$, i.e. $V\Gamma\colon\mathbb{R}\times TM\to\mathbb{R}\times TTM$. With regard to (4.4)

$$\mathcal{V}\Gamma=(\mathrm{id}_{\mathbb{R}},\kappa_M)\circ V\Gamma$$

is the so-called *vertical prolongation* of Γ realizing the only connection on $\varrho:=\pi\circ\tau_Y:V_\pi Y\to\mathbb{R}$ naturally generated by Γ [13], [14]. In fact, in view of (4.1), $\mathcal{V}\Gamma$ is a connection on $\pi_1\colon\mathbb{R}\times TM\to\mathbb{R}$, locally

$$(t,q^{\sigma},q^{\sigma}_{(1)})\overset{\mathcal{V}\Gamma}{\longmapsto}\left(t,q^{\sigma},q^{\sigma}_{(1)},\Gamma^{\sigma},\frac{\partial\Gamma^{\sigma}}{\partial q^{\lambda}}q^{\lambda}_{(1)}\right).$$

In particular, for time-independent v in (4.1) one gets $V\Gamma=(\mathrm{id}_{\mathbb{R}},Tv)$ and

$$\mathcal{V}\Gamma=(\mathrm{id}_{\mathbb{R}},v^c),$$

where v^c is the *complete lift* of v (see e.g. [6]).

Applying analogous ideas and isomorphisms $J^2\pi_1 \cong \mathbb{R} \times T^2TM$ together with the canonical involution $\kappa_M^{(2)} \colon T^2TM \to TT^2M$ we obtain the *vertical prolongation* $\mathcal{V}\Gamma^{(2)} : J^1\pi_1 \to J^2\pi_1$ as a naturally determined 2-connection on π_1; in coordinates

$$(t, q^\sigma, q^\sigma_{(1)}, \dot{q}^\sigma, \dot{q}^\sigma_{(1)}) \overset{\mathcal{V}\Gamma^{(2)}}{\longmapsto} \left(t, q^\sigma, q^\sigma_{(1)}, \dot{q}^\sigma, \dot{q}^\sigma_{(1)}, \Gamma^\sigma_{(2)}, \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda} \dot{q}^\lambda + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} \dot{q}^\lambda_{(1)} \right).$$

5. Connections on $\pi_{1,0} \colon \mathbb{R} \times TM \to \mathbb{R} \times M$

The first jet prolongation $J^1\pi_{1,0}$ of $\pi_{1,0}$ is the manifold of 1-jets of (local) connections on π with fibered coordinates $(t, q^\sigma, q^\sigma_{(1)}, z^\sigma, z^\sigma_\lambda)$, where

$$z^\sigma(j^1_y\Gamma) = \frac{\partial \Gamma^\sigma}{\partial t}(y), \quad z^\sigma_\lambda(j^1_y\Gamma) = \frac{\partial \Gamma^\sigma}{\partial q^\lambda}(y).$$

Since our particular concern in this paper is with the relations between autonomous and time-dependent situations, the following fact appears to be of importance. An immediate verification shows that there is a canonical inclusion

$$\mathbb{R} \times J^1\tau_M \overset{\varkappa}{\hookrightarrow} J^1\pi_{1,0} \tag{5.1}$$

over $J^1\pi = \mathbb{R} \times TM$. For any pair $(t, j^1_x v) \in \mathbb{R} \times J^1\tau_M$ we have $\varkappa(t, j^1_x v) = j^1_{(t,x)}\Gamma$, where Γ and v are identified by (4.2). Local equations for $\varkappa(\mathbb{R} \times J^1\tau_M) \subset J^1\pi_{1,0}$ are thus

$$z^\sigma = 0. \tag{5.2}$$

A *connection* on $\pi_{1,0}$ is a section $\Xi \colon J^1\pi \to J^1\pi_{1,0}$. The *horizontal form* of Ξ is $h_\Xi = D_{\Xi 0} \otimes \mathrm{d}t + D_{\Xi\lambda} \otimes \mathrm{d}q^\lambda$, where the *absolute derivatives*

$$D_{\Xi 0} = \frac{\partial}{\partial t} + \Xi^\sigma \frac{\partial}{\partial q^\sigma_{(1)}} \quad , \quad D_{\Xi\lambda} = \frac{\partial}{\partial q^\lambda} + \Xi^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}}$$

are the generators of the $\pi_{1,0}$-horizontal $(m+1)$-dimensional distribution H_Ξ realizing a decomposition $TJ^1\pi = H_\Xi \oplus V_{\pi_{1,0}}J^1\pi$. Notice that evidently

$$V_{\pi_{1,0}}J^1\pi \cong \mathbb{R} \times V_{\tau_M}TM.$$

The *integral sections* of a connection Ξ on $\pi_{1,0}$ are (local) connections on π satisfying $j^1\Gamma = \Xi \circ \Gamma$, which locally means a first-order system of PDE of the form

$$\frac{\partial \Gamma^\sigma}{\partial t} = \Xi^\sigma(t, q^\varrho, \Gamma^\varrho), \quad \frac{\partial \Gamma^\sigma}{\partial q^\lambda} = \Xi^\sigma_\lambda(t, q^\varrho, \Gamma^\varrho), \tag{5.3}$$

where Ξ^σ, Ξ^σ_λ are the components of Ξ.

Following (5.2), any connection Λ on τ_M can be considered as a connection on $\pi_{1,0}$ of the particular type

$$\Xi = (\mathrm{id}_{\mathbb{R}}, \Lambda)\colon \mathbb{R} \times TM \to \varkappa(\mathbb{R} \times J^1\tau_M), \tag{5.4}$$

whose components are $\Xi^\sigma = 0$, $\Xi^\sigma_\lambda = \Lambda^\sigma_\lambda(q^\varrho, q^\varrho_{(1)})$. The corresponding horizontal form is now

$$h_\Xi = \frac{\partial}{\partial t} \otimes \mathrm{d}t + \left(\frac{\partial}{\partial q^\lambda} + \Lambda^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}} \right) \otimes \mathrm{d}q^\lambda = I_{T\mathbb{R}} + h_\Lambda,$$

and the integral sections can be identified with vector fields on M (cf. Sec. 2).

The *deformations* of connections on $\pi_{1,0}$ are soldering forms on $\pi_{1,0}$, i.e. the sections

$$\varphi\colon J^1\pi \to V_{\pi_{1,0}} J^1\pi \otimes \pi^*_{1,0}(T^*Y)$$

of the vector bundle associated to $(\pi_{1,0})_{1,0}\colon J^1\pi_{1,0} \to J^1\pi$; locally

$$\varphi = \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \left(\varphi^\sigma \,\mathrm{d}t + \varphi^\sigma_\lambda \,\mathrm{d}q^\lambda \right). \tag{5.5}$$

By [7], all natural soldering forms on $\pi_{1,0}$ are of the form

$$k_1 J + k_2 C \otimes \mathrm{d}t, \qquad k_1, k_2 \in \mathcal{F}(\mathbb{R}) \tag{5.6}$$

and the key-role played by

$$S = J - C \otimes \mathrm{d}t$$

will become apparent below.

According to [20], for any soldering form φ (5.5) the *φ-torsion* of the connection Ξ on $\pi_{1,0}$ is defined by

$$\mathcal{T}_\varphi = [h_\Xi, \varphi].$$

For $\varphi = S$ the corresponding S-torsion will be called a *torsion*, locally

$$\mathcal{T} = \frac{\partial \Xi^\sigma_\lambda}{\partial q^\nu_{(1)}} \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}q^\nu \wedge \mathrm{d}q^\lambda + \left(\Xi^\sigma_\lambda - \frac{\partial \Xi^\sigma_\lambda}{\partial q^\nu_{(1)}} q^\nu_{(1)} - \frac{\partial \Xi^\sigma}{\partial q^\lambda_{(1)}} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}t \wedge \mathrm{d}q^\lambda. \tag{5.7}$$

Evidently, for Ξ given by (5.4) we can see that the torsions of Ξ and Λ coincide if and only if Λ is linear.

Let ζ be an arbitrary semispray on $\mathbb{R}\times TM$. Then $i_\zeta\mathcal{T}$ is a soldering form on $\pi_{1,0}$, locally expressed by

$$(5.8)\quad i_\zeta\mathcal{T} = \left(\frac{\partial\Xi^\sigma}{\partial q^\nu_{(1)}}q^\nu_{(1)} + \frac{\partial\Xi^\sigma_\lambda}{\partial q^\nu_{(1)}}q^\nu_{(1)}q^\lambda_{(1)} - \Xi^\sigma_\lambda q^\lambda_{(1)}\right)\frac{\partial}{\partial q^\sigma_{(1)}}\otimes \mathrm{d}t + \left(\Xi^\sigma_\lambda - \frac{\partial\Xi^\sigma_\nu}{\partial q^\lambda_{(1)}}q^\nu_{(1)} - \frac{\partial\Xi^\sigma}{\partial q^\lambda_{(1)}}\right)\frac{\partial}{\partial q^\sigma_{(1)}}\otimes \mathrm{d}q^\lambda.$$

The *tension* of a connection Ξ on $\pi_{1,0}$ can be defined as a soldering form

$$\mathcal{H} = \mathcal{L}_C h_\Xi$$

locally expressed by

$$(5.9)\qquad \mathcal{H} = \left(\frac{\partial\Xi^\sigma}{\partial q^\nu_{(1)}}q^\nu_{(1)} - \Xi^\sigma\right)\frac{\partial}{\partial q^\sigma_{(1)}}\otimes \mathrm{d}t + \left(\frac{\partial\Xi^\sigma_\lambda}{\partial q^\nu_{(1)}}q^\nu_{(1)} - \Xi^\sigma_\lambda\right)\frac{\partial}{\partial q^\sigma_{(1)}}\otimes \mathrm{d}q^\lambda.$$

By means of $\mathcal{H}$ the family of *linear connections* on the vector bundle $\pi_{1,0}$ can be characterized; namely, Ξ is linear if and only if its tension vanishes, which by (5.9) means $\Xi^\sigma = \Psi^\sigma_\nu(t,q^\varrho)q^\nu_{(1)}$, $\Xi^\sigma_\lambda = \Psi^\sigma_{\lambda\nu}(t,q^\varrho)q^\nu_{(1)}$ with $\Psi^\sigma_\nu = \partial\Xi^\sigma/\partial q^\nu_{(1)}$ and $\Psi^\sigma_{\lambda\nu} = \partial\Xi^\sigma_\lambda/\partial q^\nu_{(1)}$. Globally, $\Xi\colon J^1\pi \to J^1\pi_{1,0}$ is a linear fibered morphism between $\pi_{1,0}$ and $(\pi_{1,0})_1$ over Y. In particular, any linear connection Λ on τ_M defines a linear connection Ξ on $\pi_{1,0}$ by (5.4), i.e. $\Psi^\sigma_{\lambda\nu} = \Lambda^\sigma_{\lambda\nu}$.

According to the results of [8], there is a unique natural transformation of $J^1\pi_{1,0}$ into $J^1\pi_1$ (in fact, into $J^2\pi$) over $J^1\pi$, which is of the form

$$j^1_y\Gamma \xmapsto{\mathbf{f}_0} J^1(\Gamma,\mathrm{id}_\mathbb{R})\circ\Gamma(y),$$

where $J^1(\Gamma,\mathrm{id}_\mathbb{R})$ means the first jet prolongation of the morphism Γ over $\mathbb{R}$. Equivalently,

$$j^1_{(t,x)}\Gamma \xmapsto{\mathbf{f}_0} j^2_t\gamma,$$

where γ is the maximal integral section of Γ passing through $y=(t,x)$. In coordinates

$$(5.10)\qquad q^\sigma_{(2)}\circ\mathbf{f}_0 = z^\sigma + z^\sigma_\lambda q^\lambda_{(1)}\ .$$

In particular, due to (5.1) the only natural transformation $\mathbf{f}^M_0$ of $J^1\tau_M$ into T^2M over TM is determined as

$$j^1_x v \xmapsto{\mathbf{f}^M_0} Tv\circ v$$

and presented here in the notation of Sections 2, 3: $\ddot{x}^i \circ \mathbf{f}_0^M = \dot{x}^i_j \dot{x}^j$. As an immediate consequence, there is a unique naturally determined 2-connection $\Gamma^{(2)} = \mathbf{f}_0 \circ \Xi$ for any connection Ξ on $\pi_{1,0}$, called *characteristic* to Ξ, whose components are

$$\Gamma^\sigma_{(2)} = \Xi^\sigma + \Xi^\sigma_\lambda q^\lambda_{(1)}. \tag{5.11}$$

Evidently, the generator $D_{\Gamma^{(2)}}$ of the *characteristic distribution* $H_{\Gamma^{(2)}}$ is just the semispray *associated* to Ξ (cf. [6]). In particular, (5.11) implies (3.3).

The relationship between Ξ on $\pi_{1,0}$ and its characteristic 2-connection $\Gamma^{(2)}$ on π is based on the essential property of the corresponding horizontal distributions which reads $H_{\Gamma^{(2)}} \subset H_\Xi$. This 'horizontal' considerations lead to a description of some indirect integration methods for the connections studied (we refer to [17] for full details).

First, the integral sections of Ξ are foliated (and thus can be 'glued together') by the integral manifolds of $H_{\Gamma^{(2)}}$, called *characteristics*, which are just the first jet prolongations of the integral sections of $\Gamma^{(2)}$. Secondly, if Γ is a (local) integral section of Ξ then

$$J^1(\Gamma, \mathrm{id}_\mathbb{R}) \circ \Gamma = \Gamma^{(2)} \circ \Gamma \tag{5.12}$$

on the domain of Γ; in other words, Γ represents a (local) first-order system whose prolongation is just $\Gamma^{(2)}$. According to [17], any integral section Γ of Ξ is a *field of geodesics* of the characteristic $\Gamma^{(2)}$ which means in case of Ξ integrable (see Sec. 6) that each integral section of $\Gamma^{(2)}$ is locally an integral section of a certain integral section Γ of Ξ. Roughly speaking, a second-order problem for geodesics of $\Gamma^{(2)}$ can be reduced to a first-order problem for integral sections of Ξ. In the autonomous situation (5.4) the fields of geodesics are exactly the vector fields parallel to Λ; in fact, (5.12) reads $Tv \circ v = w_\Lambda \circ v$.

The geometry of connections on $\pi_{1,0}$ is rich in structures related, the above 'horizontal' ideas can be supplemented by some 'vertical' ones. While the above considerations had to do with integral sections themselves, the following ones are closely related to the infinitesimal symmetries (cf. Sec. 7).

By [31], any connection Ξ on $\pi_{1,0}$ can be identified with an $f(3,-1)$-*structure* $F_\Xi = 2h_\Xi - h_{\Gamma^{(2)}} - I$ of rank $2m$ on $J^1\pi$, where $\Gamma^{(2)}$ is the characteristic connection and $F^2_\Xi = v_{\Gamma^{(2)}}$. The eigenspaces of F_Ξ are generated by the projections

$$I - F^2_\Xi = h_{\Gamma^{(2)}}, \qquad \frac{1}{2}(F^2_\Xi + F_\Xi) = h_\Xi - h_{\Gamma^{(2)}}, \qquad \frac{1}{2}(F^2_\Xi - F_\Xi) = v_\Xi,$$

hence the decomposition of $TJ^1\pi$ generated by Ξ through F_Ξ is

$$TJ^1\pi = V_{\pi_{1,0}} J^1\pi \oplus H_{F_\Xi} \oplus H_{\Gamma^{(2)}}.$$

The m-dimensional $H_{F_\Xi} = \mathrm{Im}(h_\Xi - h_{\Gamma^{(2)}}) = \mathrm{span}\{D_{\Xi\lambda}\}$ is in accordance with [6] called *strong horizontal*, which means

$$V_{\pi_{1,0}} J^1\pi \oplus H_{F_\Xi} = V_{\pi_1} J^1\pi.$$

Evidently

$$H_{F_\Xi} = \mathrm{Im}\, h_\Xi\big|_{\pi^*_{1,0}(V_\pi Y)},$$

where h_Ξ is now regarded as a horizontal lift $\pi^*_{1,0}(TY) \to TJ^1\pi$ (rather than the projector on $TJ^1\pi$), which corresponds to the autonomous situation (5.4), where H_{F_Ξ} coincides with H_Λ.

Notice finally that in terms of the characteristic connection, (5.8) can be rewritten to

$$i_\zeta \mathcal{T} = \Big(2\Xi^\sigma + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} q^\lambda_{(1)} - 2\Gamma^\sigma_{(2)}\Big) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}t + \Big(2\Xi^\sigma_\lambda - \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}\Big) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}q^\lambda,$$

which becomes important below.

6. Integrals of 2-connections

As already announced, the search for integrable connections on $\pi_{1,0}$ with a common characteristic 2-connection on π might be of interest for the integration of the corresponding second-order ODE system. In this respect, if $\Gamma^{(2)}$ is a 2-connection on π then any Ξ on $\pi_{1,0}$ whose characteristic connection is $\Gamma^{(2)}$ will be called *associated* to $\Gamma^{(2)}$ and each integrable Ξ associated to $\Gamma^{(2)}$ will be called an *integral* of $\Gamma^{(2)}$.

First we recall a *local* result of [17]. Suppose we are given a system $\{a^1, \dots, a^m\}$ of independent first integrals of $H_{\Gamma^{(2)}}$ on some open $W \subset J^1\pi$, $W \subset \pi^{-1}_{1,0}(V)$, where (V, ψ) is a fibered chart on Y. If in these coordinates $\det(\partial a^\sigma / \partial q^\lambda_{(1)}) \neq 0$ on W, then by $H_\Xi = \mathrm{Anih}\{\mathrm{d}a^1, \dots, \mathrm{d}a^m\}$ an integral Ξ on $\Gamma^{(2)}$ on W is defined whose components are

$$\Xi^\sigma = -\widetilde{A}^\sigma_\nu \frac{\partial a^\nu}{\partial t}, \quad \Xi^\sigma_\lambda = -\widetilde{A}^\sigma_\nu \frac{\partial a^\nu}{\partial q^\lambda},$$

where $(\widetilde{A}^\sigma_\lambda)$ is the inverse matrix to $(A^\sigma_\lambda) = (\partial a^\sigma / \partial q^\lambda_{(1)})$.

In case of *global integrals*, the task splits into two parts; first, global connections on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ must be determined and secondly, their integrability should be studied. In what follows we proceed analogously to [9]. We have to start with a natural vector bundle morphism

$$\bar{\mathbf{f}}_0 \colon V_{\pi_{1,0}} J^1\pi \otimes \pi^*_{1,0}(T^*Y) \to V_{\pi_{1,0}} J^1\pi \otimes \pi^*_{1,0}(T^*\mathbb{R})$$

over $J^1\pi$ induced by (5.10) on the associated vector bundles. This morphism maps deformations of connections on $\pi_{1,0}$ to deformations of 2-connections on π; in coordinates

$$\varphi^{\sigma}_{(2)} \circ \bar{\mathbf{f}}_0 = \varphi^{\sigma} + \varphi^{\sigma}_{\lambda} q^{\lambda}_{(1)}. \tag{6.1}$$

If Ξ_0 is a connection on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ then evidently by $h_{\Xi_0} + \varphi$ another such connection is defined if and only if

$$\varphi \in \ker \bar{\mathbf{f}}_0, \tag{6.2}$$

and any soldering form φ on $\pi_{1,0}$ satisfying (6.2) will be called *admissible*. Due to (5.6), all *natural admissible* soldering forms on $\pi_{1,0}$ are of the form $\varphi = kS$, $k \in \mathcal{F}(\mathbb{R})$. According to [6],

$$h_{\Xi_0} = \frac{1}{2}\left(h_{\Gamma^{(2)}} + I - \mathcal{L}_{D_{\Gamma^{(2)}}} S\right) \tag{6.3}$$

is a horizontal form of a connection Ξ_0 on $\pi_{1,0}$ associated to $\Gamma^{(2)}$. Following the above ideas we can state that the family of connections on $\pi_{1,0}$ *naturally associated* to a 2-connection $\Gamma^{(2)}$ on π is defined by

$$h_{\Xi} = h_{\Xi_0} + kS, \qquad k \in \mathcal{F}(\mathbb{R}). \tag{6.4}$$

In coordinates, the components of Ξ are

$$\Xi^{\sigma}_{\lambda} = \frac{1}{2}\frac{\partial \Gamma^{\sigma}_{(2)}}{\partial q^{\lambda}_{(1)}} + k(t)\delta^{\sigma}_{\lambda}, \quad \Xi^{\sigma} = \Gamma^{\sigma}_{(2)} - \Xi^{\sigma}_{\lambda} q^{\lambda}_{(1)}. \tag{6.5}$$

This result can be compared with [30] and [9]; in fact, if we view $K^*(t) = 2k(t)$ as a component of a linear connection K^* on $\tau^*_{\mathbb{R}}$, it is easy to verify that the connection (6.4) coincides with the so-called *natural dynamical connection* of type Ω [30] for any volume form Ω on $\mathbb{R}$ which is a (global) integral section of K^*. Moreover, denoting by K the *dual* connection to K^* on $\tau_{\mathbb{R}}$ (i.e. $K(t) = -K^*(t)$), we get (6.4) as the only connection naturally assigned to the pair $\Gamma^{(2)}$, K in the sense of [9].

Next, the role of torsions and related structures can be discussed. Denote by $\mathcal{T}_0$ and $\mathcal{H}_0$ the torsion and the tension of the connection Ξ_0 given by (6.3). By (5.7), (5.8) and (5.9) one gets

$$\mathcal{T}_0 = \frac{1}{2}\frac{\partial^2\Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}\partial q^\nu_{(1)}}\frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}q^\nu \wedge \mathrm{d}q^\lambda = 0,$$

$$\mathcal{H}_0 = \left(\frac{\partial\Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}q^\lambda_{(1)} - \Gamma^\sigma_{(2)} - \frac{1}{2}\frac{\partial^2\Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}\partial q^\nu_{(1)}}q^\lambda_{(1)}q^\nu_{(1)}\right)\frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}t$$
$$+ \frac{1}{2}\left(\frac{\partial^2\Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}\partial q^\nu_{(1)}}q^\nu_{(1)} - \frac{\partial\Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}}\right)\frac{\partial}{\partial q^\sigma_{(1)}} \otimes \mathrm{d}q^\lambda.$$

Let now φ be an arbitrary admissible soldering form on $\pi_{1,0}$. Then for the connection Ξ on $\pi_{1,0}$ defined by

$$h_\Xi = h_{\Xi_0} + \varphi \tag{6.6}$$

we have $\mathcal{T} = \mathcal{T}_0 + [\varphi, S]$, $i_\zeta\mathcal{T} = 2\varphi$, $\mathcal{H} = \mathcal{H}_0 + \mathcal{L}_C\varphi$. Hence, the following assertion holds.

Proposition 6.1. *Let $\Gamma^{(2)}$ be a 2-connection on π and φ an admissible soldering form on $\pi_{1,0}$. Then there is a unique connection Ξ on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ such that its torsion $\mathcal{T}$ satisfies $i_\zeta\mathcal{T} = 2\varphi$.*

By (6.6), this connection is defined just by

$$h_\Xi = \frac{1}{2}\left(h_{\Gamma^{(2)}} + I - \mathcal{L}_{D_{\Gamma^{(2)}}}S\right) + \varphi.$$

Now it is easy to see that if the torsion of a connection Ξ on $\pi_{1,0}$ vanishes, then $\Xi = \Xi_0$ for $\Gamma^{(2)}$ being the characteristic connection to Ξ.

Let now $w = q^\sigma_{(1)}\partial/\partial q^\sigma + w^\sigma(q^\nu, q^\nu_{(1)})\partial/\partial q^\sigma_{(1)}$ be a semispray on TM and $D_{\Gamma^{(2)}} = \partial/\partial t + w$ a semispray on $\mathbb{R}\times TM$ defining a 2-connection $\Gamma^{(2)}$ on π. Following the above approach, a search for a connection Λ on τ_M associated to w is equivalent to a search for a connection Ξ of type (5.4) associated to $\Gamma^{(2)}$. From (6.6) we easily deduce that φ must satisfy

$$\varphi^\sigma = \frac{1}{2}\frac{\partial w^\sigma}{\partial q^\lambda_{(1)}}q^\lambda_{(1)} - w^\sigma$$

or equivalently

$$\varphi^\sigma_\lambda q^\lambda_{(1)} = w^\sigma - \frac{1}{2}\frac{\partial w^\sigma}{\partial q^\lambda_{(1)}}q^\lambda_{(1)},$$

which coincides with (3.6).

Following [29], certain 'homogeneous' considerations can be presented. First, in accordance with the autonomous situation and due to the underlying structures, a *spray connection* on π can be defined as a 2-connection $\Gamma^{(2)}$ on π whose components are homogeneous of order two in $q^{\lambda}_{(1)}$, i.e.

$$\frac{\partial \Gamma^{\sigma}_{(2)}}{\partial q^{\lambda}_{(1)}} q^{\lambda}_{(1)} = 2\Gamma^{\sigma}_{(2)}. \tag{6.7}$$

Throughout the paper, the smoothness on the zero section is assumed, hence (6.7) means that the functions $\Gamma^{\sigma}_{(2)}$ are quadratic in $q^{\lambda}_{(1)}$:

$$\Gamma^{\sigma}_{(2)} = \frac{1}{2} \frac{\partial^2 \Gamma^{\sigma}_{(2)}}{\partial q^{\lambda}_{(1)} \partial q^{\nu}_{(1)}} q^{\lambda}_{(1)} q^{\nu}_{(1)}.$$

It is easy to verify that if Ξ is linear then its characteristic $\Gamma^{(2)}$ is a spray connection if and only if $\Xi^{\sigma} = 0$ and conversely, the connection (6.3) associated to a spray connection is linear with

$$\Xi^{\sigma} = 0, \qquad \Psi^{\sigma}_{\lambda\nu} = \frac{1}{2} \frac{\partial^2 \Gamma^{\sigma}_{(2)}}{\partial q^{\lambda}_{(1)} \partial q^{\nu}_{(1)}}.$$

Consequently, if $\Gamma^{(2)}$ is a spray connection on π then (6.3) is the unique linear connection on $\pi_{1,0}$ associated to $\Gamma^{(2)}$.

The *integrability* of a connection on an arbitrary fibered manifold (and thus of the corresponding equations) means equivalently the involutiveness of the corresponding horizontal distribution. Accordingly , the integrability conditions can be expressed among other by the vanishing of the Lie bracket of the absolute derivatives with respect to the connection. For a connection Ξ on $\pi_{1,0}$ it means

$$[D_{\Xi 0}, D_{\Xi \lambda}] = 0 \tag{6.8}$$

$$[D_{\Xi \sigma}, D_{\Xi \lambda}] = 0. \tag{6.9}$$

7. Symmetries

In what follows we deal with (infinitesimal) symmetries of the ODE studied, i.e. with vector fields as generators of groups of transformations invariant with respect to solutions. For the general case of PDE represented by connections we refer to [31] and references therein.

An *(infinitesimal) symmetry* of a connection Γ on π (or equivalently of the time dependent vector field (4.2)) can be defined as a vector field

$$\zeta = \zeta^0 \frac{\partial}{\partial t} + \zeta^\sigma \frac{\partial}{\partial q^\sigma} \tag{7.1}$$

on Y which is Γ-related to its first prolongation

$$\mathcal{J}^1\zeta = \zeta^0 \frac{\partial}{\partial t} + \zeta^\sigma \frac{\partial}{\partial q^\sigma} + \left(D(\zeta^\sigma) - D(\zeta^0) q^\sigma_{(1)} \right) \frac{\partial}{\partial q^\sigma_{(1)}}$$

on $J^1\pi$, i.e. $\mathcal{J}^1\zeta \circ \Gamma = T\Gamma \circ \zeta$. In other words, $\mathcal{J}^1\zeta$ is a *contact* vector field (a symmetry of the Cartan distribution) tangent to $\Gamma(Y)$ and thus an *exterior* symmetry of the equation $\Gamma(Y) \subset J^1\pi$.

It is easy to see that an arbitrary Γ-horizontal vector field $\zeta = f D_\Gamma$, $f \in \mathcal{F}(Y)$, is a symmetry, which immediately means that ζ on Y is a symmetry of Γ if and only if

$$\mathcal{J}^1(v_\Gamma \circ \zeta) \circ \Gamma = V\Gamma \circ v_\Gamma(\zeta). \tag{7.2}$$

In coordinates, (7.2) is represented by a system of PDE

$$\frac{\partial \varphi^\sigma}{\partial t} + \frac{\partial \varphi^\sigma}{\partial q^\lambda}\Gamma^\lambda = \frac{\partial \Gamma^\sigma}{\partial q^\lambda}\varphi^\lambda \tag{7.3}$$

for the family of *generating functions* $\varphi^\sigma = \zeta^\sigma - \Gamma^\sigma \zeta^0$ on Y, $\sigma = 1, \ldots, m$. Using the horizontal form of Γ, (7.3) can be expressed by

$$\mathcal{L}_{v_\Gamma(\zeta)} h_\Gamma = 0 \tag{7.4}$$

and in terms of D_Γ it reads

$$[\zeta, D_\Gamma] = -D_\Gamma(\zeta^0) D_\Gamma. \tag{7.5}$$

The last relation says that the symmetries of Γ could be defined directly as the symmetries of the horizontal distribution H_Γ.

Since $\mathcal{L}_{h_\Gamma(\zeta)} h_\Gamma = 0$ if and only if ζ is π-projectable, the symbol of v_Γ in (7.4) can be omitted under the same assumption. In view of the fact that the vanishing of $\mathcal{L}_\zeta h_\Gamma$ is equivalent to $T\alpha_t \circ h_\Gamma = h_\Gamma \circ T\alpha_t$ for the flow $\{\alpha_t\}$ of ζ, a *projectable* vector field on Y is a symmetry of Γ if and only if its flow permutes the integral sections of Γ. Moreover, if such a symmetry is horizontal then it moves integral sections along themselves while π-*vertical* symmetries (time-dependent fields on M, see Sec. 4) permute integral sections without changing their parametrization . In this case (7.5) reads $[\zeta, \partial/\partial t + v] = 0$.

An intrinsic role both of vertical prolongations of connections and of strong horizontal distributions appears in case of vertical symmetries, the set of which we denote by $\mathrm{Sym}_v(\Gamma)$. Actually, applying the ideas of Sec. 4, (7.2) means

$$\mathcal{J}^1\zeta \circ \Gamma = \mathcal{V}\Gamma \circ \zeta \tag{7.6}$$

for vertical ζ. Consequently, in terms of integral sections of Γ and $\mathcal{V}\Gamma$ one gets that for any $\zeta \in \mathrm{Sym}_v(\Gamma)$, a section γ of π is an integral section of Γ if and only if $\xi = \zeta \circ \gamma$ is an integral section of $\mathcal{V}\Gamma$. Moreover, $\zeta \in \mathrm{Sym}_v(\Gamma)$ if and only if ξ is an integral section of $\mathcal{V}\Gamma$ for each integral section γ of Γ.

Let Ξ be a connection on $\pi_{1,0}$ and ζ a vertical vector field on Y. Then $\mathcal{J}^1\zeta \in H_{F_\Xi}$ locally means

$$D(\zeta^\sigma) = \Xi^\sigma_\lambda \zeta^\lambda, \tag{7.7}$$

and by (5.3), (7.3) and (7.7) the strong horizontal distribution H_{F_Ξ} contains first prolongations of vertical symmetries of integral sections of Ξ.

Let $\Gamma^{(2)}$ be a 2-connection on π. Since it is a particular (holonomic) type of a connection on π_1, a vector field $\zeta^{(1)} = \zeta^0 \partial/\partial t + \zeta^\sigma \partial/\partial q^\sigma + \zeta^\sigma_{(1)} \partial/\partial q^\sigma_{(1)}$ on $J^1\pi$ will be called a *first-order symmetry* of $\Gamma^{(2)}$ (or of the corresponding semispray $D_{\Gamma^{(2)}}$) if

$$\mathcal{L}_{v_{\Gamma^{(2)}}(\zeta^{(1)})} h_{\Gamma^{(2)}} = 0 \tag{7.8}$$

or equivalently

$$[\zeta^{(1)}, D_{\Gamma^{(2)}}] = -D_{\Gamma^{(2)}}(\zeta^0) D_{\Gamma^{(2)}}. \tag{7.9}$$

Hence the first-order symmetries of $\Gamma^{(2)}$ are just the symmetries of $H_{\Gamma^{(2)}}$. In coordinates,

$$\begin{aligned} \zeta^\sigma_{(1)} &= D_{\Gamma^{(2)}}(\zeta^\sigma) - q^\sigma_{(1)} D_{\Gamma^{(2)}}(\zeta^0) \\ D^2_{\Gamma^{(2)}}(\varphi^\sigma) &= \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda} \varphi^\lambda + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} D_{\Gamma^{(2)}}(\varphi^\lambda) \end{aligned} \tag{7.10}$$

for $\varphi^\sigma = \zeta^\sigma - q^\sigma_{(1)}\zeta^0$. Again, by (7.8) π_1-projectable fields are first-order symmetries of $\Gamma^{(2)}$ if and only if their flows permute 1-jets of integral sections of $\Gamma^{(2)}$.

Within the context of the equations studied, we are mainly interested in symmetries acting on Y. Consequently, a vector field ζ (7.1) on Y is called a *(zero-order) symmetry* of $\Gamma^{(2)}$ if its first prolongation $\mathcal{J}^1\zeta$ is the first-order one. The relation (7.9) now reads

$$[\mathcal{J}^1\zeta, D_{\Gamma^{(2)}}] = -D(\zeta^0)D_{\Gamma^{(2)}}, \tag{7.11}$$

and (7.10) holds trivially. Accordingly, a $\pi_{1,0}$*-projectable* first-order symmetry $\zeta^{(1)}$ is just the prolongation $\mathcal{J}^1\zeta$ of the symmetry $\zeta = T\pi_{1,0}(\zeta^{(1)})$.

Let ζ be π*-projectable*. Then it is a symmetry of $\Gamma^{(2)}$ if and only if the flow of $\mathcal{J}^1\zeta$ permutes first jets of integral sections of $\Gamma^{(2)}$. Due to the definition of $\mathcal{J}^1\zeta$, ζ is a symmetry if and only if its flow permutes the integral sections of $\Gamma^{(2)}$ in themselves.

Suppose the symmetries to be π*-vertical* and denote their set by $\mathrm{Sym}_v(\Gamma^{(2)})$. Analogously to (7.6) one sees that $\zeta \in \mathrm{Sym}_v(\Gamma^{(2)})$ if and only if

$$\mathcal{J}^2\zeta \circ \Gamma^{(2)} = \mathcal{V}\Gamma^{(2)} \circ \mathcal{J}^1\zeta,$$

where $\mathcal{J}^2\zeta = J^2(\zeta, \mathrm{id}_\mathbb{R})$ is the second prolongation of ζ. Consequently, γ is an integral section of $\Gamma^{(2)}$ if and only if $\xi = \zeta \circ \gamma$ is an integral section of $\mathcal{V}\Gamma^{(2)}$ and $\zeta \in \mathrm{Sym}_v(\Gamma^{(2)})$ if and only if ξ is an integral section of $\mathcal{V}\Gamma^{(2)}$ for each integral section γ of $\Gamma^{(2)}$.

The presented classification enables us to describe very naturally the interrelations between (vertical) symmetries of a 2-connection $\Gamma^{(2)}$ (second-order ODE) and its field of geodesics Γ (first-order ODE) (see (5.12)). First, since any integral section of Γ is an integral section of $\Gamma^{(2)}$, if ζ is a symmetry of $\Gamma^{(2)}$ then its restriction to the domain is a symmetry of Γ. Secondly, by virtue of the fact that

$$J^1(\Gamma, \mathrm{id}_\mathbb{R}) \circ \Gamma = \Gamma^{(2)} \circ \Gamma \qquad \text{if and only if} \qquad J^1(\mathcal{V}\Gamma, \mathrm{id}_\mathbb{R}) \circ \mathcal{V}\Gamma = \mathcal{V}\Gamma^{(2)} \circ \mathcal{V}\Gamma$$

one easily deduces that a symmetry of Γ is a symmetry of its prolongation $\Gamma^{(2)} \circ \Gamma$.

Recall the autonomous situation. In this case the considerations describe the generators of groups of transformations invariant with respect to the *graphs* of geodesics. For example, according to (7.9) a vector field $\zeta^{(1)}$ on TM is a first-order symmetry of a semispray w on TM if $[\zeta^{(1)}, w] = 0$ while according to (7.11) a vector field ζ on M is a (zeroth-order) symmetry of w if $[\zeta^c, w] = 0$.

Notice finally that comparing the above classification e.g. with [23] or [6], the first-order symmetries of a 2-connection are the so-called *dynamical symmetries* of the corresponding semispray while the zero-order ones are nothing but the *Lie symmetries.*

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