PRECOVERS AND GOLDIE'S TORSION THEORY

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Abstract. Recently, Rim and Teply [8], using the notion of τ-exact modules, found a necessary condition for the existence of τ-torsionfree covers with respect to a given hereditary torsion theory τ for the category R-mod of all unitary left R-modules over an associative ring R with identity. Some relations between τ-torsionfree and τ-exact covers have been investigated in [5]. The purpose of this note is to show that if $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ is Goldie's torsion theory and $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class whenever $\tau \geqslant \sigma$. Further, it is shown that $\mathscr{F}_\sigma$ is a cover class if and only if σ is of finite type and, in the case of non-singular rings, this is equivalent to the fact that $\mathscr{F}_\tau$ is a cover class for all hereditary torsion theories $\tau \geqslant \sigma$.

Keywords: hereditary torsion theory, Goldie's torsion theory, non-singular ring, precover class, cover class

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In what follows, R stands for an associative ring with identity and R-mod denotes the category of all unitary left R-modules. The basic properties of rings and modules can be found in [1]. A class $\mathscr{G}$ of modules is called *abstract*, if it is closed under isomorphic copies, *co-abstract*, if its members are pairwise non-isomorphic and *complete* with respect to a given property, if every module with this property is isomorphic to a member of the class $\mathscr{G}$.

Recall that a *hereditary torsion theory* $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ for the category R-mod consists of two abstract classes $\mathscr{T}_\tau$ and $\mathscr{F}_\tau$, the *τ-torsion class* and the *τ-torsionfree class*, respectively, such that $\operatorname{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}_\tau$ and $F \in \mathscr{F}_\tau$, the class $\mathscr{T}_\tau$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathscr{F}_\tau$ is closed under submodules, extensions and arbitrary direct products

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and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}_\tau$ and $F \in \mathscr{F}_\tau$. For two hereditary torsion theories τ and τ' the symbol $\tau \leqslant \tau'$ means that $\mathscr{T}_\tau \subseteq \mathscr{T}_{\tau'}$ and consequently $\mathscr{F}_{\tau'} \subseteq \mathscr{F}_\tau$. Associated with each hereditary torsion theory τ is the *Gabriel filter* $\mathscr{L}_\tau$ of left ideals of R consisting of all left ideals $I \leqslant R$ with $R/I \in \mathscr{T}_\tau$. Recall that τ is said to be of *finite type*, if $\mathscr{L}_\tau$ contains a cofinal subset $\mathscr{L}_\tau'$ of finitely generated left ideals. A submodule N of the module M is called *τ-closed* (or *τ-pure*), if the factor-module M/N belongs to $\mathscr{F}_\tau$. A module M is said to be *τ-noetherian*, if the set of all τ-closed submodules of M satisfies the maximum condition. A module Q is said to be *τ-injective*, if it is injective with respect to all short exact sequences $0 \to A \to B \to C \to 0$, where $C \in \mathscr{T}_\tau$. Further, a hereditary torsion theory τ is called *exact*, if $E(Q)/Q$ is τ-torsionfree τ-injective, $E(Q)$ being the injective hull of Q, whenever Q is a τ-torsionfree τ-injective module. If, in addition, τ is of finite type, then it is called *perfect*. For more details on torsion theories we refer to [7] or [6].

For a module M, the *singular submodule* $Z(M)$ consists of all elements $a \in M$, the *annihilator left ideal* $(0\colon a) = \{r \in R;\ ra = 0\}$ of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$, where $\mathscr{T}_\sigma = \{M \in R\text{-mod};\ Z(M/Z(M)) = M/Z(M)\}$ and $\mathscr{F}_\sigma = \{M \in R\text{-mod};\ Z(M) = 0\}$. If the ring R is σ-torsionfree, $Z(R) = 0$, then R is called *non-singular*. Note that in this case the Gabriel filter $\mathscr{L}_\sigma$ consists of essential left ideals only.

If $\mathscr{G}$ is an abstract class of modules, then a homomorphism $\varphi\colon G \to M$ is called a *$\mathscr{G}$-precover* of the module M, if $G \in \mathscr{G}$ and every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through φ, i.e. there exists a homomorphism $g\colon F \to G$ such that $\varphi g = f$. Moreover, a $\mathscr{G}$-precover φ of M is said to be a *$\mathscr{G}$-cover*, if each endomorphism f of G with $\varphi f = \varphi$ is an automorphism of the module G. An abstract class $\mathscr{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathscr{G}$-precover ($\mathscr{G}$-cover). It is well-known that an $\mathscr{F}_\tau$-precover $\varphi\colon G \to M$ is an $\mathscr{F}_\tau$-cover if and only if $\operatorname{Ker}\varphi$ contains no non-zero submodule τ-closed in G. For more details concerning the theory of precovers and covers we refer to [10].

It is well-known (see e.g. [7; Proposition 42.9]) that a hereditary torsion theory τ is of finite type if and only if any directed union of τ-torsionfree τ-injective modules is τ-injective and that this condition is sufficient for the existence of τ-torsionfree covers (see [9] for the τ-torsionfree rings and [2] for the general case). On the other hand, in [8] a necessary condition has been presented saying that the directed union of τ-exact submodules of a given module is τ-injective. By a *τ-exact* module we mean any τ-torsionfree module, every τ-torsionfree homomorphic image of which is τ-injective. The purpose of this note is to prove that for Goldie's torsion theory σ the finite type condition is necessary and sufficient for the existence of σ-torsionfree covers. Moreover, if $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class whenever

$\tau \geqslant \sigma$ and the same holds for cover classes provided that the ring R is non-singular. More precisely, we are going to prove the following two theorems.

Theorem 1. *Let $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ be Goldie's torsion theory for the category* R-mod. *If $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class for any hereditary torsion theory $\tau \geqslant \sigma$.*

Theorem 2. *Let $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ be Goldie's torsion theory for the category* R-mod. *The following conditions are equivalent:*

(i) *$\mathscr{F}_\sigma$ is a cover class;*
(ii) *σ is of finite type;*
(iii) *σ is perfect.*

If, moreover, the ring R is non-singular ($Z(R) = 0$), then these conditions are equivalent to the following three conditions:

(iv) *every non-zero left ideal of R contains a finitely generated essential left ideal;*
(v) *${}_RR$ is σ-noetherian;*
(vi) *for every hereditary torsion theory $\tau \geqslant \sigma$ the class $\mathscr{F}_\tau$ is a cover class.*

We start with some preliminary lemmas, the symbol σ will always denote Goldie's torsion theory.

Lemma 1. *Let $\tau \geqslant \sigma$ be a hereditary torsion theory for the category* R-mod. *Then*

(i) *a module $Q \in \mathscr{F}_\tau$ is τ-injective if and only if it is injective;*
(ii) *a submodule $K \leqslant Q$ with $Q \in \mathscr{F}_\tau$ injective is τ-closed if and only if it is injective. In this case the factor-module Q/K is also injective.*

P r o o f. (i) If $Q \in \mathscr{F}_\tau$ is τ-injective and $E(Q)$ is the injective hull of Q, then $E(Q)/Q \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$ by [7; Corollary 44.3]. In view of the obvious fact $E(Q)/Q \in \mathscr{T}_\sigma$ we have $Q = E(Q)$. The converse is obvious.

(ii) If K is τ-closed in Q, then $Q/K \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$. Hence K has no proper essential extension in Q and consequently it is injective. The rest is clear. □

Lemma 2. *Let $\tau \geqslant \sigma$ be a hereditary torsion theory for the category* R-mod. *If every module has an $\mathscr{F}_\tau$-cover, then every directed union of τ-torsionfree injective modules is τ-torsionfree injective.*

P r o o f. Let $K = \bigcup_{\alpha \in \Lambda} K_\alpha$ be a directed union of τ-torsionfree injective modules, let $M = E(K)$ be the injective hull of K and let $\varphi\colon G \to M/K$ be an $\mathscr{F}_\tau$-cover of the module M/K. Denoting by $\pi_\alpha\colon M/K_\alpha \to M/K$ the corresponding natural projections, there are homomorphisms $f_\alpha\colon M/K_\alpha \to G$ such that $\varphi f_\alpha = \pi_\alpha$ for every $\alpha \in \Lambda$. Obviously, $\operatorname{Ker} f_\alpha \subseteq K/K_\alpha$ and we are going to show that the equality holds for each $\alpha \in \Lambda$. If not, then $K_\beta/K_\alpha \nsubseteq \operatorname{Ker} f_\alpha$ for some $\alpha, \beta \in \Lambda$ and so $0 \neq f_\alpha(K_\beta/K_\alpha) \cong K_\beta/L_\beta \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$ yields according to Lemma 1 that $0 \neq f_\alpha(K_\beta/K_\alpha) \subseteq \operatorname{Ker} \varphi$ is injective. This contradicts the fact that φ is an $\mathscr{F}_\tau$-cover of the module M/K and consequently $\operatorname{Im} f_\alpha \cong M/K \in \mathscr{F}_\tau$ for each $\alpha \in \Lambda$. Thus $M/K \in \mathscr{F}_\sigma \cap \mathscr{T}_\sigma = 0$, $M = K$ and we are through. □

Lemma 3. *Let $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ be an arbitrary hereditary torsion theory for the category R-mod. The following conditions are equivalent:*

(i) *every module has a τ-torsionfree precover;*

(ii) *every injective module has a τ-torsionfree precover;*

(iii) *every injective module has an injective τ-torsionfree precover.*

Proof. For an arbitrary injective module M we obviously have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & M \\ {\scriptstyle\iota}\downarrow & & \| \\ E(G) & \xrightarrow{\psi} & M \end{array}$$

where ι is the inclusion map of G into its injective hull $E(G)$ and φ is an $\mathscr{F}_\tau$-precover of the module M. Then ψ is obviously an $\mathscr{F}_\tau$-precover of M and consequently (ii) implies (iii).

Assuming (iii) let us consider the pullback diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ G & \xrightarrow{\psi} & E(M) \end{array}$$

where $M \in R$-mod is arbitrary and ψ is an $\mathscr{F}_\tau$-precover of $E(M)$ with G injective. Clearly, i is injective, hence $F \in \mathscr{F}_\tau$ and the pullback property yields that φ is an $\mathscr{F}_\tau$-precover of the module M. The rest is clear. □

Lemma 4. *Let $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ be a hereditary torsion theory for the category R-mod. A homomorphism $\varphi\colon G \to M$ with $G \in \mathscr{F}_\tau$ and M injective is an $\mathscr{F}_\tau$-precover of the module M if and only if to each homomorphism $f\colon Q \to M$ with $Q \in \mathscr{F}_\tau$ injective, there exists a homomorphism $g\colon Q \to G$ such that $\varphi g = f$.*

P r o o f. Only the sufficiency requires verification. So, let us consider the commutative diagram

$$\begin{array}{ccccc} E(F) & = & E(F) & \xleftarrow{\;i\;} & F \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\;\varphi\;} & M & = & M \end{array}$$

with the given φ, M injective and $f\colon F \to M$, $F \in \mathscr{F}_\tau$, arbitrary. Then there is $h\colon E(F) \to M$ with $hi = f$, M being injective, and $g\colon E(F) \to G$ with $\varphi g = h$ by the definition of a precover. Thus $\varphi(gi) = hi = f$ and the proof is complete. $\square$

P r o o f (of T h e o r e m 1). Let λ be an arbitrary infinite cardinal and let $\mathfrak{M}_\lambda$ be any complete co-abstract set of modules of cardinalities at most λ. For any $M \in \mathfrak{M}_\lambda$ we fix an $\mathscr{F}_\sigma$-precover $\varphi_M\colon G_M \to M$ and denote by κ the first cardinal with $\kappa > |G_M|$ for each $M \in \mathfrak{M}_\lambda$.

Further, let $Q \in \mathscr{F}_\tau$ be an arbitrary injective module with $|Q| \geqslant \kappa$ and let $K \leqslant Q$ be its submodule such that $|Q/K| \leqslant \lambda$. Then, obviously, $Q \in \mathscr{F}_\sigma$ and consequently, by the above part, the factor-module Q/K has an $\mathscr{F}_\sigma$-precover $\varphi\colon G \to Q/K$ with $|G| < \kappa$. Thus, there is a homomorphism $f\colon Q \to G$ such that $\varphi f = \pi$, π being the canonical projection $Q \to Q/K$. Now $\operatorname{Ker} f = L$ is contained in K and it is a direct summand of Q by Lemma 1 (ii) owing to the fact that $Q/L \cong \operatorname{Im} f \in \mathscr{F}_\sigma$. Moreover, $|Q/L| = |\operatorname{Im} f| \leqslant |G| < \kappa$.

Now let $M \in R$-mod be an arbitrary injective module, $\lambda = \max(|M|, \aleph_0)$, and let κ be the cardinal corresponding to λ by the beginning of this proof. Further, let $\mathfrak{N}_\kappa$ be any complete co-abstract set of τ-torsionfree injective modules of cardinalities less than κ. We put $G = \bigoplus_{Q \in \mathfrak{N}_\kappa} Q^{(\operatorname{Hom}(Q,M))}$ and $\varphi\colon G \to M$ will denote the corresponding natural evaluation map. To verify that φ is a τ-torsionfree precover of the module M we shall use Lemma 4. So, let $Q \in \mathscr{F}_\tau$ be an arbitrary injective module and let $f\colon Q \to M$ be an arbitrary homomorphism. For $|Q| < \kappa$ there exists an isomorphic copy of Q lying in $\mathfrak{N}_\kappa$ and the existence of the homomorphism $g\colon Q \to G$ with $\varphi g = f$ can be easily verified. In the opposite case, for $|Q| \geqslant \kappa$, denoting $K = \operatorname{Ker} f$ we have $|Q/K| = |\operatorname{Im} f| \leqslant |M| \leqslant \lambda$. Thus, by the above part, there is a direct summand L of Q contained in K and such that $|Q/L| < \kappa$. Moreover, f naturally induces the homomorphism $\bar{f}\colon Q/L \to M$ such that $\bar{f}\pi = f$, $\pi\colon Q \to Q/L$ being the canonical projection. Thus there is $\bar{g}\colon Q/L \to G$ with $\varphi\bar{g} = \bar{f}$ by the previous case, so $\varphi(\bar{g}\pi) = \bar{f}\pi = f$ and to complete the proof it suffices now to apply Lemma 3. $\square$

P r o o f (of T h e o r e m 2). (i) implies (ii). It suffices to use Lemma 2 and [7; Proposition 42.9].

(ii) implies (i). This has been proved in [9] in the case of a faithful torsion theory and in [2] in the general case.

(ii) is equivalent to (iii). This is obvious, σ being exact by Lemma 1 (see also [7; Corollary 44.3]).

Assume now that the ring R is non-singular.

(ii) implies (iv). Since R is non-singular, the Gabriel filter $\mathscr{L}_\sigma$ consists of essential left ideals only, and consequently every essential left ideal contains an essential finitely generated left ideal by the hypothesis. So, let $0 \neq I \leqslant R$ be an arbitrary non-essential left ideal of R and let $J \leqslant R$ be any left ideal maximal with respect to $I \cap J = 0$. Then $I \oplus J$ is essential in R and consequently there is a finitely generated left ideal $K = \sum\limits_{i=1}^{n} Ra_i \subseteq I \oplus J$ essential in R. Now $a_i = b_i + c_i$, $b_i \in I$, $c_i \in J$, $i = 1, \ldots, n$, and it remains to verify that the left ideal $\sum\limits_{i=1}^{n} Rb_i$ is essential in I. However, for an arbitrary element $0 \neq u \in I$ we have $0 \neq ru = \sum\limits_{i=1}^{n} r_i a_i = \sum\limits_{i=1}^{n} r_i b_i + \sum\limits_{i=1}^{n} r_i c_i$ for suitable elements $r, r_1, \ldots, r_n \in R$, and consequently, $0 \neq ru = \sum\limits_{i=1}^{n} r_i b_i$, as we wished to show.

(iv) is equivalent to (v). See [7; Proposition 20.1].

(iv) implies (vi). Let $I \in \mathscr{L}_\tau$ be arbitrary and let $K \leqslant I$ be a finitely generated left ideal essential in I. Then $I/K \in \mathscr{T}_\sigma \subseteq \mathscr{T}_\tau$, hence $K \in \mathscr{L}_\tau$ and the torsion theory τ is of finite type. Now it suffices to use [2].

(vi) implies (i). This is trivial. $\square$

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