INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

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Preprint No. 42-2013

PRAHA 2013

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November 22, 2013

I dedicate this paper to my friend and frequent coauthor Christof Eck who died recently merely 43 years old. He is badly missing by his relatives, friends, colleagues and students.

Abstract. For the static contact problem with limited interpenetration with the obstacle the existence of solutions is proved. The frictionless case is studied at first, then the problem with Coulomb friction is investigated as well. The body has nowhere a Dirichlet boundary value condition prescribed. In both cases, if the prescribed bound of the interpenetratiom tends to 0, the solutions tend to a solution of the appropriate unilateral contact problem.

Keywords. Unilateral contact problem with finite penetration, semicoercivity, problem without friction, problem with Coulomb friction, existence theorem, approximation, local regularity of solutions.

MS 2010 classification. 35Q74, 74H20, 74M10, 74M15.

1 Introduction and basic notation

Contact problems represent an important theme of applied mathematics with many applications mostly in different mechanical problems. First, the unilateral problems respecting the impenetrability of Mass (Signorini problems) have been studied. Then the problems with normal compliance have been introduced. Usually an infinite penetration of the body into the obstacle has been allowed there which is obviously physically unrealistic.

In [4] a model allowing some, but a priori limited interpenetration of the body and the obstacle has been introduced. The given limit of the interpenetration there is not reachable, hence the variational formulation of its frictionless version has the form of an equation. Both frictionless and frictional problem has been analysed there under the assumption of the existence of a nondegenerate part of the boundary of the body, where the displacement is prescribed.

In this paper we extend the results of [4] to the case, where such a Dirichlet boundary condition is nowhere required. Hence it represents a paralel to the paper [1], where the similar problem was treated with the Signorini condition on the contact part of the boundary.

In the sequel by $H^\gamma(M), \gamma > 0$, the Sobolev spaces of the Hilbert type are denoted provided γ is an integer. For other γ it denotes respective Sobolev-Slobodetskii spaces. Here M is a domain (open connected) set in $\mathbb{R}^N$, its boundary or its part. For a linear operator (form) $\mathscr{L}$ acting on a space X, $\ker \mathscr{L} = \{x \in X;\ \mathscr{L}(x) = 0\}$. By $\operatorname{int} M$ the interior and by $\operatorname{cl} M$ the closure of a set M in a topological space are denoted. To distinguish strong and weak convergence, the notation $\to$ and $\rightharpoonup$ will be used, respectively.

*The work presented here was partially supported by the Grant Agency of the Czech Republic under the grant P201/12/0671 and by RVO 67985840.

2 The static problem without friction

A contact problem of a body whose reference configuration is given by a bounded domain $\Omega \subset \mathbb{R}^d$ with dimension $d \geq 2$ is considered. The boundary $\partial\Omega \equiv \Gamma$ of the body is Lipschitz and consists of a contact part Γ_C with a positive surface measures and a possible part Γ_N on which the boundary traction is prescribed such that the parts are disjoint. On Γ_C there is a contact with a rigid foundation, the measure of Γ_N may be even zero. By $\boldsymbol{u}$ the displacement field, by $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor with coefficients $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_{x_i} u_j + \partial_{x_j} u_i)$, and by $\boldsymbol{\sigma}$ the stress tensor are denoted. The linear constitutive law

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{1}$$

with a possibly space–dependent tensor $\mathscr{A} = (a_{ijk\ell})_{i,j,k,\ell=1}^d$ is assumed. The fourth order tensor has measurable entries, is symmetric,

$$a_{ijk\ell} = a_{k\ell ij} = a_{jik\ell}, \quad a_{ijk\ell} \in L_\infty(\Omega) \text{ for every } i,j,k,\ell \in \{1,\dots,N\}, \tag{2}$$

positively definite and bounded in the sense

$$\sum_{i,j,k,\ell=1}^{d} a_{ijk\ell}(\boldsymbol{x})\, \xi_{ij}\, \xi_{k\ell} \geq a_0 |\boldsymbol{\xi}|^2 \text{ and } \sum_{i,j,k,\ell=1}^{d} a_{ijk\ell}(\boldsymbol{x})\, \xi_{ij}\, \eta_{k\ell} \leq A_0 |\boldsymbol{\xi}||\boldsymbol{\eta}| \tag{3}$$

for every symmetric tensors $\boldsymbol{\xi} = (\xi_{ij})_{i,j=1}^d$, $\boldsymbol{\eta} = (\eta_{ij})_{i,j=1}^d$ with norm $|\boldsymbol{\xi}| = \sqrt{\sum_{i,j=1}^d |\xi_{ij}|^2}$ and constants $a_0, A_0 > 0$ independent of $\boldsymbol{x} \in \Omega$. The normal component of the boundary traction is denoted by $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the unit outward normal. The contact model shall be a normal compliance law of the type

$$\sigma_\nu(\boldsymbol{u}) = -p(u_\nu - g)$$

with some function $p : \mathbb{R} \to \overline{\mathbb{R}_+} = [0, +\infty]$. The natural assumptions for p are $p(y) = 0$ for $y \leq \alpha$ with a constant α, p is non-decreasing, $\lim_{y\to\beta-} p(y) = +\infty$ with a $\beta > \alpha$, and $p(y) = +\infty$ for $y \geq \beta$. The third requirement here means that the interpenetration in the normal compliance model is limited by β. The value of α may describe the contact of the first asperities, the value of β the total flattening of the boundary such that no further interpenetration is possible. However, we require only that

$$\begin{aligned} &p : \mathbb{R} \to \overline{\mathbb{R}_+}, \quad p_{|(-\infty,\beta)} \in AC(-\infty,\beta), \quad \lim_{y\to-\infty} p(y) = 0, \\ &\lim_{y\nearrow\beta} p(y) = +\infty, \quad p \text{ is non-decreasing and } \int_{-\infty}^{y} p(s)\, ds < +\infty, \text{ for } y < \beta. \end{aligned} \tag{4}$$

where the AC requirement signifies the absolute continuity on each closed bounded subinterval of $(-\infty, \beta)$. Even this requirement can be weakened in this section, in fact only the existence of a positive sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $\lambda_n \searrow 0$, the derivatives $p'(\beta - \lambda_n)$ exist for each $n \in \mathbb{N}$ and $p'(\beta - \lambda_n) \nearrow +\infty$ is needed. However, this is not true for the problem studied in the next section, the AC requirement is strictly needed there.

The *classical formulation* of the contact problem with this contact response function follows: *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ *such that*

$$-\operatorname{Div}(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega, \tag{5}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{b} \quad \text{on } \Gamma_N, \tag{6}$$

$$\sigma_\nu(\boldsymbol{u}) = -p(u_\nu - g), \tag{7}$$

$$\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = 0 \quad \text{on } \Gamma_C. \tag{8}$$

Here Div denotes the matrix divergence, $(\operatorname{Div} A)_i = \sum_{j=1}^{d} \partial_{x_j} a_{ij}$, and the subscript τ refers to the tangential component, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$. For the data $\boldsymbol{f}$, $\boldsymbol{b}$ and g the validity of the assumption

$$\text{(9)} \qquad \boldsymbol{f} \in \boldsymbol{H}^1(\Omega)^*, \quad \boldsymbol{b} \in \boldsymbol{H}^{1/2}(\Gamma_N)^*, \quad 0 \le g \in H^{1/2}(\Gamma_C)$$

is required. Here and in the sequel the d–dimensional vectors, vector functions $\mathbb{R}^d \to \mathbb{R}^d$ and their spaces are denoted by bold letters, i. e. for a suitable manifold M and $\gamma > 0$, $\boldsymbol{H}^\gamma(M) \equiv H^\gamma(M;\mathbb{R}^d) \equiv (H^\gamma(M))^d$ etc. We further assume that Ω is a bounded domain with a Lipschitz boundary.

To introduce the weak formulation of the classical problem stated above, the set

$$\text{(10)} \qquad \operatorname{dom} p := \{v \in H^{1/2}(\Gamma_C);\ p(v) \in H^{1/2}(\Gamma_C)^*\}$$

is needed. Observe that for each $v \in \operatorname{dom} p$, $p(v)$ belongs to $L_1(\Gamma_C)$ which is obvious from taking $\langle p(v), 1\rangle_{\Gamma_C}$ which is the $L_1(\Gamma_C)$ norm of $p(v)$.

The weak formulation of (5)–(8) is given by the following variational equation

Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $u_\nu - g \in \operatorname{dom}(p)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\text{(11)} \qquad \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_\Omega + \langle p(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} = \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle$$

Here $\boldsymbol{\ell}$ denotes the functional $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{b}, \boldsymbol{v}\rangle_{\Gamma_N}$. Let us recall the standard week formulation of the normal boundary traction

$$\text{(12)} \qquad \langle \sigma_\nu(\boldsymbol{u}), w_\nu\rangle_{\Gamma_C} = \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_\Omega - \langle \boldsymbol{\ell}, \boldsymbol{w}\rangle \ \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \text{ with } \boldsymbol{w}_t = 0 \text{ on } \Gamma.$$

As in [4] we introduce a sequence of approximate problems via an appropriate sequence of approximations of the function p. Due to the assumptions to the function p there is a sequence $\lambda_k \searrow 0$ such that the derivatives $p'(\beta - \lambda_k) \nearrow +\infty$ for $k \to +\infty$. Let us define the sequence of approximate functions

$$\text{(13)} \qquad p_k(y) = \begin{cases} p(y) & \forall y \le \beta - \lambda_k, \\ \min\{p(y), p(\beta - \lambda_k) + p'(\beta - \lambda_k)(y - \beta + \lambda_k)\} & \forall y > \beta - \lambda_k. \end{cases}$$

It obviously holds $p_k \le p_{k+1} \le p,\ k \in \mathbb{N}$ and

$$\text{(14)} \qquad p_k(s) = p_k(\beta) + p'(\beta - \lambda_k)(s - \beta) \text{ for any } s \ge \beta \text{ and any } k \in \mathbb{N}.$$

In both the classical and variational formulation of the approximate problems we simply replace the function p by the function p_k. Let us introduce

$$\text{(15)} \qquad \mathscr{J} : \boldsymbol{v} \mapsto \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega - \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle + \int_{\Gamma_C} P(v_\nu - g)\, dx_s \text{ with } P : s \mapsto \int_{-\infty}^{s} p(t)\, dt,$$

$$\text{(16)} \qquad \mathscr{J}_k : \boldsymbol{v} \mapsto \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega - \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle + \int_{\Gamma_C} P_k(v_\nu - g)\, dx_s \text{ with } P_k : s \mapsto \int_{-\infty}^{s} p_k(t)\, dt \quad \text{for all } k \in \mathbb{N}.$$

Moreover, let us introduce the finite dimensional space $\mathscr{R}$ as the set of rigid motions (the kernel of the strain tensor considered as a linear operator, cf. [3] or [5]) and the set $\mathscr{K} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); v_n \le 0 \text{ a.e. in } \Gamma_C\}$. To be able to ensure the solvability of the variational equation (11) as well as their approximations, the following requirement will be needed:

$$\text{(17)} \qquad \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle < 0 \ \forall \boldsymbol{v} \in \mathscr{K} \cap \mathscr{R} \setminus \{0\}.$$

This condition physically means that the given forces press the body towards the obstacle and it ensures the coercivity of the functionals $\mathscr{J}$, $\mathscr{J}_k$, $k \in \mathbb{N}$. Indeed the following lemma holds:

Lemma 1 *Under the above stated assumptions to Ω, its partition of the boundary, the assumptions* (1), (2), (3), (4), (9) *and the pressure condition* (17) *we have*

$$\liminf_{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\to+\infty} \frac{\mathscr{J}(\boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} > 0. \tag{18}$$

The same holds for the functional $\mathscr{J}_k$, $k \in \mathbb{N}$.

The *proof* of the assertion is made by contradiction. Since $P_k \le P_{k+1} \le P$ for any $k \in \mathbb{N}$, it is enough to prove it for $k = 1$. Without a loss of generality we may assume that $p'(\beta - \lambda_1) = 1$. Let there is a sequence $\{\boldsymbol{v}_m\} \subset \boldsymbol{H}^1(\Omega)$ with $\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \to +\infty$ such that

$$\mathscr{J}_1(\boldsymbol{v}_m)/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \to c \le 0. \tag{19}$$

Since p_1 is nonnegative on $\mathbb{R}$ and the relation (14) is true, then for each $t \ge \beta$ it holds $P_1(t) \ge |\beta - t|^2/2$. Let $\mathscr{Q}$ denote the $\boldsymbol{H}^1(\Omega)$-orthogonal complement to $\mathscr{R}$ in $\boldsymbol{H}^1(\Omega)$. Using the decomposition $\boldsymbol{v}_m = \boldsymbol{y}_m + \boldsymbol{z}_m$ with $\boldsymbol{y}_m = \pi_{\mathscr{Q}}\boldsymbol{v}_m$, $\boldsymbol{z}_m = \pi_{\mathscr{R}}\boldsymbol{v}_m$ and π_X the appropriate projections for $X = \mathscr{Q}$, $\mathscr{R}$ we get

$$\lim_{m\to+\infty} \frac{\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{y}_m), \boldsymbol{\varepsilon}(\boldsymbol{y}_m)\rangle_\Omega}{\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)}} = \lim_{m\to+\infty}\left(\frac{\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{y}_m), \boldsymbol{\varepsilon}(\boldsymbol{y}_m)\rangle_\Omega}{\|\boldsymbol{y}_m\|_{\boldsymbol{H}^1(\Omega)}} \frac{\|\boldsymbol{y}_m\|_{\boldsymbol{H}^1(\Omega)}}{\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)}}\right) \le \|\boldsymbol{\ell}\|_{\boldsymbol{H}^1(\Omega)^*}. \tag{20}$$

This yields

$$\lim_{m\to+\infty} \frac{\|\boldsymbol{y}_m\|_{\boldsymbol{H}^1(\Omega)}}{\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)}} = 0. \tag{21}$$

This is obvious if the sequence $\{\boldsymbol{y}_m\}$ is bounded. If there is a possible subsequence of $\{\boldsymbol{y}_m\}$ such that its norm tends to $+\infty$, we shall denote it by $\{\boldsymbol{y}_m\}$ again and we get (21) from (20) and the coercivity of the employed bilinear form, hence $\{\boldsymbol{z}_m\}$ is unbounded. For $\boldsymbol{s}_m = \boldsymbol{v}_m/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)}$ there is a subsequence tending weakly to some $\boldsymbol{s}_0$ in $\boldsymbol{H}^1(\Omega)$. The above used decomposition, the relation (21) and the finite dimension of $\mathscr{R}$ give that this convergence is even strong in $\boldsymbol{H}^1(\Omega)$, $\boldsymbol{s}_0 \in \mathscr{R}$ and $\|\boldsymbol{s}_0\|_{\boldsymbol{H}^1(\Omega)} = 1$. Obviously $s_{m\nu} \to s_{0\nu}$ a.e. on Γ_C. If $s_{0\nu} > 0$ on a set of a positive measure in Γ_C, then $v_{m\nu} \to +\infty$ there, $P_1(v_{m\nu} - g))/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \ge |v_{m\nu} - \beta - g|^2/(2\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)})$ for m big enough and thus $P_1(v_{m\nu} - g))/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \to +\infty$ there too which contradicts to (19). Hence $s_0 \in \mathscr{K}$, but to hold (19) it must be $\langle \boldsymbol{\ell}, \boldsymbol{s}_0\rangle \ge 0$ which contradicts to (17). □

Let us remark that the condition (17) cannot be required in case in which there is a nonzero $\boldsymbol{z} \in \mathscr{R} \cap \mathscr{K}$ such that $-\boldsymbol{z} \in \mathscr{R} \cap \mathscr{K}$. For such elements $z_\nu \equiv 0$ on Γ_C, hence their addition to the argument do not influence the value of the operator p. The set $\mathscr{R}_0 := \{\boldsymbol{z} \in \mathscr{R}; \boldsymbol{z}, -\boldsymbol{z} \in \mathscr{K}\}$ is a subspace of $\mathscr{R}$. We can modify the pressure condition (17) in such a way that

$$\langle \boldsymbol{\ell}, \mathscr{R}_0\rangle = \{0\} \;\&\; \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle > 0 \;\forall v \in \mathscr{K} \cap \mathscr{R} \cap \mathscr{R}_0^\perp \setminus \{0\}, \tag{22}$$

i.e. only such given forces which press the body towards the obstacle and simultaneously do not influence the tangentional rigid motions are admissible. It is easy to modify the proof of Lemma 1 to prove the coercivity of $\mathscr{J}$ and $\mathscr{J}_k$, $k \in \mathbb{N}$, on $\mathscr{R}_0^\perp$.

The functionals $\mathscr{J}$, $\mathscr{J}_k$, $k \in \mathbb{N}$ are obviously convex and lower semicontinuous, their coercivity yields the existence of their minimizers. It is clear that this minimizer is a weak solution of the corresponding problem because of the obvious fact that the directional derivative of the operator P, P_k at a point $v \in \operatorname{dom} p$, $v \in \operatorname{dom} p_k$ and a direction w is $\langle p(v), w\rangle_{\Gamma_C}$,

$\langle p_k(v), w\rangle_{\Gamma_C}$, respectively, (cf. [4]). Moreover, the set of minimizers of all those functionals is bounded in $\boldsymbol{H}^1(\Omega)$ due to Lemma 1. Hence, if we have a sequence $\{\boldsymbol{u}_k\}$ which are minimizers of $\mathscr{J}_k$, there must be a subsequence having an accumulation point $\boldsymbol{u}_0 \in \boldsymbol{H}^1(\Omega)$. The monotone convergence theorem yields $\mathscr{J}_k(\boldsymbol{v}) \to \mathscr{J}(\boldsymbol{v})$ for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and for any minimizer $\boldsymbol{u}$ of $\mathscr{J}$ it holds $\mathscr{J}_k(\boldsymbol{u}_k) \le \mathscr{J}_k(\boldsymbol{u})$. As $\boldsymbol{u}_k \to \boldsymbol{u}$ in $\boldsymbol{H}^{1-\delta}(\Omega)$ for any $\delta > 0$, then $u_{k\nu} \to u_{0\nu}$ almost everywhere on Γ_C. Then by Fatou lemma

$$\int_{\Gamma_C} P(u_{0\nu} - g)\, dx_s \le \liminf_{k\to+\infty} \int_{\Gamma_C} P_k(u_{k\nu} - g)\, dx_s \Rightarrow \mathscr{J}(\boldsymbol{u}_0) \le \liminf_{k\to+\infty} \mathscr{J}_k(\boldsymbol{u}_k) \tag{23}$$

where the lower semicontinuity of the appropriate quadratic form related with $\mathscr{A}$ has beeen employed in the last inequality. From (23) we have $\mathscr{J}(\boldsymbol{u}_0) \le \mathscr{J}(\boldsymbol{u})$, hence $\boldsymbol{u}_0$ is a minimizer of $\mathscr{J}$ and $\mathscr{J}(\boldsymbol{u}_0) = \mathscr{J}(\boldsymbol{u})$. Moreover,

$$\limsup_{k\to+\infty} \mathscr{J}_k(\boldsymbol{u}_k) \le \mathscr{J}(\boldsymbol{u}_0) \Rightarrow \mathscr{J}(\boldsymbol{u}_0) = \lim_{k\to+\infty} \mathscr{J}_k(\boldsymbol{u}_k). \tag{24}$$

This yields $P_k(u_{k\nu} - g) \to P(u_{0\nu} - g)$ a.e. in Γ_C and $\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}(\boldsymbol{u}_k)\rangle_\Omega \to \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle_\Omega$ hence $\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}_0)\rangle_\Omega \to 0$ from which the strong convergence of $\pi_{\mathscr{Q}}\boldsymbol{u}_k$ to $\pi_{\mathscr{Q}}\boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$ follows. Because of the finite dimension of $\mathscr{R}$ we have finally $\boldsymbol{u}_k \to \boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$.

The following theorem holds:

Theorem 2 *Under the introduced assumption on Ω, the parts of its boundary and the assumptions* (1), (2). (3), (4), (9) *and* (17) *or* (22) *there exists a weak solution to the problem* (5–8). *Two solutions may differ only by an element of $\mathscr{R} \cap \ker \boldsymbol{\ell}$ such that the operator $p(u_\nu - g)$ is the same for both solutions. If the function p happens to be strictly increasing, then the solution is unique. The same facts hold for the approximate problems defined by means of the approximate function p_k. Any sequence of the solutions of the approximate problem having a weak accumulation point in $\boldsymbol{H}^1(\Omega)$ must tend there strongly to a solution of the original problem. If* (22) *holds, the problem is solvable, too, and the sum of any solution with an arbitrary element of $\mathscr{R}_0$ is again a solution of it. To get the convergence of a sequence of solutions of the approximate problems to a solution of the original one, their projections to $\mathscr{R}_0$ must converge.*

Proof of the "uniqueness": By the standard way we can see for a couple of such solutions $\boldsymbol{u}^{(i)}$, $i = 1, 2$, that $\pi_{\mathscr{Q}}\boldsymbol{u}^{(1)} = \pi_{\mathscr{Q}}\boldsymbol{u}^{(2)}$ and moreover $\langle p(u_\nu^{(1)} - g) - p(u_\nu^{(2)} - g), u_\nu^{(1)} - u_\nu^{(2)}\rangle_{\Gamma_C} = 0$. This and the monotonicity of p yields that $u_\nu^{(1)} = u_\nu^{(2)}$ or $p(u_\nu^{(1)} - g) = p(u_\nu^{(2)} - g)$ which must be true also in the first case. If p happens to be strictly increasing, this yields $u_n^{(1)} = u_n^{(2)}$. Condition (17) does not allow nonzero elements of $\mathscr{R}$ with vanishing normal part on the contact boundary. Hence in this special case the solution must be unique. In general we know only that $0 = \langle \boldsymbol{\ell}, \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}\rangle = \langle \boldsymbol{\ell}, \pi_{\mathscr{R}}\boldsymbol{u}^{(1)} - \pi_{\mathscr{R}}\boldsymbol{u}^{(2)}\rangle$. The same argument holds also for the approximate problems due to the monotonicity of p_k.

Under the condition (22) the values of the functionals $\mathscr{J}$, $\mathscr{J}_k$, $k \in \mathbb{N}$ do not depend on the elements of $\mathscr{R}_0$, but all above stated facts are true on its orthogonal complement. □

Let us remark that a nontrivial space $\mathscr{R}_0$ occurs whenever Γ_C is a segment or a part of a (hyper)plane (in the dependence on the dimension d) or is composed of components such that all of this has such a character. Observe that for strictly increasing p the construction of p_k ensures that it is strictly increasing, too. In this case all the problems have unique solution and the strong $\boldsymbol{H}^1(\Omega)$ convergence of the whole $\{\boldsymbol{u}_k\}$ to the solution $\boldsymbol{u}$ is valid.

Let us introduce

$$\mathscr{K}_g := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); v_n \le g \text{ a.e. in } \Gamma_C\}. \tag{25}$$

In this sense, the former introduced $\mathscr{K}$ is in fact $\mathscr{K}_0$. Under the assumption of Theorem 2 the following proposition holds:

Proposition 3 *Let us assume that p is such that $p \equiv 0$ on $(-\infty, 0]$ and $p > 0$ on $(0, +\infty)$. Let us have for any such $\beta > 0$ a fixed problem (11) called $\mathscr{P}_\beta$. Let a sequence $\{\beta_m\}$ be such that $\beta_m \searrow 0$ and let $\{\boldsymbol{u}_m\} \subset \boldsymbol{H}^1(\Omega)$ be a sequence of respective solutions to the problems $\mathscr{P}_{\beta_m}$. Under the validity of (17) there exists a subsequence having a weak limit $\boldsymbol{u}_0$ in that space, $\boldsymbol{u}_0$ solves the following Signorini problem*

$$\boldsymbol{u} \in \mathscr{K}_g \text{ and } \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_\Omega \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \forall \boldsymbol{v} \in \mathscr{K}_g \tag{26}$$

and the convergence of the sequence must be even strong in $\boldsymbol{H}^1(\Omega)$. If (22) holds, we can take a fixed $\pi_{\mathscr{R}_0}(u_\beta)$, $\beta > 0$ and the assertion remains true, the limit has obviously the same projection to $\mathscr{R}_0$.

Proof. Defining $\mathscr{J}_\beta$ as in (15) for $p = p_\beta$, we have for any $\beta > 0$ and any solution of $\mathscr{P}_\beta$ that $\mathscr{J}_\beta(u_\beta) \leq \mathscr{J}_\beta(0) = 0$. Moreover, the following estimate must hold

$$\|\pi_{\mathscr{Q}} \boldsymbol{u}_\beta\|_{\boldsymbol{H}^1(\Omega)} + \|p_\beta(u_{\beta\nu} - g)\|_{H^{1/2}(\Gamma_C)^*} \leq C \tag{27}$$

with C independent of $\beta > 0$. Indeed, by putting $v = 0$ in (11) and using $p(-g) = 0$ we get from monotonicity of p_β that $\langle p_\beta(u_{\beta\nu} - g), u_{\beta\nu}\rangle_{\Gamma_C} \geq 0$. This and (3) yields the uniform estimate of the first summand. The estimate of the second one is for the solution of (11) straightforward. Under assumption (17) we can now repeat the proof of Lemma 1 and prove the uniform coercivity of $\mathscr{J}_{\beta_m}$ on $\boldsymbol{H}^1(\Omega)$ by contradiction. The sequence $\boldsymbol{v}_m$ chosen to satisfy

$$\mathscr{J}_{\beta_m}(\boldsymbol{v}_m)/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)} \to c \leq 0 \text{ as } m \to +\infty \tag{28}$$

is such that $\boldsymbol{v}_m \in \mathrm{dom}(p_{\beta_m})$, i.e. $v_{m\nu} < \beta_m + g$ a.e. on Γ_C. Then any accumulation point of the sequence $\boldsymbol{s}_m = \boldsymbol{v}_m/\|\boldsymbol{v}_m\|_{\boldsymbol{H}^1(\Omega)}$, $m \in \mathbb{N}$ must belong to $\mathscr{K}$, the convergence of the appropriate subsequence to it is strong, therefore its $\boldsymbol{H}^1(\Omega)$ norm equals 1 from the same reason as in that proof. The assumption (28) contradicts the assumption (17) for each such accumulation point and we are done. Hence the set of all solutions of problems $\mathscr{P}_{\beta_m}$, $m \in \mathbb{N}$ is bounded. The reflexivity of $\boldsymbol{H}^1(\Omega)$ yealds the existence of a subsequence having its weak limit we denote as $\boldsymbol{u}_0$. Since $u_{m\nu} < g + \beta_m$ a.e. on Γ_C, $\boldsymbol{u}_0$ must belong to $\mathscr{K}_g$. Then there is a subsequence (we denote it by β_m again) such that $p_{\beta_m}(u_{m\nu} - g)) \rightharpoonup \theta$ in $H^{1/2}(\Gamma_C)^*$ and $\langle p_{\beta_m}(u_{m\nu} - g)), u_{m\nu} - g)\rangle_{\Gamma_C} \to \Theta$ which is nonnegative because of the monotonicity of all p_{β_m} and the fact that $p_{\beta_m}(0) = 0$, $m \in \mathbb{N}$. We pass to the limit $m \to \infty$ in (11) for $\beta = \beta_m$. Using the weak lower semicontinuity of the quadratic form associated with the bilinear form $\mathscr{A}$ we find

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_0)\rangle_\Omega + \langle \theta, v_\nu - g\rangle_{\Gamma_C} - \Theta \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}_0\rangle. \tag{29}$$

For the test function $\boldsymbol{v} = \boldsymbol{u}_0$ in (11) we obtain that

$$\langle \theta, u_{0\nu} - g\rangle_{\Gamma_C} \geq \Theta. \tag{30}$$

On the other hand, for each $\boldsymbol{v} \in \mathscr{K}_g$ we have $p_{\beta_m}(v_\nu - g) = 0$ and therefore, via the monotonicity of p_{β_m},

$$\begin{aligned} 0 &\leq \lim_{m \to +\infty} \langle p_{\beta_m}(u_{m\nu} - g) - p_{\beta_m}(u_{0\nu} - g), u_{m\nu} - u_{0\nu}\rangle_{\Gamma_C} \\ &= \Theta - \langle \theta, u_{0\nu} - g\rangle_{\Gamma_C}. \end{aligned} \tag{31}$$

Since (30) and (31) yield $\Theta = \langle\theta, u_{0\nu} - g\rangle_{\Gamma_C}$ (observe that since $\theta \geq 0$ and $u_0 \leq g$, this implies $\Theta \leq 0$, therefore it is 0), we have proved that $\boldsymbol{u}_0$ is a solution of (26). Since such a subsequence can be extracted from any subsequence of the original $\{\beta_m\}$, the whole sequence must have the above qualities and we are done.

Moreover, if we take $\boldsymbol{u}_0 - \boldsymbol{u}_m$ as a test function in equation (11) for $\beta = \beta_m$, it gives us that $\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_m), \boldsymbol{\varepsilon}(\boldsymbol{u}_m - \boldsymbol{u}_0)\rangle_\Omega \to 0$. The ellipticity of $\mathscr{A}$ then imply the strong convergence of $\pi_{\mathscr{Q}}\boldsymbol{u}_m$ to $\pi_{\mathscr{Q}}\boldsymbol{u}_0$ in $\boldsymbol{H}^1(\Omega)$ while the convergence of the projections to $\mathscr{R}$ is strong due to the finite dimension of $\mathscr{R}$.

For the case (22) we can make all above estimates with the given choice of $\pi_{\mathscr{R}_0}(u_\beta)$ as well and all above taken consequences remain valid. □

3 The static problem with Coulomb friction

Here we solve the problem in which the condition (8) is replaced by

$$\begin{aligned} &\boldsymbol{u}_\tau = 0 \Rightarrow |\boldsymbol{\sigma}_\tau| \leq \mathscr{F}(\boldsymbol{u})|\sigma_\nu|, \\ &\boldsymbol{u}_\tau \neq 0 \Rightarrow \boldsymbol{\sigma}_\tau = -\mathscr{F}(\boldsymbol{u})|\sigma_\nu|\frac{\boldsymbol{u}_\tau}{|\boldsymbol{u}_\tau|} \end{aligned} \tag{32}$$

which is the classical Coulomb friction law with the (given) coefficient of friction $\mathscr{F}$. The conditions (5)–(7) remain unchanged. Due to (7) the term $|\sigma_\nu|$ in (32) can be replaced by $p(u_\nu - g)$.

The weak formulation is based on the following pointwise weak formulation of the friction law:

$$\boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) + \mathscr{F}(\boldsymbol{u})|\sigma_\nu|\big(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\big) \geq 0 \quad \text{for every } \boldsymbol{v} \in \mathbb{R}^d.$$

It is given by the variational inequality
Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ with $u_\nu - g \in \operatorname{dom}(p)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned} &\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\rangle_\Omega + \langle p(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} \\ &\qquad + \langle \mathscr{F}(\boldsymbol{u})\, p(u_\nu - g), |\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\rangle_{\Gamma_C} \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle \end{aligned} \tag{33}$$

We shall solve the problem under the introduced assumption on Ω, the parts of its boundary and the assumptions (1), (2), (3), (4), (9) and (17) or (22). Moreover, we shall require that Γ_C is of the class $C^{2+\delta}$ for some $\delta > 0$. The nonnegative coefficient of friction $\mathscr{F}$ may depend on the space variable and both on the tangential and normal component of the displacement $\boldsymbol{u}$ on Γ_C, the last one may describe its dependence on the normal indentation. Moreover, $\mathscr{F}$ satisfies the Carathéodory condition on $\Gamma_C \times \mathbb{R}^d$, $\Gamma_{\mathscr{F}} := \operatorname{cl}\{x \in \Gamma_C; \exists [y, z] \in \mathbb{R}^{d-1} \times \mathbb{R}, \mathscr{F}(x, y, z) \neq 0\} \subset \operatorname{int} \Gamma_C$ and $\|\mathscr{F}\|_{L_\infty(\Gamma_C)} \leq C_{\mathscr{F}}$ for the constant $C_{\mathscr{F}}$ precised later. These assumptions are standard and have been used for the Signorini problem just in [1] and then in [3].

However, to prove (7) after the all limit processes, the following additional assumption to the surface geometry of Γ_C is needed (cf [4], Lemma 6): Γ_C is composed from at most final amount of component with a positive mutual distance. The relative boundary of each of those components is Lipschitz.

3.1 Approximations of the frictional contact problem and regularity of their solutions

Since the first paper proving the solvability of a unilateral contact problem with Coulomb friction [6] it is well known that there is mostly necessary to approximate suitably the original problem and to prove a certain reguarity of solutions to that problem to be able to find a solution of the original problem by means of the appropriate limit process. To construct such approximate problems we simply replace the function p by the functions p_k, $k \in \mathbb{N}$, introduced in (13). However, due to the presence of the nondifferentiable friction term the variational formulations of these problems preserve still the form of variational inequalities. Therefore we regularize the norm $|\cdot|$ by a smooth convex approximation Φ_μ with the properties $|\Phi_\mu(\boldsymbol{v}) - |\boldsymbol{v}|| \leq \mu$, $\Phi_\mu(\boldsymbol{v}) \geq 0$ and $|\nabla \Phi_\mu(\boldsymbol{v})| \leq 1$ for every $\boldsymbol{v} \in \mathbb{R}^d$. This leads to the corresponding variational inequality:

Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned} &\langle \mathscr{A} \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\rangle_\Omega + \langle p_k(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} \\ &\quad + \langle \mathscr{F}(\boldsymbol{u})\, p_k(u_\nu - g), \Phi_\mu(\boldsymbol{v}_\tau) - \Phi_\mu(\boldsymbol{u}_\tau)\rangle_{\Gamma_C} \geq \langle \boldsymbol{\ell}, \boldsymbol{v}-\boldsymbol{u}\rangle \end{aligned} \tag{34}$$

Since Φ_μ is convex with continuous derivatives, this variational inequality is equivalent to the variational equation

Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned} &\langle \mathscr{A} \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega + \langle p_k(u_\nu - g), v_\nu\rangle_{\Gamma_C} \\ &\quad + \langle \mathscr{F}(\boldsymbol{u})\, p_k(u_\nu - g) \nabla \Phi_\mu(\boldsymbol{u}_\tau), \boldsymbol{v}_\tau\rangle_{\Gamma_C} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle \end{aligned} \tag{35}$$

Unlike the coercive problem solved in [4] we are not able to use Lemma 3.1.2 from [3] directly, because its proof is based on the Schauder fixed–point theorem and to this some kind of coercivity and unique solvability of a suitable approximate problem is needed. Hence we prove an analogous assertion here for this case. Let us assume that (17) is satisfied. We replace in the fricton term $\mathscr{F}(\boldsymbol{u}) p_k(u_\nu - g)$ by $\mathscr{F}(\boldsymbol{w}) p_k(w_\nu - g)$ for some given $\boldsymbol{w}$ (like it is in problems with Tresca friction, where $\mathscr{F}(w) p_k(w_\nu - g)$ is a given friction force. For such approximation of (35) the solution is found as a minimum of the continuous functional $v \mapsto \mathscr{J}_k(v) + \int_{\Gamma_C} \mathscr{F}(\boldsymbol{w}) p_k(w_\nu - g) \Phi_\mu(\boldsymbol{v}_\tau)\, dx + \eta \|\pi_{\mathscr{R}} \boldsymbol{v}\|_{\boldsymbol{L}_2(\Omega)}$ for any $\eta > 0$. Unlike the previous section, *all the projections here and in the sequel are meant as $\boldsymbol{L}_2(\Omega)$–projections.* In Lemma 1 we have proved the uniform coercivity of the convex functional $\mathscr{J}_k$, the remaining terms are non-negative and convex, too. The additional term with $\pi_{\mathscr{R}} \boldsymbol{v}$ makes the functional strictly convex on $\boldsymbol{H}^1(\Omega)$, hence there is a unique minimizer of such functional. Moreover, just from the uniform coercivity of $\mathscr{J}_k$ all minimizers of such functionals are uniformly bounded in $\boldsymbol{H}^1(\Omega)$ independently of η and $\boldsymbol{w}$. The operator $\mathscr{I}$ mapping any $\boldsymbol{w} \in \boldsymbol{L}_2(\Gamma_C)$ to the solution of this problem is well defined. Therefore, taking in mind that the trace operator acts from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{H}^{1/2}(\Gamma)$, we can use the Schauder fixed point technique to prove the existence of the following problem

Find $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned} &\langle \mathscr{A} \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega + 2\eta \langle \pi_{\mathscr{R}} u, \pi_{\mathscr{R}} v\rangle_\Omega + \langle p_k(u_\nu - g), v_\nu\rangle_{\Gamma_C} \\ &\quad + \langle \mathscr{F}(\boldsymbol{u})\, p_k(u_\nu - g) \nabla \Phi_\mu(\boldsymbol{u}_\tau), \boldsymbol{v}_\tau\rangle_{\Gamma_C} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle \end{aligned} \tag{36}$$

For any sequence $\eta_m \to 0$ we can find a subsequence such that the appropriate solutions $\boldsymbol{u}_m$ of (36) tend weakly to some limit $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$. Then the traces converge strongly in $\boldsymbol{L}_q(\Gamma)$ with $q \in [1, 2 + 2/(d-2))$ if $d \geq 3$ and $q \in [1, \infty)$ for $d = 2$. Due to the linear growth of the function p_k at $+\infty$ and the properties of coefficient of friction $\mathscr{F}$ it is easy to see that such $\boldsymbol{u}$ solves the problem (35).

The case of condition (22) is similar, because the functionals $\mathscr{J}_k$ do not depend on elements of $\mathscr{R}_0$ at all (the tensors and the normal component of their traces on Γ_C vanish). However, since these elements may influence the frictional term via the tangential displacement on the boundary, we construct in this case the *coercive approximation* of the problem in the following way: For a nonnegative parameter ζ we look for $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ such that for every $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$:

$$\begin{aligned} 2\zeta\langle \pi_{\mathscr{R}_0}\boldsymbol{u}, \boldsymbol{v}\rangle_\Omega + \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_\Omega + \langle p_k(u_\nu - g), v_\nu\rangle_{\Gamma_C} \\ + \langle \mathscr{F}(\boldsymbol{u})\, p_k(u_\nu - g)\nabla\Phi_\mu(\boldsymbol{u}_\tau), \boldsymbol{v}_\tau\rangle_{\Gamma_C} = \langle \boldsymbol{\ell}, \boldsymbol{v}\rangle. \end{aligned} \tag{37}$$

This problem is related with the functional $\boldsymbol{v} \mapsto \mathscr{J}_k(\boldsymbol{v}) + \zeta\|\pi_{\mathscr{R}_0}\boldsymbol{v}\|^2_{\boldsymbol{L}_2(Q)}$ and this is coercive on $\boldsymbol{H}^1(\Omega)$ uniformly with respect to k. Although problems described by (37) and by (36) look very similar, observe that the introduction of η was only to solve the auxiliary problem (35) while the parameter ζ will be kept to the very end of the solving of the problem in question. Then for a fixed parametrer ζ we can proceed with the same fixed-point and strict convexification idea as in the case of (17) to prove after similar considerations the solvability of (37).

This, the equivalence of the appropriate problem with the variational inequality for Φ_μ and the non-negativity of the friction term then yields

Lemma 4 *Under assumptions of Theorem 2 and the assumptions to the coefficient of friction* $\mathscr{F}$ *stated in the introductory part of this section there exists a solution of the variational equation* (35) *if* (17) *is satisfied. In the case of* (22) *there exists a solution of* (37). *In both cases the* $\boldsymbol{H}^1(\Omega)$ *norms of all solutions are bounded by a constant independent of the smoothing parameter* μ *and the approximation parameter* k.

The limit process with the parameter $\mu \searrow 0$ is, on the base of estimates of Lemma 4, similar to the process with $\eta \searrow 0$. So we can extend the validity of Lemma 4 to the problem (33) with p_k instead of p if condition (17) is satisfied. Under condition (22) the term $2\zeta\langle \pi_{\mathscr{R}_0}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_\Omega$ must be added there, hence the inequality has the form

$$\begin{aligned} 2\zeta\langle \pi_{\mathscr{R}_0}\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle_\Omega + \langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u})\rangle_\Omega + \langle p_k(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} \\ + \langle \mathscr{F}(\boldsymbol{u})\, p_k(u_\nu - g), |\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\rangle_{\Gamma_C} \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \forall v \in \boldsymbol{H}^1(\Omega). \end{aligned} \tag{38}$$

However, the passage from the problem with the approximate functions p_k to the original problem with the function p is a completely different story. While by the use of functions p_k the friction term represents a compact perturbation of the frictionless problem only, this is certainly *not true* for the problem with p. To be able to perform this limit procedure for the friction term, we need the regularity of traces of solutions of the problem with p_k, because with the a priori estimate in $\boldsymbol{H}^1(\Omega)$ and its trace consequences we have only weak convergences in its components which are not enough to justify the convergence of their product. Such proof of regularity needs some more regularity of the imput data, namely

$$a_{ijkl} \text{ are Lipschitz on } \Omega',\ g \in H^1(\Gamma_C) \text{ and } \boldsymbol{f} \in \boldsymbol{H}^{-1/2}(\Omega'), \tag{39}$$

where the open set $\Omega' \subset \Omega$ is such that $\Gamma_C \subset \partial\Omega'$. It has been performed in [4] with the help of localization, local rectification of the boundary and translation method in detail. Since in this point there is no difference between our problem solved and the coercive problem solved there, we omit here the cumbersome proof and postpone the kind reader to quite extensive appendices of [4]. We only remark that such estimates do not depend on the additional norm in the case of the validity of (22), because the depend only on the local behaviour of the first derivatives of the solution. The derivation of such regularity result requires to have $\|\mathscr{F}\|_{L_\infty(\Gamma_C\times\mathbb{R}^d)} < C_{\mathscr{F}}$ which constant is given by two special trace-type estimates valid for some localized version of the elasticity system on a half-space $Q = \mathbb{R}^{d-1} \times \mathbb{R}_+$ with the boundary $S = \mathbb{R}^{d-1} \times \{0\}$. The estimates are formulated for for the energy norm

$$\|\boldsymbol{u}\|_{\mathscr{A}} = \big(\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_Q\big)^{1/2}$$

and a suitable extension operator $\mathscr{E} : \boldsymbol{H}^{1/2}(S) \to \boldsymbol{H}^1(\Omega)$. They are given by:

$$\|\boldsymbol{u}_\tau\|^2_{\boldsymbol{H}^{1/2}(S)} \le C_0^{(1)} \|\boldsymbol{u}\|^2_{\mathscr{A}} + c_1 \|\boldsymbol{u}\|^2_{\boldsymbol{L}_2(S)} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(Q) \tag{40}$$

$$\|\mathscr{E}\boldsymbol{w}\|^2_{\mathscr{A}} \le C_0^{(2)} \|\boldsymbol{w}\|^2_{H^{1/2}(S)} + c_2 \|\boldsymbol{w}\|^2_{L_2(S)} \; \forall \boldsymbol{w} \in \boldsymbol{H}^{1/2}(S) \text{ with } \boldsymbol{w}_\tau = 0 \text{ on } S. \tag{41}$$

The upper bound for the admissible coefficient of friction is

$$C_{\mathscr{F}} = \sqrt{C_0^{(1)} C_0^{(2)}}. \tag{42}$$

Proposition 5 *Let the assumptions of Theorem 2 with the assumption* (17)*, moreover, the assumption* (39) *and the assumptions about* $\mathscr{F}$ *from the introductory part of this section be valid with the constant* $C_{\mathscr{F}}$ *defined in* (42)*. Then for any* C^2*–smooth cut-off function* ρ *with*

$$\Gamma_{\mathscr{F}} \subset \operatorname{supp} \rho \cap \partial\Omega \subset \Gamma_C$$

and every solution $\boldsymbol{u}$ *of problem* (33) *with the compliance function* p_k *it holds*

$$\|\rho \boldsymbol{u}\|_{\boldsymbol{H}^1(\Gamma_c)} + \|\rho p_k(u_\nu - g)\|_{L_2(\Gamma_c)} \le C_\rho \tag{43}$$

with a constant C_ρ *independent of* k*. For the case of* (22) *we prove the same for the variational problem* (38)*.*

Remark. The upper bound $C_{\mathscr{F}}$ for the coefficient of friction can be calculated for various cases; for isotropic material with Poisson ratio ν it takes the values

$$\mathscr{F} = \begin{cases} \dfrac{\sqrt{3-4\nu}}{2-2\nu} & \text{for } d = 2, \\ \sqrt{\dfrac{3-4\nu}{4-4\nu}} & \text{for } d \ge 3, \end{cases}$$

see [3], Formula (3.1.40). A more general formula to the first one valid for 2D orthotropic material has been derived in [2] (formula (17) there). The bound for a general material is $\sqrt{3a_0/(4A_0)}$ for any dimension.

3.2 Existence of solutions to the contact problem with friction

Let us concentrate at first to the case of the validity of (17). The estimate of Lemma 4 and the estimate (43) yield that there is a subsequence denoted again by $\{p_k\}$ such that the solutions $\boldsymbol{u}_k$ tend weakly to some $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$, $\rho\boldsymbol{u}_k \rightharpoonup \rho\boldsymbol{u}$ in $\boldsymbol{H}^1(\Gamma_C)$ and $\rho p_k(u_{k\nu} - g) \rightharpoonup \rho p_k(u_\nu - g)$ in $L_2(\Gamma_C)$. In fact, in general there is some limit $\rho\theta$ of the last sequence, but under the additional assumption to the geometry of Γ_C introduced in the introductory part of this section $\theta = p(u_\nu - g)$, cf. Lemma 6 of [4]. By the standard compact imbedding theorem

$$\rho\boldsymbol{u}_k \to \rho\boldsymbol{u} \text{ in } \boldsymbol{L}_{\hat{p}}(\Gamma_{\mathscr{F}}) \text{ for any } \hat{p} \in \left[2 + \frac{4}{d-3}\right) \text{ for any } d > 3, \; \text{for any } \hat{p} \in [2, \infty) \text{ for } d = 2, 3, \text{ and for } \hat{p} = \infty \text{ if } d = 2. \tag{44}$$

Moreover, $\mathscr{F}(\boldsymbol{u}_k) \to \mathscr{F}(\boldsymbol{u})$ in $L_{\hat{q}}(\Gamma_{\mathscr{F}})$ for any $\hat{q} \in [1, \infty)$. Assume a fixed function $\rho \in C^2(\operatorname{cl}\Omega)$ is such that $\rho(\operatorname{cl}\Omega) = [0, 1]$, $\rho_{|\Gamma_{\mathscr{F}}} \equiv 1$ and $\operatorname{dist}(\operatorname{supp}\rho, \Gamma_N) > 0$. For a general function $\boldsymbol{w} \in$

$\boldsymbol{H}^1(\Omega)$ we take the test function $\boldsymbol{v} = \boldsymbol{u}_k + \rho\boldsymbol{w}$ in (33) with the function p_k. The above mentioned convergences are sufficient to prove that (33) with the function p is satisfied for the test function $\boldsymbol{v} = \boldsymbol{u} + \rho\boldsymbol{w}$. The friction term vanishes if we take $\boldsymbol{v} = \boldsymbol{u}_k + (1-\rho)\boldsymbol{w}$ as the test function in (33) with p_k and it becomes an equation. Thanks to Lemma 6 in [4] the limit process proves the validity of (33) with p for the function $\boldsymbol{u} + (1-\rho)\boldsymbol{w}$, thus $\boldsymbol{u}$ is a weak solution of the original problem.

If (22) holds, we prove similarly the existence of solutions to the problem

$$\begin{aligned} 2\zeta\langle\pi_{\mathscr{R}_0}\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u}\rangle_\Omega + \langle\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\rangle_\Omega + \langle p(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} \\ + \langle\mathscr{F}(\boldsymbol{u})\, p(u_\nu - g), |\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\rangle_{\Gamma_C} \ge \langle\boldsymbol{\ell}, \boldsymbol{v}-\boldsymbol{u}\rangle,\ \boldsymbol{v}\in\boldsymbol{H}^1(\Omega). \end{aligned} \tag{45}$$

For the solutions of the problem with $\zeta > 0$ we have the ζ–independent $\boldsymbol{H}^1(\Omega)$–estimate of their $\pi_{\mathscr{R}_0^\perp}$ projections and also the estimate (43) does not depend on ζ. Let a sequence $\zeta_k \searrow 0$. Then we have two possibilities:

1. The sequence $\{\boldsymbol{u}_k\}$ of the corresponding solutions to (45) is bounded in $\boldsymbol{H}^1(\Omega)$. Then there is a subsequence tending to a limit $\boldsymbol{u}$ there and with the help of (44) and Lemma 6 of [4] we prove that $\boldsymbol{u}$ is a solution of (33).

2. $\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} \to +\infty$. Then there is a subsequence we shall index again by k such that $\boldsymbol{u}_k/\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} \to \boldsymbol{q}$ in $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{q}\in\mathscr{R}_0$. Taking $\boldsymbol{v} = 0$ in (45), multiplying it by $\|\boldsymbol{u}_k\|^{-1}_{\boldsymbol{H}^1(\Omega)}$ and passing to the limit $k\to+\infty$ we get

$$\lim_{k\to+\infty}\langle\mathscr{F}(\boldsymbol{u}_k)\, p(u_{k\nu} - g), |\boldsymbol{u}_{k\tau}/\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}|\rangle_{\Gamma_C} = 0 \text{ and } \lim_{k\to+\infty}\|\zeta_k\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)} = 0. \tag{46}$$

Due to the a estimate (43) we may assume $\mathscr{F}(\boldsymbol{u}_k)\, p(u_{k\nu} - g) \rightharpoonup S_0$ in $L_2(\Gamma_C)$. We prove by contradiction that $S_0 = 0$. Let $S_0 > 0$ on a set M with a positive surface measure in $\Gamma_{\mathscr{F}}$. For the elements of $\mathscr{R}_0$ normal components of their traces vanish on Γ_C and from (46) also $\boldsymbol{q}_\tau$ is zero on M. Since $\boldsymbol{q}$ is an element of $\mathscr{R}$, $\boldsymbol{q} : \boldsymbol{x}\mapsto\mathbb{B}\boldsymbol{x}+\boldsymbol{r}$, $\boldsymbol{x}\in\operatorname{cl}\Omega$, with a constant $\boldsymbol{r}\in\mathbb{R}^d$ and an antisymmetric matrix $\mathbb{B} = -\mathbb{B}^\perp\in\mathbb{R}^{d,d}$. As there must be a point $\boldsymbol{x}\in M$ and its open convex neighbourhood $\mathscr{U}$ such that $\operatorname{mes}_{d-1}(M\cap\mathscr{U}) > 0$, by taking differences $\boldsymbol{q}(\boldsymbol{y}) - \boldsymbol{q}(\boldsymbol{x})$ for $\boldsymbol{y}\in M\cap\mathscr{U}$ we deduce $\mathbb{B} = 0$ but then also $\boldsymbol{r} = 0$ which contradicts the fact that $\|\boldsymbol{q}\|_{\boldsymbol{H}^1(\Omega)} = 1$. Via the limit procedure in (38) we get then that the limit $\boldsymbol{u}$ satisfies for any $\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)$ the inequality

$$\langle\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\rangle_\Omega + \langle p(u_\nu - g), v_\nu - u_\nu\rangle_{\Gamma_C} \ge \langle\boldsymbol{\ell}, \boldsymbol{v}-\boldsymbol{u}\rangle$$

which is identical to (11). However, we need the identification $S_0 = \mathscr{F}(\boldsymbol{u})p(u_\nu - g)$ to be able to say that in this case the solution of (33) happens to have zero friction, i.e. we need $\mathscr{F}(\boldsymbol{u}_k)\to\mathscr{F}(\boldsymbol{u})$. This is obvious only if $\mathscr{F}$ does not depend on the tangential part of the solution, because its normal part does not depend on elements of $\mathscr{R}_0$.

We have proved the following theorem:

Theorem 6 *Let the assumptions on Ω, the parts of its boundary extended by the assumption to Γ_C in the introductory part of this section, the assumptions* (1), (2), (3), (4), (9), (39) *and the assumptions to the coefficient of friction $\mathscr{F}$ stated in the introductory part of this section be satisfied. Moreover, let assumption* (17) *be satisfied. Then there exists a weak solution to the problem* (5, 6, 7, 32). *If the condition* (22) *is satisfied, the same holds under the additional condition that $\mathscr{F}$ does not depend on the tangential part of the solution.*

Let us remark that the identification problem in the case of (22) does not occur at the Signorini problem with the Coulomb friction, cf. [3], page 193 including Theorem 3.2.7 there.

With the same assumption to p as in Proposition 3 we can again prove that for any sequence $\beta_k \searrow 0$ there is its subsequence tending to a solution $\boldsymbol{u} \in \mathscr{K}_g$ of the unilateral problem

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_\Omega + \langle \mathscr{F}|\sigma_\nu(u_\nu)|, |\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\rangle_{\Gamma_C} \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \forall \boldsymbol{v} \in \mathscr{K}_g. \tag{47}$$

Let us treat the assertion under the validity of (17) at first. Because of the non-negativity of the friction term, the $\boldsymbol{H}^1(\Omega)$ estimates of solutions to problems with p_{β_k} remain independent of β_k. Then the same is true for the estimates (43). The reflexivity of the employed spaces yields the existence of a subsequence, we denote it $\{\beta_k\}$ again, such that it has a weak limit $\boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ and the traces in (43) converge in their corresponding spaces. As in the proof of Proposition 3 we prove that $\boldsymbol{u} \in \mathscr{K}_g$. Denoting θ the weak limit of $p_{\beta_k}(u_{k\nu} - g)$, we get from the monotonicity of β_k that for any $\boldsymbol{v} \in \mathscr{K}_g$ for the term $d(\boldsymbol{u}, \boldsymbol{v}) \equiv \langle \theta, v_\nu - u_\nu\rangle_{\Gamma_C} \leq 0$. The use of the decomposition of the test functions with the help of the fixed cut–off function ρ, the appropriate limit process and Lemma 6 of [4] gives us the variational inequality

$$\langle \mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})\rangle_\Omega + d(\boldsymbol{u}, \boldsymbol{v}) + \langle \mathscr{F}|\sigma_\nu(u_\nu)|, |\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|\rangle_{\Gamma_C} \geq \langle \boldsymbol{\ell}, \boldsymbol{v} - \boldsymbol{u}\rangle \ \forall \boldsymbol{v} \in \mathscr{K}_g.$$

and we are done.

In the case of (22) we have again the projections of solutions to the space $\mathscr{R}_0$ unestimated. Hence we proceed as in the proof of Theorem 6. If they happen to be bounded, we can proceed as in the previous case. If they are not bounded, we get as in that proof $\mathscr{F}(\boldsymbol{u})\sigma_\nu(\boldsymbol{u}) = 0$. We have proved

Proposition 7 *Under assumption of Theorem 6 and Propositon 3 for any sequence $\beta_k \searrow 0$ there is a subsequence and subsequence of weak solutions to the corresponding normal compliance problems such that they converge to a weak solution to the Signorini problem with Coulomb friction.*

Acknowledgement. I am deeply grateful to Jana Stará for reading this manuscript and making valuable suggestions to its improvement.

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