UNIFORM CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES

Péter Kórus, Szeged

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Abstract. It is a classical problem in Fourier analysis to give conditions for a single sine or cosine series to be uniformly convergent. Several authors gave conditions for this problem supposing that the coefficients are monotone, non-negative or more recently, general monotone. There are also results for the regular convergence of double sine series to be uniform in case the coefficients are monotone or general monotone double sequences. In this paper we give new sufficient conditions for the uniformity of the regular convergence of sine-cosine and double cosine series, which are necessary as well in case the coefficients are non-negative. The new results also bring necessary and sufficient conditions for the uniform regular convergence of double trigonometric series in complex form.

Keywords: sine series, cosine series, double sine series, sine-cosine series, double cosine series, uniform convergence, regular convergence, general monotone sequence, general monotone double sequence, supremum bounded variation

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1. Background: single trigonometric series

Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of complex numbers and consider the series

$$\sum_{k=1}^{\infty} c_k \cos kx, \tag{1.1}$$

$$\sum_{k=1}^{\infty} c_k \sin kx. \tag{1.2}$$

The uniform convergence of the above series has been considered by many authors. Chaundy and Jolliffe proved the following basic theorem for sine series.

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Theorem A ([1]). *If $\{c_k\}$ is non-negative, decreasing to zero, then* (1.2) *converges uniformly in x if and only if*

$$kc_k \to 0 \quad \text{as } k \to \infty. \tag{1.3}$$

It is our main goal to relax the monotonicity condition on the coefficients. Several classes of sequences have been introduced to generalize Theorem A, for historical examples see [11]. The most recent classes are MVBVS, SBVS and SBVS_2. We define these classes in the context of β-general monotone sequences (see [2], [8]): $\{c_k\}$ is a *general monotone sequence with majorant* β, in symbols: $\{c_k\} \in \mathrm{GM}(\beta)$, if

$$\sum_{k=n}^{2n-1} |\Delta c_k| \leqslant C\beta_n$$

where $\Delta c_k = c_k - c_{k+1}$, C is a positive constant independent of n and $\beta = \{\beta_k\}_{k=1}^{\infty}$ is a positive sequence. For β-general monotone sequences, the following theorem was proved in [9].

Theorem B. *Let $\{c_k\} \in \mathrm{GM}(\beta)$. If*

$$n\beta_n \to 0 \text{ as } n \to \infty, \tag{1.4}$$

then (1.2) *is uniformly convergent in x.*

For the uniform convergence of (1.2), a necessary condition was given in case of special $\mathrm{GM}(\beta)$ classes mentioned before. We detail these classes and the adequate result.

The class MVBVS was introduced in [11] (see also [9]). We say that $\{c_k\}$ is a *mean value bounded variation sequence*, in symbols: $\{c_k\} \in \mathrm{MVBVS}$, if $\{c_k\} \in \mathrm{GM}(\beta)$ with

$$\beta_n = \frac{1}{n} \sum_{k=[n/\lambda]}^{[\lambda n]} |c_k|.$$

The definition of SBVS is the following (see [3], [4]). A $\{c_k\}$ sequence is a *supremum bounded variation sequence*, in symbols: $\{c_k\} \in \mathrm{SBVS}$, if $\{c_k\} \in \mathrm{GM}(\beta)$ with

$$\beta_n = \frac{1}{n} \sup_{m \geqslant [n/\lambda]} \sum_{k=m}^{2m} |c_k|.$$

The class SBVS_2 was defined in [4]. A $\{c_k\}$ sequence is a *supremum bounded variation sequence of second type*, in symbols: $\{c_k\} \in \mathrm{SBVS}_2$, if $\{c_k\} \in \mathrm{GM}(\beta)$ with

$$\beta_n = \frac{1}{n} \sup_{m \geqslant b(n)} \sum_{k=m}^{2m} |c_k|,$$

where $\{b(k)\}_{k=1}^{\infty}$ tends to infinity. In this case, it is easy to see that without loss of generality, we can assume that $\{b(k)\}$ is non-decreasing or $b(k) \leqslant k$. It was proved in [4] that $\mathrm{MVBVS} \subsetneq \mathrm{SBVS} \subsetneq \mathrm{SBVS}_2$. Hence SBVS_2 is the largest known class for which the Chaundy-Jolliffe theorem can be generalized appropriately, as the next theorem shows.

Theorem C. *Let* $\{c_k\} \subset \mathbb{C}$ *belong to* SBVS_2.
(i) *If* (1.3) *is satisfied, then* (1.2) *is uniformly convergent in* x.
(ii) *Conversely, if* $\{c_k\}$ *is a non-negative sequence and* (1.2) *converges uniformly in* x*, then* (1.3) *holds.*

For cosine series it is obvious that in case of non-negative $\{c_k\}$ coefficients, the uniform convergence of (1.1) is equivalent to the convergence of $\sum c_k$. If we do not require the non-negativity condition, then the situation is less obvious. The following theorem was proved in [2].

Theorem D. *Let* $\{c_k\} \in \mathrm{GM}(\beta)$. *If* (1.4) *is satisfied, then* (1.1) *is uniformly convergent in* x *if and only if*

$$\sum_{k=1}^{\infty} c_k \text{ converges.} \tag{1.5}$$

It is useful to formulate an analogous result to Theorem C for cosine series as well.

Theorem E. *Let* $\{c_k\} \subset \mathbb{C}$ *belong to* SBVS_2.
(i) *If* (1.3) *and* (1.5) *are satisfied, then* (1.1) *is uniformly convergent in* x.
(ii) *Conversely, if* $\{c_k\}$ *is non-negative and* (1.1) *converges uniformly in* x*, then* (1.3) *and* (1.5) *hold.*

Proof. Part (i): The uniform convergence of (1.1) can be deduced from Theorem D, since for any $\{c_k\} \in \mathrm{SBVS}_2$, (1.3) implies (1.4):

$$n\beta_n = \sup_{m \geqslant b(n)} \sum_{k=m}^{2m} |c_k| \leqslant \sup_{m \geqslant b(n)} \frac{1}{m} \sum_{k=m}^{2m} k|c_k| \to 0 \quad \text{as } n \to \infty.$$

For the reader's convenience, we present a short proof of Part (i) without the use of Theorem D. Let us suppose $\{c_k\}$ belongs to SBVS_2 with a constant C and $\{b(k)\}$, moreover, let (1.3) and (1.5) hold. Set $\varepsilon > 0$ arbitrarily. Then by (1.3) and (1.5) there exists $n_1 = n_1(\varepsilon)$ such that for any $n_1 \leqslant n \leqslant N$ we have

$$\left|\sum_{k=n}^{N} c_k\right| \leqslant \varepsilon, \quad n|c_n| \leqslant \varepsilon. \tag{1.6}$$

We will prove the validity of the inequality

$$|s(n,N,x)| := \left|\sum_{k=n}^{N} c_k \cos kx\right| \leqslant (8\pi C + 4\pi + 2)\varepsilon \tag{1.7}$$

for any $x \in [0,\pi]$ and $n_0 \leqslant n \leqslant N$ where n_0 is the number for which any $n \geqslant n_0$ satisfies $n \geqslant n_1$ and $b(n) \geqslant n_1$.

For $x = 0$, (1.6) immediately implies (1.7). Now let us suppose $x \in (0,\pi]$ is arbitrary and set $\nu := [1/x]$. First, for any $n_0 \leqslant n \leqslant N < \nu$ (if there are such n and N) we have

$$\begin{aligned}
|s(n,N;x)| &= \left|\sum_{k=n}^{N} c_k - \sum_{k=n}^{N} c_k(1-\cos kx)\right| \leqslant \left|\sum_{k=n}^{N} c_k\right| + \left|\sum_{k=n}^{N} 2c_k \sin^2\frac{kx}{2}\right| \\
&\leqslant \varepsilon + \sum_{k=n}^{N} 2|c_k| \sin\frac{kx}{2} \leqslant \varepsilon + x\sum_{k=n}^{\nu-1} k|c_k| \leqslant \varepsilon + \frac{1}{\nu}(\nu-1)\varepsilon \leqslant 2\varepsilon.
\end{aligned}$$

Secondly, for $\max\{n_0,\nu\} \leqslant n \leqslant N$, estimate as follows:

$$\begin{aligned}
|s(n,N;x)| &\leqslant \sum_{k=n}^{N-1} |\Delta c_k||D_k(x)| + |c_n||D_{n-1}(x)| + |c_N||D_N(x)| \\
&\leqslant \frac{\pi}{x}\left(\sum_{k=n}^{\infty} |\Delta c_k| + |c_n| + |c_N|\right) \leqslant \pi(\nu+1)\sum_{k=n}^{\infty} |\Delta c_k| + 2\pi(\nu+1)\frac{\varepsilon}{\nu},
\end{aligned}$$

where $D_k(x) = \sum\limits_{l=1}^{k} \cos lx$ is the Dirichlet kernel for which it is known that $|D_k(x)| \leqslant \pi/x$ for any $k \geqslant 1$ and $0 < x \leqslant \pi$. We have for $n_0 \leqslant n$ that

$$\begin{aligned}
\sum_{k=n}^{\infty} |\Delta c_k| &= \sum_{r=0}^{\infty} \sum_{k=2^r n}^{2^{r+1}n-1} |\Delta c_k| \leqslant C\sum_{r=0}^{\infty} \frac{1}{2^r n} \sup_{m \geqslant b(2^r n)} \sum_{k=m}^{2m} |c_k| \\
&\leqslant \frac{C\varepsilon}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} \sup_{m \geqslant b(2^r n)} \sum_{k=m}^{2m} \frac{1}{k} \leqslant \frac{2C\varepsilon}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} = \frac{4C\varepsilon}{\nu}.
\end{aligned}$$

Hence

$$|s(n,N;x)| \leqslant (8\pi C + 4\pi)\varepsilon.$$

At last, for $n_0 \leqslant n < \nu \leqslant N$, just combine the previous results:

$$|s(n,N;x)| \leqslant |s(n,\nu-1;x)| + |s(\nu,N;x)| \leqslant (8\pi C + 4\pi + 2)\varepsilon.$$

Finally, we have got (1.7) for any $x \in [0,\pi]$ and $n_0 \leqslant n \leqslant N$, which is enough for the proof due to Cauchy's criterion.

Part (ii): (1.5) comes from the convergence of (1.1) at $x = 0$. On the other hand, by [9, Lemma 2.1] we have

$$c_n \leqslant \frac{C}{n} \sup_{m \geqslant b(n)} \sum_{k=m}^{2m} c_k + \frac{1}{n} \sum_{k=n}^{2n} c_k,$$

hence

$$nc_n \leqslant C \sup_{m \geqslant b(n)} \sum_{k=m}^{2m} c_k + \sum_{k=n}^{2n} c_k.$$

This inequality together with (1.5) gives (1.3). □

We assure the reader that there exists a sequence $\{c_k\} \in \mathrm{SBVS}_2$ which satisfies (1.3) and (1.5) but $\sum c_k$ is not absolutely convergent (in the case $\sum c_k$ is absolutely convergent, (1.1) is absolutely and uniformly convergent). Consider the sequence

$$c_k := \begin{cases} (-1)^r 2^{-r}(r+1)^{-1} & \text{if } \ 2^r \leqslant k \leqslant 2^{r+1}-1, \ r = 0,1,\dots \\ 0 & \text{else.} \end{cases}$$

Then $kc_k \to 0$, $\sum c_k = \ln 2$, $\sum |c_k| = \infty$, moreover, $\{c_k\} \in \mathrm{SBVS}_2$ can be easily seen. Part (i) of Theorem E implies the uniform convergence of (1.1) with the above coefficients. Part (ii) only indicates that if a cosine series with non-negative SBVS_2 coefficients is uniformly convergent, then (1.3) and (1.5) are necessary conditions, too.

2. Double trigonometric series

Let $\{c_{jk}\}_{j,k=1}^{\infty}$ be a double sequence of complex numbers. Consider the double series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk}\cos jx\cos ky, \tag{2.1}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk}\sin jx\cos ky, \tag{2.2}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk}\sin jx\sin ky. \tag{2.3}$$

We will study the uniform convergence of the above series in (x, y) in the regular sense. First we recall that a double series $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} z_{jk}$ of complex numbers $\{z_{jk}\}_{j,k=1}^{\infty}$ converges regularly if the sums $\sum_{j=1}^{m}\sum_{k=1}^{n} z_{jk}$ converge to a finite number as m and n tend to infinity independently of each other, moreover, the row series $\sum_{j=1}^{\infty} z_{jn}$, $n = 1, 2, \dots$ and column series $\sum_{k=1}^{\infty} z_{mk}$, $m = 1, 2, \dots$ are convergent. Or equivalently (see [7]), if for any $\varepsilon > 0$ there exists a positive number $m_0 = m_0(\varepsilon)$ such that

$$\left|\sum_{j=m}^{M}\sum_{k=n}^{N} z_{jk}\right| \leqslant \varepsilon \tag{2.4}$$

holds for any m, n, M, N for which $m + n > m_0$, $1 \leqslant m \leqslant M$ and $1 \leqslant n \leqslant N$.

We will use the usual notation

$$\begin{aligned}
&\Delta_{10}c_{jk} := c_{jk} - c_{j+1,k},\ \Delta_{01}c_{jk} := c_{jk} - c_{j,k+1},\\
&\Delta_{11}c_{jk} := \Delta_{10}(\Delta_{01}c_{jk}) = \Delta_{01}(\Delta_{10}c_{jk}) = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.
\end{aligned}$$

A monotonically decreasing double sequence $\{c_{jk}\}_{j,k=1}^{\infty}$ is a sequence of real numbers for which

$$\Delta_{10}c_{jk} \geqslant 0, \quad \Delta_{01}c_{jk} \geqslant 0, \quad \Delta_{11}c_{jk} \geqslant 0, \quad j, k = 1, 2, \dots.$$

Most results are about the uniform regular convergence of the double sine series (2.3). The basic one is due to Zhak and Shneider, and is analogous to Theorem A.

Theorem F ([10]). *If $\{c_{jk}\}_{j,k=1}^{\infty}$ is a non-negative, monotonically decreasing double sequence, then* (2.3) *is uniformly regularly convergent in* (x,y) *if and only if*

$$jkc_{jk} \to 0 \quad \text{as } j+k \to \infty. \tag{2.5}$$

The most recent general monotone double sequence classes are MVBVDS, SBVDS_1 and SBVDS_2 (see [5], [6]). We remind the reader of the definitions of these classes.

Definition. A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to class MVBVDS (*mean value bounded variation double sequences*), if there exist constants C and $\lambda \geqslant 2$, depending only on $\{c_{jk}\}$, for which

$$\sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| \leqslant \frac{C}{m} \sum_{j=[m/\lambda]}^{[\lambda m]} |c_{jn}|, \quad m \geqslant \lambda,\ n \geqslant 1,$$
$$\sum_{k=n}^{2n-1} |\Delta_{01}c_{mk}| \leqslant \frac{C}{n} \sum_{k=[n/\lambda]}^{[\lambda n]} |c_{mk}|, \quad m \geqslant 1,\ n \geqslant \lambda,$$
$$\sum_{j=m}^{2m-1}\sum_{k=n}^{2n-1} |\Delta_{11}c_{jk}| \leqslant \frac{C}{mn} \sum_{j=[m/\lambda]}^{[\lambda m]}\sum_{k=[n/\lambda]}^{[\lambda n]} |c_{jk}|, \quad m,n \geqslant \lambda.$$

Definition. $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ is said to be a *supremum bounded variation double sequence of the first type*, in symbols: $\{c_{jk}\} \in \mathrm{SBVDS}_1$, if there exist constants C and integer $\lambda \geqslant 2$ and sequences $\{b_1(l)\}_{l=1}^{\infty}$, $\{b_2(l)\}_{l=1}^{\infty}$, $\{b_3(l)\}_{l=1}^{\infty}$ such that each one converges (not necessarily monotonically) to infinity, all of them depend only on $\{c_{jk}\}$, and

$$\sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| \leqslant \frac{C}{m}\Bigg(\max_{b_1(m)\leqslant M\leqslant \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}|\Bigg), \quad m \geqslant \lambda,\ n \geqslant 1, \tag{2.6}$$

$$\sum_{k=n}^{2n-1} |\Delta_{01}c_{mk}| \leqslant \frac{C}{n}\Bigg(\max_{b_2(n)\leqslant N\leqslant \lambda b_2(n)} \sum_{k=N}^{2N} |c_{mk}|\Bigg), \quad m \geqslant 1,\ n \geqslant \lambda, \tag{2.7}$$

$$\sum_{j=m}^{2m-1}\sum_{k=n}^{2n-1} |\Delta_{11}c_{jk}| \leqslant \frac{C}{mn}\Bigg(\sup_{M+N\geqslant b_3(m+n)} \sum_{j=M}^{2M}\sum_{k=N}^{2N} |c_{jk}|\Bigg), \quad m,n \geqslant \lambda. \tag{2.8}$$

Definition. $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ is said to be a *supremum bounded variation double sequence of the second type*, shortly $\{c_{jk}\} \in \mathrm{SBVDS}_2$, if there exist constants C and integer $\lambda \geqslant 1$ and $\{b(l)\}_{l=1}^{\infty}$ converging monotonically to infinity, depending only on

$\{c_{jk}\}$, such that

$$\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \leqslant \frac{C}{m}\Big(\sup_{M \geqslant b(m)} \sum_{j=M}^{2M} |c_{jn}|\Big), \quad m \geqslant \lambda,\ n \geqslant 1,$$

$$\sum_{k=n}^{2n-1} |\Delta_{01} c_{mk}| \leqslant \frac{C}{n}\Big(\sup_{N \geqslant b(n)} \sum_{k=N}^{2N} |c_{mk}|\Big), \quad m \geqslant 1,\ n \geqslant \lambda,$$

$$\sum_{j=m}^{2m-1}\sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| \leqslant \frac{C}{mn}\Big(\sup_{M+N \geqslant b(m+n)} \sum_{j=M}^{2M}\sum_{k=N}^{2N} |c_{jk}|\Big), \quad m,n \geqslant \lambda.$$

It was proved in [5] that MVBVDS $\subsetneq$ SBVDS_1 $\subsetneq$ SBVDS_2. The latest generalization of Theorem F is the summation of Theorems 1 and 2 of [5].

Theorem G.

(i) *If* $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ *belongs to the class* SBVDS_2 *and* (2.5) *holds, then the regular convergence of the double sine series* (2.3) *is uniform in* (x, y).

(ii) *Conversely, if* $\{c_{jk}\}_{j,k=1}^{\infty}$ *is non-negative and belongs to* SBVDS_1 *and the regular convergence of* (2.3) *is uniform in* (x, y), *then* (2.5) *is satisfied.*

For double cosine series with non-negative $\{c_{jk}\}$ coefficients, the regular convergence of (2.1) is uniform in (x, y) if and only if $\sum\sum c_{jk}$ is regularly convergent. In the next section we will give conditions for the sine-cosine and double cosine series with general monotone coefficients to be uniformly convergent in the regular sense.

3. Main results on double series

We prove three results for double trigonometric series with coefficients from the class SBVDS_1 (or one of its subclasses). The conditions we give for uniform convergence are sufficient for coefficients of complex numbers and are necessary for non-negative coefficients. However, all our attempts have failed so far to modify the proofs of the sufficiency parts of the theorems to extend the sufficiency results for SBVDS_2, unlike the case of double sine series.

Theorem 1. *Suppose that* $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ *belongs to the class* SBVDS_1.

(i) *If* (2.5) *holds and there exists an* $m_1 \geqslant 1$ *such that*

$$\max_{m<m_1} \sum_{k=n}^{\infty} c_{mk} \to 0 \quad \text{and} \quad \sup_{m \geqslant m_1} \sum_{k=n}^{\infty} m|c_{mk}| \to 0 \quad \text{as } n \to \infty, \tag{3.1}$$

then the regular convergence of the sine-cosine series (2.2) *is uniform in* (x, y).

(ii) *Conversely, if* $\{c_{jk}\}_{j,k=1}^{\infty}$ *is non-negative and the regular convergence of* (2.2) *is uniform in* (x,y)*, then* (2.5) *holds and* (3.1) *is satisfied for any* m_1.

Theorem 2. *Suppose that* $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ *belongs to the class* SBVDS_1.

(i) *If* (2.5) *holds,*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} \text{ converges regularly} \tag{3.2}$$

and there exists an $m_1 \geqslant 1$ *such that*

$$\sup_{m \geqslant m_1} \sum_{k=n}^{\infty} m|c_{mk}| \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \sup_{n \geqslant m_1} \sum_{j=m}^{\infty} n|c_{jn}| \to 0 \quad \text{as } m \to \infty, \tag{3.3}$$

then the regular convergence of the double cosine series (2.1) *is uniform in* (x,y).

(ii) *Conversely, if* $\{c_{jk}\}_{j,k=1}^{\infty}$ *is non-negative and the regular convergence of* (2.1) *is uniform in* (x,y)*, then* (2.5), (3.2) *hold and* (3.3) *is satisfied for any* m_1.

Corollary 1. *Suppose that* $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ *belongs to the class* SBVDS_1.

(i) *If* (2.5), (3.2) *hold and* (3.3) *is satisfied for an* m_1*, then the regular convergence of*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk} \mathrm{e}^{\mathrm{i}jx} \mathrm{e}^{\mathrm{i}ky} \tag{3.4}$$

is uniform in (x,y).

(ii) *Conversely, if* $\{c_{jk}\}_{j,k=1}^{\infty}$ *is non-negative and the regular convergence of* (3.4) *is uniform in* (x,y)*, then* (2.5), (3.2) *hold and* (3.3) *is satisfied for any* m_1.

To show the usability of our results, we will give an example for a double sequence $\{c_{jk}\} \in \mathrm{SBVDS}_1$ which satisfies (2.5), (3.2) and (3.3) but $\sum\sum |c_{jk}| = \infty$. It can be seen that

$$c_{jk} := \begin{cases} (-1)^r 2^{-2r}(r+1)^{-1} & \text{if} \quad 2^r \leqslant j,k \leqslant 2^{r+1}-1, \ r = 0,1,\dots \\ 0 & \text{else} \end{cases}$$

is such a sequence. We can apply part (i) of Theorem 1, Theorem 2 or Corollary 1, hence we have the uniform convergence of (2.1), (2.2) and (3.4) with the above coefficients. Part (ii) only shows the necessity of the sufficient conditions in the special case when the coefficients are non-negative and from SBVDS_1.

4. Auxiliary results

To prove Theorems 1–2 and Corollary 1 we need three lemmas, which are used in the investigations of the regular convergence of double sine series as well. The proofs of these assertions can be found in [5].

Lemma 1 ([5, Lemma 1]). *If $\{c_{jk}\} \subset \mathbb{C}$ is such that conditions* (2.5) *and* (2.8) *are satisfied, then*

$$mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} c_{jk}| \to 0 \quad \text{as } m+n \to \infty \text{ and } m,n \geqslant \lambda.$$

Lemma 2 ([5, Lemma 2]). *Under the conditions of Lemma* 1, *we have*

$$mn \sum_{j=m}^{\infty} \sup_{k\geqslant n} |\Delta_{10} c_{jk}| \to 0, \quad mn \sum_{k=n}^{\infty} \sup_{j\geqslant m} |\Delta_{01} c_{jk}| \to 0$$

as $m+n \to \infty$ and $m,n \geqslant \lambda$.

Lemma 3 ([5, Lemma 3]). *If $\{c_{jk}\}$ is non-negative and belongs to the class* SBVDS_1 *with C, λ and $\{b_1(l)\}$, $\{b_2(l)\}$, $\{b_3(l)\}$ then for any $m,n \geqslant \lambda$ we have*

$$\begin{aligned} mnc_{mn} \leqslant C \sup_{M+N\geqslant b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} + C \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n} c_{jk} \\ + C \sum_{j=m}^{2m} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 2 \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{jk}. \end{aligned} \tag{4.1}$$

5. Proofs of the main results

P r o o f of Theorem 1. Let $\{c_{jk}\} \subset \mathbb{C}$ belong to the class SBVDS_1 with C, λ and $\{b_1(l)\}$, $\{b_2(l)\}$, $\{b_3(l)\}$.

Part (i): Our proof is analogous to the proof of the first part of [6, Theorem 1]. First, we can see that the single series

$$\sum_{j=1}^{\infty} c_{jn} \sin jx, \quad n = 1, 2, \ldots, \qquad \sum_{k=1}^{\infty} c_{mk} \cos ky, \quad m = 1, 2, \ldots \tag{5.1}$$

are uniformly convergent in (x,y) in consequence of Theorems C and E which can be applied since $\{c_{jn}\}_{j=1}^{\infty} \in \mathrm{SBVS}_2$ for any n, $\{c_{mk}\}_{k=1}^{\infty} \in \mathrm{SBVS}_2$ for any m and (2.5), (3.1) are satisfied. Secondly, let $\varepsilon > 0$ be arbitrarily fixed. We will prove that for any $M \geqslant m > m_0$, $N \geqslant n > m_0$ and any (x,y) we have

$$|s(m,M;n,N;x,y)| \leqslant (9\pi^2 + 8\pi + 4\pi\lambda C + 2)\varepsilon \tag{5.2}$$

where

$$s(m,M;n,N;x,y) := \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky$$

and $m_0 = m_0(\varepsilon) > \max\{m_1, \lambda\}$ is the natural number which satisfies for any $m, n > m_0$,

$$mn|c_{mn}| \leqslant \varepsilon, \quad mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} c_{jk}| \leqslant \varepsilon, \tag{5.3}$$
$$mn \sum_{j=m}^{\infty} \sup_{k \geqslant n} |\Delta_{10} c_{jk}| \leqslant \varepsilon, \quad mn \sum_{k=n}^{\infty} \sup_{j \geqslant m} |\Delta_{01} c_{jk}| \leqslant \varepsilon$$

for any $m \geqslant m_1$,

$$\sum_{k=n}^{\infty} m|c_{mk}| \leqslant \varepsilon, \tag{5.4}$$

and, in addition, for any $m > m_0$, $b_1(m) \geqslant m_1$ is satisfied. The existence of m_0 is justified by (2.5), (3.1), Lemmas 1–2 and the fact that $b_1(l)$ tends to infinity. fact that if a series is convergent, then the arithmetic mean of it also does converge to the same limit.

For $x = 0$ and arbitrary y, (5.2) is trivial. For $y = 0$ and $0 < x \leqslant \pi$, set $\mu = \mu(x) := [1/x]$, where $[\,\cdot\,]$ means the integer part of a real number. Then for any $m_0 < m \leqslant M \leqslant \mu$ and $m_0 < n \leqslant N$, by (5.4) we have

$$|s(m,M;n,N;x,0)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \right| \leqslant x \sum_{j=m}^{M} \sum_{k=n}^{N} j|c_{jk}| \leqslant \frac{1}{\mu} \sum_{j=m}^{\mu} \varepsilon \leqslant \varepsilon$$

and for any $\max\{m_0, \mu\} < m \leqslant M$ and $m_0 < n \leqslant N$, by (2.6) and (5.4),

$$
\begin{aligned}
|s(m, M; n, N; x, 0)| &\leqslant \sum_{k=n}^{N} \Bigl(\sum_{j=m}^{M-1} |\Delta_{10} c_{jk}| |\widetilde{D}_j(x)| + |c_{Mk}| |\widetilde{D}_M(x)| + |c_{mk}| |\widetilde{D}_{m-1}(x)| \Bigr) \\
&\leqslant \frac{\pi}{x} \sum_{k=n}^{\infty} \Bigl(\sum_{j=m}^{\infty} |\Delta_{10} c_{jk}| + |c_{Mk}| + |c_{mk}| \Bigr) \\
&\leqslant \pi \sum_{k=n}^{\infty} \Bigl(m \sum_{r=0}^{\infty} \sum_{j=2^r m}^{2^{r+1}m-1} |\Delta_{10} c_{jk}| + M|c_{Mk}| + m|c_{mk}| \Bigr) \\
&\leqslant 2\pi\varepsilon + \pi \sum_{k=n}^{\infty} \Bigl(m \sum_{r=0}^{\infty} \frac{C}{2^r m} \max_{b_1(m) \leqslant m' \leqslant \lambda b_1(m)} \sum_{j=m'}^{2m'} |c_{jk}| \Bigr) \\
&\leqslant 2\pi\varepsilon + \pi C \sum_{k=n}^{\infty} \sum_{r=0}^{\infty} \frac{1}{2^r} \sum_{j=b_1(m)}^{2\lambda b_1(m)} |c_{jk}| \leqslant 2\pi\varepsilon + 2\pi C \sum_{k=n}^{\infty} \sum_{j=b_1(m)}^{2\lambda b_1(m)} |c_{jk}| \\
&\leqslant 2\pi\varepsilon + 4\pi\lambda C \frac{1}{2\lambda b_1(m)} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{\infty} j|c_{jk}| \leqslant 2\pi\varepsilon + 4\pi\lambda C\varepsilon
\end{aligned}
$$

where $\widetilde{D}_j(x) = \sum\limits_{l=1}^{j} \sin lx$ is the conjugate Dirichlet kernel, for which it is known that $|\widetilde{D}_j(x)| \leqslant \pi/x$ for any $j \geqslant 1$ and $0 < x \leqslant \pi$. If we combine the above estimates, we get (5.2) for $y = 0$. To consider the remaining case when $0 < x, y \leqslant \pi$, set

$$
\mu = \mu(x) := \Bigl[\frac{1}{x}\Bigr], \quad \nu = \nu(y) := \Bigl[\frac{1}{y}\Bigr].
$$

We investigate the four basic cases as in the proof of [6, Theorem 1] but for the double series (2.2). We will use in every case the validity of the four inequalities in (5.3) to estimate.

Case (a): $m_0 < m \leqslant M \leqslant \mu$ and $m_0 < n \leqslant N \leqslant \nu$.

$$
\begin{aligned}
|s(m, M; n, N; x, y)| &\leqslant |s(m, M; n, N; x, 0)| + \Bigl| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \Bigr| \\
&\leqslant \varepsilon + \sum_{j=m}^{M} \sum_{k=n}^{N} 2|c_{jk}| \sin jx \sin \frac{ky}{2} \leqslant \varepsilon + xy \sum_{j=m}^{M} \sum_{k=n}^{N} jk|c_{jk}| \\
&\leqslant \varepsilon + \frac{1}{\mu\nu} \sum_{j=m}^{\mu} \sum_{k=n}^{\nu} jk|c_{jk}| \leqslant 2\varepsilon.
\end{aligned}
$$

Case (b): $\max\{m_0, \mu\} < m \leqslant M$ and $m_0 < n \leqslant N \leqslant \nu$.

$$|s(m,M;n,N;x,y)| \leqslant |s(m,M;n,N;x,0)| + \sum_{k=n}^{N} 2\sin^2\frac{ky}{2}\Big|\sum_{j=m}^{M} c_{jk}\sin jx\Big|$$

$$\leqslant 2\pi\varepsilon + 4\pi\lambda C\varepsilon + y\sum_{k=n}^{N} k\Big(\sum_{j=m}^{M-1}|\Delta_{10}c_{jk}||\widetilde{D}_j(x)| + |c_{Mk}||\widetilde{D}_M(x)| + |c_{mk}||\widetilde{D}_{m-1}(x)|\Big)$$

$$< 2\pi\varepsilon + 4\pi\lambda C\varepsilon + \frac{\pi}{x\nu}\sum_{k=n}^{\nu} k\sum_{j=m}^{M-1}|\Delta_{10}c_{jk}| + 2\sup_{j\geqslant m} k|c_{jk}|$$

$$\leqslant 2\pi\varepsilon + 4\pi\lambda C\varepsilon + \frac{\pi}{\nu}\sum_{k=n}^{\nu} km\sum_{j=m}^{\infty}|\Delta_{10}c_{jk}| + 2\sup_{j\geqslant m} jk|c_{jk}|$$

$$\leqslant 5\pi\varepsilon + 4\pi\lambda C\varepsilon.$$

Case (c): $m_0 < m \leqslant M \leqslant \mu$ and $\max\{m_0, \nu\} < n \leqslant N$.

$$|s(m,M;n,N;x,y)| \leqslant \sum_{j=m}^{M}\sin jx\Big|\sum_{k=n}^{N} c_{jk}\cos ky\Big| \leqslant x\sum_{j=m}^{M} j\Big|\sum_{k=n}^{N} c_{jk}\cos ky\Big|$$

$$\leqslant \frac{1}{\mu}\sum_{j=m}^{\mu} j\Big(\sum_{k=n}^{N-1}|\Delta_{01}c_{jk}||D_k(y)| + |c_{jN}||D_N(y)| + |c_{jn}||D_{n-1}(y)|\Big)$$

$$\leqslant \frac{\pi}{y\mu}\sum_{j=m}^{\mu} j\sum_{k=n}^{N-1}|\Delta_{01}c_{jk}| + 2\sup_{k\geqslant n} j|c_{jk}|$$

$$\leqslant \frac{\pi}{\mu}\sum_{j=m}^{\mu} jn\sum_{k=n}^{\infty}|\Delta_{01}c_{jk}| + 2\sup_{k\geqslant n} jk|c_{jk}| \leqslant 3\pi\varepsilon.$$

is known that $|D_k(x)| \leqslant \pi/y$ for any $k \geqslant 1$ and $0 < y \leqslant \pi$.

Case (d): $\max\{m_0, \mu\} < m \leqslant M$ and $\max\{m_0, \nu\} < n \leqslant N$. This time, similarly to the proof of [6, Theorem 1]—except that we replace each $\widetilde{D}_l(y)$ by $D_l(y)$—double summation by parts gives us

$$|s(m,M;n,N;x,y)| \leqslant \pi^2\Big(mn\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}|\Delta_{11}c_{jk}| + 2mn\sum_{j=m}^{\infty}\sup_{k\geqslant n}|\Delta_{10}c_{jk}|$$
$$+ 2mn\sum_{k=n}^{\infty}\sup_{j\geqslant m}|\Delta_{01}c_{jk}| + 4\sup_{j\geqslant m, k\geqslant n} jk|c_{jk}|\Big) \leqslant 9\pi^2\varepsilon.$$

If we summarize Cases (a)–(d), we get (5.2). To complete the proof of part (i) just consider the end of the proof of [6, Theorem 1].

Part (ii): Suppose that $\{c_{jk}\}$ is non-negative and let $\varepsilon > 0$ be arbitrarily fixed. Using the form (2.4) for the uniform regular convergence of (2.2), we find that there exists an integer $m_0 = m_0(\varepsilon)$ for which

$$|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \leqslant \varepsilon \tag{5.5}$$

holds for any $m + n > m_0$ and any (x, y). Since $\{b_i(l)\}_{l=1}^{\infty}$ converges to infinity for $i = 1, 2, 3$, there exists an m_1 such that for any m, n: $m + n > m_1$ implies $m + n > m_0$, $b_3(m + n) > m_0$, $b_1(m) + n > m_0$ and $m + b_2(n) > m_0$. Set

$$x_1(m) = \frac{\pi}{4m}, \quad x_2(m) = \frac{\pi}{4\lambda m}.$$

Then

$$\begin{aligned} \sin(jx_1(m)) &\geqslant \sin \frac{\pi}{4} \qquad \text{if } m \leqslant j \leqslant 2m; \\ \sin(jx_2(m)) &\geqslant \sin \frac{\pi}{4\lambda} \qquad \text{if } m \leqslant j \leqslant 2\lambda m. \end{aligned}$$

By (2.6), we have for any $m \geqslant \lambda$ and $k \geqslant 1$ that

$$mc_{mk} \leqslant \sum_{j=m+1}^{2m} \left(c_{jk} + \sum_{l=m}^{j-1} |\Delta_{10} c_{lk}| \right) \leqslant \sum_{j=m}^{2m} c_{jk} + C \max_{b_1(m) \leqslant M \leqslant \lambda b_1(m)} \sum_{j=M}^{2M} c_{jk}. \tag{5.6}$$

Hence, by (5.5), for any $m \geqslant \lambda$ and $n > m_1$,

$$\begin{aligned} \sum_{k=n}^{N} mc_{mk} &\leqslant \sum_{j=m}^{2m} \sum_{k=n}^{N} c_{jk} + C \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{N} c_{jk} \\ &\leqslant \sin^{-1} \frac{\pi}{4} |s(m, 2m; n, N; x_1, 0)| \\ &\quad + C \sin^{-1} \frac{\pi}{4\lambda} |s(b_1(m), 2\lambda b_1(m); n, N; x_2, 0)| \\ &\leqslant \left(\sin^{-1} \frac{\pi}{4} + C \sin^{-1} \frac{\pi}{4\lambda} \right) \varepsilon, \end{aligned}$$

The previous inequality and the uniform convergence of the first $[\lambda]$ cosine series in (5.1) together with Theorem E imply (3.1) for any fixed $m_1 \geqslant \lambda$. It is easy to verify that (3.1) holds for $m_1 < \lambda$ as well since $\{c_{jk}\}$ is non-negative and λ is finite.

Moreover, by Lemma 3, we conclude that for $m, n \geqslant \lambda$, $m + n > m_1$,

$$\begin{aligned}
mnc_{mn} &\leqslant C \sup_{M+N \geqslant b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} + C \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n} c_{jk} \\
&\quad + C \sum_{j=m}^{2m} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 2 \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{jk} \\
&\leqslant C \sin^{-1} \frac{\pi}{4} \sup_{M+N \geqslant b_3(m+n)} |s(M, 2M; N, 2N; x_1, 0)| \\
&\quad + C \sin^{-1} \frac{\pi}{4\lambda} |s(b_1(m), 2\lambda b_1(m); n, 2n; x_2, 0)| \\
&\quad + C \sin^{-1} \frac{\pi}{4} |s(m, 2m; b_2(n), 2\lambda b_2(n); x_1, 0)| \\
&\quad + 2 \sin^{-1} \frac{\pi}{4} |s(m, 2m; n, 2n; x_1, 0)| \\
&\leqslant \Bigl((2C + 2) \sin^{-1} \frac{\pi}{4} + C \sin^{-1} \frac{\pi}{4\lambda}\Bigr)\varepsilon.
\end{aligned}$$

Hence (2.5) is satisfied when $j + k \to \infty$ and $j, k \geqslant \lambda$. If $j \to \infty$ and $k < \lambda$ or $j < \lambda$ and $k \to \infty$, (2.5) follows from the uniform convergence of the series in (5.1) and Theorems C and E. $\square$

P r o o f of Theorem 2. Let $\{c_{jk}\} \subset \mathbb{C}$ belong to the class SBVDS_1 with C, λ and $\{b_1(l)\}$, $\{b_2(l)\}$, $\{b_3(l)\}$.

Part (i): Our proof is analogous to the proof of Theorem 1 and [6, Theorem 1]. We can see that the single series

$$\sum_{j=1}^{\infty} c_{jn} \cos jx, \quad n = 1, 2, \ldots, \qquad \sum_{k=1}^{\infty} c_{mk} \cos ky, \quad m = 1, 2, \ldots \tag{5.7}$$

are uniformly convergent in (x, y) in consequence of Theorem E, since (2.5), (3.2) and (3.3) are satisfied. Now let $\varepsilon > 0$ be arbitrarily fixed. We will prove that

$$|s(m, M; n, N; x, y)| \leqslant (9\pi^2 + 10\pi + 8\pi\lambda C + 4)\varepsilon \tag{5.8}$$

holds for any $M \geqslant m > m_0$, $N \geqslant n > m_0$ and any (x, y), where

$$s(m, M; n, N; x, y) := \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \cos jx \cos ky$$

and $m_0 > \max\{m_1, \lambda\}$ is the natural number which satisfies the four inequalities in (5.3), (5.4),

$$\Bigl|\sum_{j=m}^{\infty}\sum_{k=n}^{\infty} jkc_{jk}\Bigr| \leqslant \varepsilon \quad \text{for any } m, n > m_0, \tag{5.9}$$

$$\sum_{j=m}^{\infty} n|c_{jn}| \leqslant \varepsilon \quad \text{for any } n \geqslant m_1, \tag{5.10}$$

and furthermore, $b_i(l) \geqslant m_1$ holds for any $l > m_0$ and $i = 1, 2$. The existence of m_0 is guaranteed by (2.5), (3.2), (3.3), Lemmas 1–2 and the fact that $b_1(l)$, $b_2(l)$ tend to infinity.

For $x = y = 0$, (5.9) immediately implies (5.8). For $y = 0$ and $0 < x \leqslant \pi$, set $\mu = \mu(x) := [1/x]$. Then for any $m_0 < m \leqslant M \leqslant \mu$ and $m_0 < n \leqslant N$, by (5.4),

$$\begin{aligned}
|s(m, M; n, N; x, 0)| &= \Bigl|\sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk}\cos jx\Bigr| \\
&\leqslant |s(m, M; n, N; 0, 0)| + \Bigl|\sum_{j=m}^{M}\sum_{k=n}^{N} 2c_{jk}\sin^2\frac{jx}{2}\Bigr| \\
&\leqslant \varepsilon + x\sum_{j=m}^{M}\sum_{k=n}^{N} j|c_{jk}| \leqslant \varepsilon + \frac{1}{\mu}\sum_{j=m}^{\mu}\varepsilon \leqslant 2\varepsilon
\end{aligned}$$

and for any $\max\{m_0, \mu\} < m \leqslant M$ and $m_0 < n \leqslant N$, after a summation by parts we get

$$|s(m, M; n, N; x, 0)| \leqslant \sum_{k=n}^{N}\Bigl(\sum_{j=m}^{M-1}|\Delta_{10}c_{jk}||D_j(x)| + |c_{Mk}||D_M(x)| + |c_{mk}||D_{m-1}(x)|\Bigr).$$

In an identical way as in the appropriate case of the proof of the previous theorem, using the inequalities (2.6) and (5.4) we obtain

$$|s(m, M; n, N; x, 0)| \leqslant 2\pi\varepsilon + 4\pi\lambda C\varepsilon.$$

Combining the above estimates we get (5.8) for $y = 0$. For $x = 0$ and arbitrary y an analogous argumentation gives

$$|s(m, M; n, N; 0, y)| \leqslant 2\varepsilon + 2\pi\varepsilon + 4\pi\lambda C\varepsilon;$$

in this case (2.7) is applied instead of (2.6) and (5.10) instead of (5.4). To consider the remaining case when $0 < x, y \leqslant \pi$, set $\mu := [1/x]$ and $\nu := [1/y]$. The four cases we need to investigate are the following.

C a s e (a): $m_0 < m \leqslant M \leqslant \mu$ and $m_0 < n \leqslant N \leqslant \nu$.

$$\begin{aligned}
|s(m,M;n,N;x,y)| &\leqslant |s(m,M;n,N;0,0)| + \Bigl|\sum_{j=m}^{M}\sum_{k=n}^{N} 2c_{jk}\sin^2\frac{jx}{2}\Bigr| \\
&\quad + \Bigl|\sum_{j=m}^{M}\sum_{k=n}^{N} 2c_{jk}\sin^2\frac{ky}{2}\Bigr| + \Bigl|\sum_{j=m}^{M}\sum_{k=n}^{N} 4c_{jk}\sin^2\frac{jx}{2}\sin^2\frac{ky}{2}\Bigr| \\
&\leqslant \varepsilon + x\sum_{j=m}^{M}\sum_{k=n}^{N} j|c_{jk}| + y\sum_{k=n}^{N}\sum_{j=m}^{M} k|c_{jk}| + xy\sum_{j=m}^{M}\sum_{k=n}^{N} jk|c_{jk}| \\
&\leqslant \varepsilon + \frac{1}{\mu}\sum_{j=m}^{\mu}\varepsilon + \frac{1}{\nu}\sum_{k=n}^{\nu}\varepsilon + \frac{1}{\mu\nu}\sum_{j=m}^{\mu}\sum_{k=n}^{\nu}\varepsilon \leqslant 4\varepsilon.
\end{aligned}$$

C a s e (b): $\max\{m_0,\mu\} < m \leqslant M$ and $m_0 < n \leqslant N \leqslant \nu$.

$$|s(m,M;n,N;x,y)| \leqslant |s(m,M;n,N;x,0)| + \sum_{k=n}^{N} 2\sin^2\frac{ky}{2}\Bigl|\sum_{j=m}^{M} c_{jk}\cos jx\Bigr|.$$

Repeating Case (b) of the proof of Theorem 1 except for replacing each conjugate Dirichlet kernel by the appropriate Dirichlet kernel, we get

$$|s(m,M;n,N;x,y)| \leqslant 5\pi\varepsilon + 4\pi\lambda C\varepsilon.$$

C a s e (c): $m_0 < m \leqslant M \leqslant \mu$ and $\max\{m_0,\nu\} < n \leqslant N$. This is the symmetric counterpart of Case (b), hence

$$|s(m,M;n,N;x,y)| \leqslant 5\pi\varepsilon + 4\pi\lambda C\varepsilon.$$

C a s e (d): $\max\{m_0,\mu\} < m \leqslant M$ and $\max\{m_0,\nu\} < n \leqslant N$. From an argumentation analogous to Case (d) of the proof of [6, Theorem 1] (except for replacing the conjugate Dirichlet kernels by Dirichlet kernels) we obtain

$$|s(m,M;n,N;x,y)| \leqslant 9\pi^2\varepsilon.$$

If we summarize Cases (a)–(d), we get (5.8) and the end of the proof of [6, Theorem 1] completes the proof of this theorem's part (i).

Part (ii): Suppose $\{c_{jk}\}$ is non-negative and let $\varepsilon > 0$ be arbitrarily fixed. From the regular convergence of (2.1) in $(x,y) = (0,0)$ we have that

$$|s(m,M;n,N;0,0)| = \sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk} \to 0 \quad \text{as } m+n \to \infty. \tag{5.11}$$

This implies (3.2). Hence by (5.6), it is clear that

$$\sup_{m\geqslant\lambda}\sum_{k=n}^{N} mc_{mk} \leqslant \sup_{m\geqslant\lambda}\left(\sum_{j=m}^{2m}\sum_{k=n}^{N} c_{jk} + C\sum_{j=b_1(m)}^{2\lambda b_1(m)}\sum_{k=n}^{N} c_{jk}\right) \to 0 \quad \text{as } n\to\infty;$$

similarly,

$$\sup_{n\geqslant\lambda}\sum_{j=m}^{M} nc_{jn} \leqslant \sup_{n\geqslant\lambda}\left(\sum_{j=m}^{M}\sum_{k=n}^{2n} c_{jk} + C\sum_{j=m}^{M}\sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk}\right) \to 0 \quad \text{as } m\to\infty,$$

hence (3.3) is satisfied for $m_1 \geqslant \lambda$. It can be easily seen that (3.3) holds for $m_1 < \lambda$ as well since $\{c_{jk}\}$ is non-negative, λ is finite and (3.2) holds. At last, by Lemma 3, the right hand side of (4.1) converges to zero as $m+n$ tends to infinity and $m, n \geqslant \lambda$. This implies that (2.5) is satisfied when $j+k\to\infty$ and $j, k \geqslant \lambda$. In the remaining cases (2.5) can be obtained from the uniform convergence of the series in (5.7) and Theorem E. □

P r o o f of Corollary 1. Part (i): Obviously,

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk}\mathrm{e}^{\mathrm{i}jx}\mathrm{e}^{\mathrm{i}ky} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{jk}(&\cos jx\cos ky - \sin jx \sin ky\\ &+ \mathrm{i}(\sin jx\cos ky + \cos jx\sin ky))\end{aligned}$$

and we can apply Theorem G and Theorems 1–2.

Part (ii): In the proof of part (ii) of Theorem 2 we only used (5.11), the uniform convergence of the row and column series in (5.7) and that $\{c_{jk}\}$ is non-negative and belongs to SBVDS_1. In this case, the regular convergence of (3.4) in $(x, y) = (0, 0)$ gives us (5.11):

$$\left|\sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk}\mathrm{e}^{\mathrm{i}j0}\mathrm{e}^{\mathrm{i}k0}\right| = \left|\sum_{j=m}^{M}\sum_{k=n}^{N} c_{jk}\right| \to 0 \quad \text{as } m+n\to\infty.$$

Furthermore, the uniform convergence of the series in (5.7) follows from the uniform convergence of the row and column series of (3.4), since

$$\Re\left\{\sum_{j=1}^{\infty} c_{jn}\mathrm{e}^{\mathrm{i}jx}\right\} = \sum_{j=1}^{\infty} c_{jn}\cos jx, \quad \Re\left\{\sum_{k=1}^{\infty} c_{mk}\mathrm{e}^{\mathrm{i}ky}\right\} = \sum_{k=1}^{\infty} c_{mk}\cos ky.$$

Hence we can just repeat the proof of part (ii) of Theorem 2. □

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Author's address: *Péter Kórus*, Bolyai Institute, University of Szeged, H-6720 Szeged, Aradi vértanúk tere 1, Hungary, e-mail: `korpet@math.u-szeged.hu`.