ON THE CLASS OF ORDER DUNFORD-PETTIS OPERATORS

Khalid Bouras, Larache, Abdelmonaim El Kaddouri, Jawad H'michane, Mohammed Moussa, Kénitra

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Abstract. We characterize Banach lattices E and F on which the adjoint of each operator from E into F which is order Dunford-Pettis and weak Dunford-Pettis, is Dunford-Pettis. More precisely, we show that if E and F are two Banach lattices then each order Dunford-Pettis and weak Dunford-Pettis operator T from E into F has an adjoint Dunford-Pettis operator T' from F' into E' if, and only if, the norm of E' is order continuous or F' has the Schur property. As a consequence we show that, if E and F are two Banach lattices such that E or F has the Dunford-Pettis property, then each order Dunford-Pettis operator T from E into F has an adjoint $T'\colon F' \to E'$ which is Dunford-Pettis if, and only if, the norm of E' is order continuous or F' has the Schur property.

Keywords: Dunford-Pettis operator, weak Dunford-Pettis operator, order Dunford-Pettis operator, order continuous norm, Schur property

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1. Introduction

The problem discussed in the article [5] was to impose conditions on Banach lattices, E and F, and the operator T from E to F for its adjoint operator T' to be weak Dunford-Pettis. In this paper, we continue our research on this way and give necessary and sufficient conditions on E, F and T to have a Dunford-Pettis adjoint operator T'. More precisely, we show that if E and F are two Banach lattices then each order Dunford-Pettis and weak Dunford-Pettis operator T from E into F has an adjoint Dunford-Pettis operator T' from F' into E' if, and only if, the norm of E' is order continuous or F' has the Schur property (Theorem 3.1). Our theorem, Theorem 3.1, appears to be a reformulation of Theorems 3.2 and 3.5 in [5] in the following sense. In the sufficient condition of Theorem 3.2 [5], the authors give the condition of AM-compactness property of spaces E and F. However, under these conditions, a positive weak Dunford-Pettis operator is an order and weak Dunford-Pettis operator. This shows that Theorem 3.2 [5] can be easily deduced from our Theorem 3.1 and the conditions that were sufficient are also necessary. Theorem 3.5 [1] which gives a necessary condition is also included in our theorem in the way that the conditions that were only necessary became also sufficient if the operator is supposed to be order Dunford-Pettis. Hence the importance of Theorem 3.1 given in this article.

2. Preliminaries and notation

In [2] K. T. Andrews said that a norm bounded subset A of a Banach space X is a Dunford-Pettis set whenever every weakly compact operator from X to an arbitrary Banach space carries A to a norm totally bounded set. Alternatively, a norm bounded subset A of a Banach lattice E is said to be a Dunford-Pettis set if every weakly null sequence (f_n) of E converges uniformly to zero on the set A, that is, $\sup_{x\in A}|f_n(x)| \to 0$ (see Theorem 5.98 of [1]). On the other hand, a Banach space X is said to have the Dunford-Pettis property if every weakly compact operator T defined on E and taking values in a Banach space F is Dunford-Pettis. For example, the Banach space ℓ^∞ has the Dunford-Pettis property but the Banach space $\ell^\infty(\ell^2_n)$ does not have the Dunford-Pettis property.

Based on the concept of Dunford-Pettis sets, the class of order Dunford-Pettis operators is defined in [4]. In fact, an operator T from a Banach lattice E into a Banach space X is said to be order Dunford-Pettis if it carries each order bounded subset of E into a Dunford-Pettis set of X, i.e., if for each $x \in E^+$, the subset $T([-x,x])$ is Dunford-Pettis in X.

Let X and Y be two Banach spaces. An operator $T\colon X \to Y$ is called a Dunford-Pettis operator if T carries weakly convergent sequences to norm convergent sequences. (Equivalently, for each weakly null sequence (x_n) we have $\lim_{n\to\infty}\|T(x_n)\| = 0$). Alternatively, an operator $T\colon X \to Y$ is a Dunford-Pettis operator if and only if T carries relatively weakly compact sets to norm totally bounded sets.

On the other hand, unlike compact operators, there are operators T from a Banach space X into another Y whose dual operators T' from Y' into X' are not Dunford-Pettis. In fact, the dual operator of the identity operator of the Banach space ℓ^1, which is the identity of the Banach space ℓ^∞, is not Dunford-Pettis.

Recall from [1] that an operator T from a Banach space X into another Y is said to be weak Dunford-Pettis if $y_n(T(x_n))$ converges to 0 whenever (x_n) converges weakly to 0 in X and (y_n) converges weakly to 0 in Y. Alternatively, T is weak

Dunford-Pettis if the composed operator $S \circ T$ is Dunford-Pettis for each weakly compact operator S from Y into G, for an arbitrary Banach space G.

The latter class of operators was connected in Theorem 5.98 of [1] with the class of the Dunford-Pettis sets.

Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be AM-compact if it carries each order-bounded subset of E onto a relatively compact subset of X. In [3], the Banach lattice E is said to have the AM-compactness property if every weakly compact operator defined on E, and taking values in a Banach space X, is AM-compact. For example, the Banach lattice $L^2([0,1])$ does not have the AM-compactness property, but ℓ^1 has the AM-compactness property.

It follows from Proposition 3.1 of [3] that a Banach lattice E has the AM-compactness property if and only if for every weakly null sequence (f_n) of E we have $|f_n| \to 0$ for $\sigma(E', E)$.

On the other hand, it is well known that there exist weak Dunford-Pettis operators whose adjoints are not Dunford-Pettis. In fact, let us consider the Banach lattice ℓ^1: its identity operator $\mathrm{Id}_{\ell^1}\colon \ell^1 \to \ell^1$ is weak Dunford-Pettis while its dual operator $\mathrm{Id}_{\ell^\infty}\colon \ell^\infty \to \ell^\infty$ is not Dunford-Pettis. Also, there exist order Dunford-Pettis operators whose adjoints are not Dunford-Pettis. In fact, as the Banach space ℓ^2 has the AM-compactness property, the identity operator Id_{ℓ^2} is order Dunford-Pettis, but its dual operator, which is the identity operator of ℓ^2, is not Dunford-Pettis (because the Banach space ℓ^2 does not have the Schur property). However, we will prove that each operator is weak Dunford-Pettis and also order Dunford-Pettis if its adjoint is.

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following condition: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. A Banach lattice E has the Schur property if each weakly null sequence in E converges to zero in the norm. For example, the Banach lattice ℓ^1 has the Schur property but the Banach lattice $L^1([0,1])$ does not have the Schur property. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.

We will use the term operator $T\colon E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T\colon E \to F$ between two Banach lattices is positive, then its adjoint

$T'\colon F' \to E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

For terminology concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. Main results

Let X and Y be two Banach spaces, and let E be a Banach lattice. We denote:

$wDP(X,Y)$, the space of all weak Dunford-Pettis operators from X into Y,
$oDP(E,Y)$, the space of all order Dunford-Pettis operators from E into Y and
$DP(X,Y)$, the space of all Dunford-Pettis operators from X into Y.

To give the proof of Proposition 3.1, we need the following lemma

Lemma 3.1. *Let A be a bounded subset of a Banach space X. If for each $\varepsilon > 0$ there exists a Dunford-Pettis set A_ε in X such that $A \subseteq A_\varepsilon + \varepsilon B_X$ (where B_X is the closed unit ball of X), then A is a Dunford-Pettis set.*

Proof. Let Y be a Banach space and let $T\colon X \to Y$ be a weakly compact operator. We have to prove that $T(A)$ is relatively compact in Y. Let $\varepsilon > 0$, then by hypothesis there exists a Dunford-Pettis subset A_ε of X such that $A \subseteq A_\varepsilon + \varepsilon B_X$, and then $T(A) \subseteq T(A_{\varepsilon.}) + \varepsilon\|T\|B_Y$. Now as A_ε is a Dunford-Pettis set, $T(A_\varepsilon)$ is relatively compact in Y and hence by Theorem 3.1 of [1], $T(A)$ is relatively compact in Y. This shows that A is a Dunford-Pettis set. □

Proposition 3.1. *Let E and F be two Banach lattices, and let X be a Banach space. Then*

(1) *$oDP(E,X)$ is a norm closed vector subspace of the space $L(E,X)$ of all operators from E into X,*
(2) *if $T\colon E \to F$ is an order Dunford-Pettis operator, then for each operator $S\colon F \to X$, the composed operator $S \circ T$ is order Dunford-Pettis,*
(3) *if $T\colon E \to F$ is an order bounded operator, then for each order Dunford-Pettis operator $S\colon F \to X$, the composed operator $S \circ T$ is order Dunford-Pettis.*

Proof. (1) Clearly, $oDP(E,X)$ is a vector subspace of $L(E,X)$. To see that $oDP(E,X)$ is also norm closed, let S be in the norm closure of $oDP(E,X)$. To this end, let x be a nonzero in E^+ and $\varepsilon > 0$. Choose some $T \in oDP(E,X)$ satisfying $\|S - T\| \leqslant \varepsilon/\|x\|$, and observe that $S([-x,x]) \subset T([-x,x]) + \varepsilon B_X$ holds. Since T is order Dunford-Pettis, $T([-x,x])$ is a Dunford-Pettis set and hence by Lemma 3.1 $S([-x,x])$ is a Dunford-Pettis set. This shows that S is order Dunford-Pettis.

(2) Let $T\colon E \to F$ be an order Dunford-Pettis operator. Then for each $x \in E^+$, $T([-x, x])$ is a Dunford-Pettis set in F and hence $S(T[-x, x])$ is a Dunford-Pettis set in X. So, $S \circ T$ is order Dunford-Pettis.

(3) Let $T\colon E \to F$ be an order bounded operator. Then for each $x \in E^+$, $T([-x, x])$ is an order interval and since S is order Dunford-Pettis, $S(T[-x, x])$ is a Dunford-Pettis set in X. Hence $S \circ T$ is order Dunford-Pettis. □

Proposition 3.2. *Let E be a Banach lattice and X a Banach space. If the norm of E is order continuous and X has the Dunford-Pettis property then each operator T from E into X is order Dunford-Pettis.*

Proof. Since the norm of E is order continuous, it follows from Theorem 2.4.3 of [7] that for each $x \in E^+$, the order interval $[-x, x]$ is weakly compact. If $T\colon E \to X$ is an operator, then $T([-x, x])$ is weakly compact in X.

On the other hand, since X has the Dunford-Pettis property, the identity operator of X is weak Dunford-Pettis and hence by Theorem 5.99 of [1], $T([-x, x])$ is a Dunford-Pettis set. This shows that T is order Dunford-Pettis. □

The following proposition gives some characterizations of order Dunford-Pettis operators

Proposition 3.3 ([4]). *Let T be an operator from a Banach lattice E into a Banach space X. Then the following assertions are equivalent:*

(1) *T is an order Dunford-Pettis operator,*
(2) *for each weakly compact operator S from X into an arbitrary Banach space Z, the composed operator $S \circ T$ is* AM-*compact,*
(3) *for each weakly null sequence (f_n) in X' we have $|T'(f_n)| \to 0$ for $\sigma(E', E)$.*

There exist operators that are not order Dunford-Pettis. In fact, the identity operator of the Banach lattice $L^2([0, 1])$ is not order Dunford-Pettis. The following result gives a characterization of a Banach lattice which has the AM-compactness property.

Proposition 3.4. *Let E be a Banach lattice. Then the following statements are equivalent:*

(1) *each positive operator from E into E is order Dunford-Pettis,*
(2) *the identity operator of E is order Dunford-Pettis,*
(3) *E has the* AM-*compactness property.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $x \in E^+$ and let $T\colon E \to X$ be a weakly compact operator where X is arbitrary Banach space.

Since the identity operator of E is an order Dunford-Pettis, $[-x, x]$ is a Dunford-Pettis set in E and hence $T([-x, x])$ is relatively compact. This shows that T is AM-compact and hence E has the AM-compactness property.

(3) $\Longrightarrow$ (1) Let $T\colon E \to E$ be a positive operator and $S\colon E \to Z$ a weakly compact operator where Z is an arbitrary Banach space. Since E has the AM-compactness property, the operator S is AM-compact and hence $S \circ T$ is AM-compact. Finally, it follows from Proposition 3.3 that T is order Dunford-Pettis. $\square$

Proposition 3.5. *Let T be an operator from a Banach lattice E into a Banach space F. If $T' \in DP(F', E')$, then $T \in oDP(E, F)$.*

Proof. Let (f_n) be a sequence of F' such that $f_n \to 0$ in the weak topology $\sigma(F', F'')$.

As the adjoint T' is Dunford-Pettis from F' into E', we deduce that $T'(f_n) \to 0$ for the norm of E' and hence $|T'(f_n)| \to 0$ for $\sigma(E', E)$. Finally, by Proposition 3.3, we deduce that T is order Dunford-Pettis. $\square$

Proposition 3.6. *Let T be an operator from a Banach lattice E into a Banach space F. If $T' \in DP(F', E')$, then $T \in wDP(E, F)$.*

Proof. Let (x_n) (resp. (f_n)) be a sequence of E (of F') such that $x_n \to 0$ in the weak topology $\sigma(E, E')$ ($f_n \to 0$ in $\sigma(F', F'')$). We have to prove that $f_n(T(x_n)) \to 0$. As (f_n) is a sequence of F' such that $f_n \to 0$ in $\sigma(F', F'')$ and hence T' is Dunford-Pettis then $T'(f_n) \to 0$ for the norm of E'.

On the other hand, since $x_n \to 0$ in the weak topology $\sigma(E, E')$ hence (x_n) is norm bounded and by the inequality $|T'(f_n)(x_n)| \leqslant \|T'(f_n)\|_{E'}$, we conclude that T is weak Dunford-Pettis. $\square$

Theorem 3.1. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

(1) *each order Dunford-Pettis and weak Dunford-Pettis operator T from E into F has an adjoint Dunford-Pettis operator T' from F' into E',*

(2) *one of the following is valid:*
   - (a) *the norm of E' is order continuous,*
   - (b) *F' has the Schur property.*

Proof. (1) $\Longrightarrow$ (2) Assume that (2) is false, i.e., the norm of E' is not order continuous and F' does not have the Schur property. We will construct an operator $T\colon E \to F$ which is weak Dunford-Pettis and order Dunford-Pettis but its adjoint $T'\colon F' \to E'$ is not Dunford-Pettis. Indeed, suppose that E' does not have an order continuous norm. By Theorem 2.4.14 of [7] we may assume that ℓ^1 is a closed sublattice of E, and it follows from Proposition 2.3.11 of [7] that there is a positive projection P from E into ℓ^1. On the other hand, since F' does not have the Schur property, there exists a weakly null sequence $(f_n) \subset F'$ such that $\|f_n\| = 1$ for all n. Moreover, there exists a sequence $(y_n) \subset F^+$ with $\|y_n\| \leqslant 1$, and an $\varepsilon > 0$ such that $|f_n(y_n)| \geqslant \varepsilon$ for all n.

Now, we consider the operator $T = S \circ P \colon E \to \ell^1 \to F$ where S is the operator defined by

$$S\colon \ell^1 \to F,\ (\alpha_n) \to \sum_n \alpha_n y_n.$$

Since ℓ^1 has the Dunford-Pettis property , the operator T is weak Dunford-Pettis.

Also, T is order Dunford-Pettis. In fact, since ℓ^1 is discrete and its norm is order continuous, it is clear that $P([-x, x])$ is relatively compact in ℓ^1. Then $T = S \circ P([-x; x])$ is relatively compact in F and hence there is a Dunford-Pettis set in F for each $x \in E_+$. Finally, we conclude that T is order Dunford-Pettis.

But the adjoint $T'\colon F' \to E'$ is not Dunford-Pettis. Indeed, the sequence (f_n) is weakly null in F'. And as the operator $P\colon E \to \ell^1$ is surjective, there exists $\delta > 0$ such that $\delta \cdot B_{\ell^1} \subset P(B_E)$ where B_H is the closed unit ball of $H = E$ or ℓ^1. Hence

$$\begin{aligned}\|T'(f_n)\| &= \sup_{x\in B_E} |T'(f_n)(x)| = \sup_{x\in B_E} |f_n(T(x))| \\ &= \sup_{x\in B_E} |f_n \circ S(P(x))| \geqslant \delta \cdot |f_n \circ S((e_n))| \geqslant \delta \cdot |f_n(y_n)| > \delta \cdot \varepsilon\end{aligned}$$

(where $(e_n)_{n=1}^{\infty}$ is the canonical basis of ℓ^1). Then $\|T'(f_n)\| > \delta \cdot \varepsilon$ for all n, and we conclude that T' is not Dunford-Pettis. This presents a contradiction.

(2; a) $\Longrightarrow$ (1) Let (f_n) be a disjoint sequence of F' such that $(f_n) \to 0$ in $\sigma(F', F'')$. We have to prove that $(T'(f_n))$ converges to 0 for the norm of E'. By using Corollary 2.7 of Dodds-Fremlin [6], it suffices to prove that $|T'(f_n)| \to 0$ in $\sigma(E', E)$ and $T'(f_n)(x_n) \to 0$ for every norm bounded disjoint sequence $(x_n) \in E_+$. In fact, as (f_n) is a weakly null sequence in F' and since T is order Dunford-Pettis we have $|T'(f_n)| \to 0$ for $\sigma(E', E)$. On the other hand, since the norm of E' is order continuous, it follows from Corollary 2.9 of Dodds and Fremlin [6] that $x_n \to 0$ in $\sigma(E, E')$. Hence, as T is a weak Dunford-Pettis operator, we obtain $T'(f_n)(x_n) = f_n(T(x_n)) \to 0$, and this proves that T' is Dunford-Pettis.

(2; b) $\Longrightarrow$ (1) Obvious. □

Corollary 3.1. *Let E and F be two Banach lattices such that E or F has the Dunford-Pettis property. Then the following assertions are equivalent:*

(1) *each order Dunford-Pettis operator T from E into F has an adjoint Dunford-Pettis operator from F' into E',*

(2) *one of the following is valid:*
   (a) *the norm of E' is order continuous,*
   (b) *F' has the Schur property.*

As consequences of Theorem 3.1 and Proposition 3.4, we obtain the following result:

Corollary 3.2. *Let E and F be two Banach lattices such that E has the* AM-*compactness property. Then the following assertions are equivalent:*

(1) *each weak Dunford-Pettis operator T from E into F has an adjoint Dunford-Pettis operator from F' into E',*
(2) *one of the following is valid:*
   (a) *the norm of E' is order continuous,*
   (b) *F' has the Schur property.*

As consequences of Theorem 3.1 and Proposition 3.2, we obtain the following result:

Corollary 3.3. *Let E and F be two Banach lattices such that the norm of E is order continuous and F has the Dunford-Pettis property. Then the following assertions are equivalent:*

(1) *each operator T from E into F has an adjoint which is Dunford-Pettis,*
(2) *one of the following is valid:*
   (a) *the norm of E' is order continuous,*
   (b) *F' has the Schur property.*

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Authors' addresses: *Khalid Bouras*, Université Abdelmalek Essaadi, Faculté polydisciplinaire, B.P. 745, Larache, Morocco, e-mail: bouraskhalid@hotmail.com; *Abdelmonaim El Kaddouri*, *Jawad H'michane*, *Mohammed Moussa*, Ibn Tofail University, P.B. 133, Kénitra, Morocco, e-mail: elkaddouri.abdelmonaim@gmail.com, hm1982jad@gmail.com, mohammed.moussa09@gmail.com.