STABILITY IN LINEAR NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE DELAYS

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Abstract. In this paper we use the fixed point method to prove asymptotic stability results of the zero solution of a generalized linear neutral difference equation with variable delays. An asymptotic stability theorem with a sufficient condition is proved, which improves and generalizes some results due to Y. N. Raffoul (2006), E. Yankson (2009), M. Islam and E. Yankson (2005).

Keywords: fixed point, stability, neutral difference equation, variable delay

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1. Introduction

The Lyapunov direct method has been one among the efficient tools for the study of stability properties of a large class of ordinary, functional, partial differential and difference equations. Nevertheless, the application of this method to problems of stability in differential and difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms ([4], [5], [8]–[10], [12], [21]). Recently, Burton, Furumochi, Zhang and others have noticed that some of these difficulties vanish or might be overcome by means of the fixed point theory (see [1], [2], [4], [5], [13], [19], [20], [23]–[25]). The application of the fixed point theory to certain problems on stability has shown a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [4]). It is also worth adding that there is a wide number of investigators working on stability theory of difference equations,

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with or without delay, who have established and proved interesting results by using other ideas than Lyapunov's method, see the papers [2], [3], [6], [11], [15]–[18], [26].

Let $a, b, c, a_j, c_j\colon \mathbb{Z}^+ \to \mathbb{R}$ and $\tau, \tau_j\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n-\tau(n) \to \infty$ and $n-\tau_j(n) \to \infty$ as $n \to \infty$. Here Δ denotes the forward difference operator $\Delta x(t) = x(n+1)-x(n)$ for any sequence $\{x(n),\ n \in \mathbb{Z}^+\}$.

In [19], Raffoul studied the equation

$$\Delta x(n) = -a(n)x(n-\tau(n)), \tag{1.1}$$

and proved the following theorem.

Theorem A (Raffoul [19]). *Suppose that $\tau(n) = r$ and $a(n+r) \neq 1$ and there exists a constant $\alpha < 1$ such that*

$$\sum_{s=n-r}^{n-1} |a(s+r)| + \sum_{s=0}^{n-1} \left(|a(s+r)| \left| \prod_{k=s+1}^{n-1} [1-a(k+r)] \right| \sum_{u=s-r}^{s-1} |a(u+r)| \right) \leqslant \alpha \tag{1.2}$$

for all $n \in \mathbb{Z}^+$, and $\prod\limits_{s=0}^{n-1} [1-a(s+r)] \to 0$ as $n \to \infty$. Then, for every small initial sequence $\psi\colon [-r,0] \cap \mathbb{Z} \to \mathbb{R}$, the solution $x(n) = x(n,0,\psi)$ of (1.1) is bounded and tends to zero as $n \to \infty$.

In [23], Yankson studied the generalization of (1.1) as follows,

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)), \tag{1.3}$$

and obtained the following theorem.

Theorem B (Yankson [23]). *Suppose that $Q(n) \neq 0$ for all $n \in [n_0,\infty) \cap \mathbb{Z}$, the inverse sequence g_j of $n-\tau_j(n)$ exists and there exists a constant $\alpha \in (0,1)$ for all $n \in [n_0,\infty) \cap \mathbb{Z}$ such that*

$$\begin{aligned} &\sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |a_j(g_j(s))| \\ &\quad + \sum_{s=n_0}^{n-1} \left(|1-Q(s)| \left| \prod_{k=s+1}^{n-1} Q(k) \right| \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j(g_j(u))| \right) \leqslant \alpha, \end{aligned} \tag{1.4}$$

where $Q(n) = 1 - \sum\limits_{j=1}^{N} a_j(g_j(n))$. Then the zero solution of (1.3) is asymptotically stable if $\prod\limits_{s=n_0}^{n-1} Q(s) \to 0$ as $n \to \infty$.

Obviously, Theorem B improves and generalizes Theorem A. On the other hand, Islam and Yankson in [13] considered the linear neutral difference equation

$$x(n+1) = a(n)x(n) + b(n)x(n-\tau(n)) + c(n)\Delta x(n-\tau(n)), \tag{1.5}$$

and obtained the following theorem.

Theorem C (Islam and Yankson [13]). *Suppose that $a(n) \neq 0$ and there exists a constant $\alpha \in (0,1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that*

$$|c(n-1)| + \sum_{s=n_0}^{n-1} |b(s) - \varphi(s)| \left| \prod_{u=s+1}^{n-1} a(u) \right| \leqslant \alpha, \tag{1.6}$$

where $\varphi(s) = c(s) - c(s-1)a(s)$. Then the zero solution of (1.5) is asymptotically stable if $\prod_{s=n_0}^{n-1} a(s) \to 0$ as $n \to \infty$.

In this paper, we consider the generalization of a linear neutral difference equation with variable delays (1.5) of the form

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \sum_{j=1}^{N} c_j(n)\Delta x(n-\tau_j(n)) \tag{1.7}$$

with the initial condition

$$x(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z}, \tag{1.8}$$

where $\psi\colon [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ is a bounded sequence and for $n_0 \geqslant 0$,

$$m_j(n_0) = \inf\{n - \tau_j(n),\ n \geqslant n_0\}, \quad m(n_0) = \min\{m_j(n_0),\ 1 \leqslant j \leqslant N\}.$$

Note that (1.7) becomes (1.5) for $N = 2$, $\tau_1 = 0$, $\tau_2 = \tau$, $a_1 = 1 - a$, $a_2 = -b$, $c_1 = 0$, $c_2 = c$. Thus, we know that (1.7) includes (1.1), (1.3) and (1.5) as special cases.

Equation (1.7) can be viewed as a discrete analogue of the linear neutral differential equation

$$x'(t) = -\sum_{j=1}^{N} a_j(t)x(t-\tau_j(t)) + \sum_{j=1}^{N} c_j(t)x'(t-\tau_j(t)). \tag{1.9}$$

In [1], the authors investigated (1.9) and obtained

Theorem D (Ardjouni and Djoudi [1]). *Suppose that τ_j is twice differentiable and $\tau_j'(t) \neq 1$ for all $t \in \mathbb{R}^+$, and there exist continuous functions $h_j\colon [m_j(t_0), \infty) \to \mathbb{R}$ for $j = 1, 2, \ldots, N$ and a constant $\alpha \in (0,1)$ such that for $t \geqslant 0$*

$$\liminf_{t\to\infty} \int_0^t H(s)\,\mathrm{d}s > -\infty,$$

and

$$\begin{aligned}\sum_{j=1}^{N} \Big|\frac{c_j(t)}{1-\tau_j'(t)}\Big| &+ \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} |h_j(s)|\,\mathrm{d}s \\ &+ \sum_{j=1}^{N} \int_0^t \mathrm{e}^{-\int_s^t H(u)\,\mathrm{d}u} |-a_j(s) + h_j(s-\tau_j(s))(1-\tau_j'(s)) - r_j(s)|\,\mathrm{d}s \\ &+ \sum_{j=1}^{N} \int_0^t \mathrm{e}^{-\int_s^t H(u)\,\mathrm{d}u} |H(s)| \Big(\int_{s-\tau_j(s)}^{s} |h_j(u)|\,\mathrm{d}u\Big)\,\mathrm{d}s \leqslant \alpha,\end{aligned}$$

where

$$H(t) = \sum_{j=1}^{N} h_j(t) \quad \text{and} \quad r_j(t) = \frac{[c_j(t)H(t) + c_j'(t)](1-\tau_j'(t)) + c_j(t)\tau_j''(t)}{(1-\tau_j'(t))^2}.$$

Then the zero solution of (1.9) *is asymptotically stable if and only if*

$$\int_0^t H(s)\,\mathrm{d}s \to \infty \text{ as } t \to \infty.$$

Our purpose here is to give, by using the contraction mapping principle, asymptotic stability results for the generalized linear neutral difference equation with variable delays (1.7). For details on the contraction mapping principle we refer the reader to [22] and for more on the calculus of difference equations, we refer the reader to [7] and [14]. The results presented in the present paper improve and generalize the main results in [13], [19], [23].

2. Main results

For a fixed n_0, we denote by $D(n_0)$ the set of bounded sequences $\psi\colon [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ with the norm $|\psi|_0 = \max\{|\psi(n)|\colon n \in [m(n_0), n_0] \cap \mathbb{Z}\}$. Also, let $(\mathbb{B}, \|.\|)$ be the Banach space of bounded sequences $x\colon [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ with the maximum norm $\|.\|$. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, a solution of (1.7) through (n_0, ψ) is a sequence $x\colon [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ such that x satisfies (1.7) on $[n_0, \infty) \cap \mathbb{Z}$ and $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$. We denote such a solution by $x(n) = x(n, n_0, \psi)$. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, there exists a unique solution $x(n) = x(n, n_0, \psi)$ of (1.7) defined on $[m(n_0), \infty) \cap \mathbb{Z}$.

Let $h_j\colon [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ be an arbitrary sequence. Rewrite (1.7) as

$$\begin{aligned}\Delta x(n) = &-\sum_{j=1}^{N} h_j(n)x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) \\ &+ \sum_{j=1}^{N} \{h_j(n-\tau_j(n)) - a_j(n)\} x(n-\tau_j(n)) \\ &+ \sum_{j=1}^{N} c_j(n)\Delta x(n-\tau_j(n)),\end{aligned} \tag{2.1}$$

where Δ_n indicates that the difference is taken with respect to n. If we let $H(n) = 1 - \sum\limits_{j=1}^{N} h_j(n)$ then (2.1) is equivalent to

$$\begin{aligned}x(n+1) = H(n)x(n) &+ \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) \\ &+ \sum_{j=1}^{N} \{h_j(n-\tau_j(n)) - a_j(n)\} x(n-\tau_j(n)) \\ &+ \sum_{j=1}^{N} c_j(n)\Delta x(n-\tau_j(n)).\end{aligned} \tag{2.2}$$

In the process, for any sequence x we denote

$$\sum_{k=a}^{b} x(k) = 0 \quad \text{and} \quad \prod_{k=a}^{b} x(k) = 1 \quad \text{for any } a > b.$$

Lemma 2.1. *Suppose that $H(n)\neq 0$ for all $n\in[n_0,\infty)\cap\mathbb{Z}$. Then x is a solution of equation* (1.7) *if and only if*

$$
\begin{aligned}
x(n) = \Big\{ & x(n_0) - \sum_{j=1}^{N} c_j(n_0-1)x(n_0-\tau_j(n_0)) \\
& - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)x(s) \Big\} \prod_{u=n_0}^{n-1} H(u) \\
& + \sum_{j=1}^{N} c_j(n-1)x(n-\tau_j(n)) + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) \\
& + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u)\{h_j(s-\tau_j(s)) - a_j(s) - \varphi_j(s)\}x(s-\tau_j(s)) \\
& - \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \{1-H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v),
\end{aligned} \tag{2.3}
$$

where

$$\varphi_j(n) = c_j(n) - c_j(n-1)H(n). \tag{2.4}$$

P r o o f. Let x be a solution of (1.7). By multiplying both sides of (2.2) by $\prod\limits_{u=n_0}^{n}[H(u)]^{-1}$ and by summing from n_0 to $n-1$ we obtain

$$
\begin{aligned}
\sum_{s=n_0}^{n-1} \Delta\Big[\prod_{u=n_0}^{s-1}[H(u)]^{-1}x(s)\Big] = & \sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}\Delta_s \sum_{j=1}^{N}\sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v) \\
& + \sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}\sum_{j=1}^{N}\{h_j(s-\tau_j(s)) - a_j(s)\}x(s-\tau_j(s)) \\
& + \sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}\sum_{j=1}^{N} c_j(s)\Delta x(s-\tau_j(s)).
\end{aligned}
$$

As a consequence, we arrive at

$$
\prod_{u=n_0}^{n-1}[H(u)]^{-1}x(n) - \prod_{u=n_0}^{n_0-1}[H(u)]^{-1}x(n_0)
$$

$$= \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}\Delta_s \sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v)$$

$$+\sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}\{h_j(s-\tau_j(s))-a_j(s)\}x(s-\tau_j(s))$$

$$+\sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=n_0}^{s}[H(u)]^{-1}c_j(s)\Delta x(s-\tau_j(s)).$$

By dividing both sides of the above expression by $\prod_{u=n_0}^{n-1}[H(u)]^{-1}$ we get

$$x(n) = x(n_0)\prod_{u=n_0}^{n-1}H(u) + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\Delta_s \sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v) \tag{2.5}$$

$$+\sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\{h_j(s-\tau_j(s))-a_j(s)\}x(s-\tau_j(s))$$

$$+\sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)c_j(s)\Delta x(s-\tau_j(s)).$$

By performing a summation by parts, we have

$$\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\Delta_s \sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v) \tag{2.6}$$

$$= \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) - \prod_{u=n_0}^{n-1}H(u)\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)x(s)$$

$$-\sum_{s=n_0}^{n-1}\{1-H(s)\}\prod_{u=s+1}^{n-1}H(u)\sum_{v=s-\tau_j(s)}^{s-1} h_j(v)x(v),$$

and

$$\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)c_j(s)\Delta x(s-\tau_j(s)) \tag{2.7}$$

$$= -c_j(n_0-1)x(n_0-\tau_j(n_0))\prod_{u=n_0}^{n-1}H(u) + c_j(n-1)x(n-\tau_j(n))$$

$$-\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\varphi_j(s)x(s-\tau_j(s)),$$

where φ_j is given by (2.4). Finally, substituting (2.6) and (2.7) into (2.5) completes the proof. □

Definition 2.2. The zero solution of (1.7) is Lyapunov stable if for any $\varepsilon > 0$ and any integer $n_0 \geqslant 0$ there exists a $\delta > 0$ such that $|\psi(n)| \leqslant \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ implies $|x(n, n_0, \psi)| \leqslant \varepsilon$ for $n \in [n_0, \infty) \cap \mathbb{Z}$.

Theorem 2.3. *Suppose that* $H(n) \neq 0$ *for all* $n \in [n_0, \infty) \cap \mathbb{Z}$, *and there exist a positive constant* M *and a constant* $\alpha \in (0, 1)$ *such that for* $n \in [n_0, \infty) \cap \mathbb{Z}$

$$\left|\prod_{u=n_0}^{n-1} H(u)\right| \leqslant M, \tag{2.8}$$

and

$$\begin{aligned}\sum_{j=1}^{N} |c_j(n-1)| &+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\ &+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \left|\prod_{u=s+1}^{n-1} H(u)\right| |h_j(s-\tau_j(s)) - a_j(s) - \varphi_j(s)| \\ &+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1-H(s)| \left|\prod_{u=s+1}^{n-1} H(u)\right| \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \leqslant \alpha.\end{aligned} \tag{2.9}$$

Then the zero solution of (1.7) *is stable.*

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$(M + \alpha M)\delta + \alpha\varepsilon \leqslant \varepsilon.$$

Let $\psi \in D(n_0)$ be such that $|\psi(n)| \leqslant \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$. Define

$$\mathbb{S}_\varepsilon = \{\varphi \in \mathbb{B} \colon \varphi(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z},\ \|\varphi\| \leqslant \varepsilon\}.$$

Then $(\mathbb{S}_\varepsilon, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the maximum norm.

Use (2.3) to define the operator $P\colon \mathbb{S}_\varepsilon \to \mathbb{B}$ by $(P\varphi)(n) = \psi(n)$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ and

$$\begin{aligned}(2.10)\qquad &(P\varphi)(n)\\ &= \Big\{\psi(n_0) - \sum_{j=1}^{N} c_j(n_0-1)\psi(n_0-\tau_j(n_0)) - \sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)\psi(s)\Big\}\\ &\quad \times \prod_{u=n_0}^{n-1} H(u) + \sum_{j=1}^{N} c_j(n-1)\varphi(n-\tau_j(n)) + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s)\\ &\quad + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1} H(u)\{h_j(s-\tau_j(s)) - a_j(s) - \varphi_j(s)\}\varphi(s-\tau_j(s))\\ &\quad - \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\{1-H(s)\}\prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v)\varphi(v)\end{aligned}$$

for $n \in [n_0, \infty) \cap \mathbb{Z}$. Clearly, $P\varphi$ is bounded by (2.9). We first show that P maps $\mathbb{S}_\varepsilon$ into $\mathbb{S}_\varepsilon$. We have

$$\begin{aligned}|(P\varphi)(n)| &\leqslant M\delta + \alpha M\delta + \Big\{\sum_{j=1}^{N}|c_j(n-1)| + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\\ &\quad + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\Big|\prod_{u=s+1}^{n-1} H(u)\Big||h_j(s-\tau_j(s)) - a_j(s) - \varphi_j(s)|\\ &\quad + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}|1-H(s)|\Big|\prod_{u=s+1}^{n-1} H(u)\Big|\sum_{v=s-\tau_j(s)}^{s-1}|h_j(v)|\Big\}\|\varphi\|\\ &\leqslant (M+\alpha M)\delta + \alpha\varepsilon \leqslant \varepsilon,\end{aligned}$$

by (2.8) and (2.9). Thus P maps $\mathbb{S}_\varepsilon$ into itself. We next show that P is a contraction. Let $\varphi_1, \varphi_2 \in \mathbb{S}_\varepsilon$, then

$$\begin{aligned}&|(P\varphi_1)(n) - (P\varphi_2)(n)|\\ &\quad\leqslant \Big\{\sum_{j=1}^{N}|c_j(n-1)| + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\\ &\qquad + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}\Big|\prod_{u=s+1}^{n-1} H(u)\Big||h_j(s-\tau_j(s)) - a_j(s) - \varphi_j(s)|\\ &\qquad + \sum_{j=1}^{N}\sum_{s=n_0}^{n-1}|1-H(s)|\Big|\prod_{u=s+1}^{n-1} H(u)\Big|\sum_{v=s-\tau_j(s)}^{s-1}|h_j(v)|\Big\}\|\varphi_1-\varphi_2\|\\ &\quad\leqslant \alpha\|\varphi_1-\varphi_2\|,\end{aligned}$$

by (2.8). This shows that P is a contraction with contraction constant α. Thus, by the contraction mapping principle ([22], p. 2), P has a unique fixed point x in $\mathbb{S}_\varepsilon$ which is a solution of (1.7) with $x=\psi$ on $[m(n_0),n_0]\cap\mathbb{Z}$ and $|x(n)|=|x(n,n_0,\psi)|\leqslant\varepsilon$ for $n\in[n_0,\infty)\cap\mathbb{Z}$. This proves that the zero solution of (1.7) is stable. □

Definition 2.4. The zero solution of (1.7) is asymptotically stable if it is Lyapunov stable and if for any integer $n_0\geqslant 0$ there exists a $\delta>0$ such that $|\psi(n)|\leqslant\delta$ for $n\in[m(n_0),n_0]\cap\mathbb{Z}$ implies $x(n,n_0,\psi)\to 0$ as $n\to\infty$.

Theorem 2.5. *Assume that the hypotheses of Theorem* 2.3 *hold. Also assume that*

$$\prod_{u=n_0}^{n-1} H(u)\to 0 \quad \text{as } n\to\infty. \tag{2.11}$$

Then the zero solution of (1.7) *is asymptotically stable.*

Proof. We have already proved that the zero solution of (1.7) is stable. For a given $\varepsilon>0$ let $\psi\in D(n_0)$ be such that $|\psi(n)|\leqslant\delta$ for $n\in[m(n_0),n_0]\cap\mathbb{Z}$ where $\delta>0$, and define

$$\begin{aligned}\mathbb{S}_\varepsilon^*=\{\varphi\in\mathbb{B}\colon\ &\varphi(n)=\psi(n) \text{ for } n\in[m(n_0),n_0]\cap\mathbb{Z},\\ &\|\varphi\|\leqslant\varepsilon \text{ and } \varphi(n)\to 0 \text{ as } n\to\infty\}.\end{aligned}$$

Define $P\colon\ \mathbb{S}_\varepsilon^*\to\mathbb{S}_\varepsilon^*$ by (2.10). From the proof of Theorem 2.3, the map P is a contraction with the contraction constant α and for every $\varphi\in\mathbb{S}_\varepsilon^*$, $\|P\varphi\|\leqslant\varepsilon$.

We next show that $(P\varphi)(n)\to 0$ as $n\to\infty$. There are five terms on the right hand side in (2.10). Denote them, respectively, by I_k, $k=1,2,\ldots,5$. It is obvious that the first term I_1 tends to zero as $t\to\infty$, by condition (2.11). Also, due to the facts that $\varphi(n)\to 0$ and $n-\tau_j(n)\to\infty$ for $j=1,2,\ldots,N$ as $n\to\infty$, the second term I_2 tends to zero as $n\to\infty$. What is left is to show that each one of the remaining terms in (2.10) goes to zero at infinity.

Let $\varphi\in\mathbb{S}_\varepsilon^*$ be fixed. For a given $\varepsilon_1\in(0,\varepsilon)$, we choose $N_0>n_0$ large enough such that $n-\tau_j(n)\geqslant N_0$, $j=1,2,\ldots,N$, implies $|\varphi(s)|<\varepsilon_1$ if $s\geqslant n-\tau_j(n)$. Therefore, the third term I_3 in (2.10) satisfies

$$\begin{aligned}|I_3|&=\Bigl|\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)\varphi(s)\Bigr|\leqslant\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)||\varphi(s)|\\ &\leqslant\varepsilon_1\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\leqslant\alpha\varepsilon_1<\varepsilon_1.\end{aligned}$$

Thus, $I_3 \to 0$ as $n \to \infty$. Now for a given $\varepsilon_2 \in (0, \varepsilon)$, there exists an $N_1 > n_0$ such that $s \geqslant N_1$ implies $|\varphi(s - \tau_j(s))| < \varepsilon_2$ for $j = 1, 2, \ldots, N$. Thus, for $n \geqslant N_1$, the term I_4 in (2.10) satisfies

$$\begin{aligned}
|I_4| &= \Big| \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \{h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)\} \varphi(s - \tau_j(s)) \Big| \\
&\leqslant \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} \Big| \prod_{u=s+1}^{n-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)| |\varphi(s - \tau_j(s))| \\
&\quad + \sum_{j=1}^{N} \sum_{s=N_1}^{n-1} \Big| \prod_{u=s+1}^{n-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)| |\varphi(s - \tau_j(s))| \\
&\leqslant \sup_{\sigma \geqslant m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} \Big| \prod_{u=s+1}^{n-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)| \\
&\quad + \varepsilon_2 \sum_{j=1}^{N} \sum_{s=N_1}^{n-1} \Big| \prod_{u=s+1}^{n-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)|.
\end{aligned}$$

By (2.11), we can find $N_2 > N_1$ such that $n \geqslant N_2$ implies

$$\begin{aligned}
&\sup_{\sigma \geqslant m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} \Big| \prod_{u=s+1}^{n-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)| \\
&= \sup_{\sigma \geqslant m(n_0)} |\varphi(\sigma)| \Big| \prod_{u=N_2}^{n-1} H(u) \Big| \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} \Big| \prod_{u=s+1}^{N_2-1} H(u) \Big| |h_j(s - \tau_j(s)) - a_j(s) - \varphi_j(s)| \\
&< \varepsilon_2.
\end{aligned}$$

Now, apply (2.9) to obtain $|I_4| < \varepsilon_2 + \alpha\varepsilon_2 < 2\varepsilon_2$. Thus, $I_4 \to 0$ as $n \to \infty$. Similarly, by using (2.9), then, if $n \geqslant N_2$, the term I_5 in (2.10) satisfies

$$\begin{aligned}
|I_5| &= \Big| \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) \varphi(v) \Big| \\
&\leqslant \sup_{\sigma \geqslant m(n_0)} |\varphi(\sigma)| \Big| \prod_{u=N_2}^{n-1} H(u) \Big| \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} |1 - H(s)| \Big| \prod_{u=s+1}^{N_2-1} H(u) \Big| \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \\
&\quad + \varepsilon_2 \sum_{j=1}^{N} \sum_{s=N_1}^{n-1} |1 - H(s)| \Big| \prod_{u=s+1}^{n-1} H(u) \Big| \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \\
&< \varepsilon_2 + \alpha\varepsilon_2 < 2\varepsilon_2.
\end{aligned}$$

Thus, $I_5 \to 0$ as $n \to \infty$. In conclusion $(P\varphi)(n) \to 0$ as $n \to \infty$, as required. Hence P maps $\mathbb{S}^*_\varepsilon$ into $\mathbb{S}^*_\varepsilon$.

By the contraction mapping principle, P has a unique fixed point $x \in \mathbb{S}_{\varepsilon}^{*}$ which solves (1.7). Therefore, the zero solution of (1.7) is asymptotically stable. $\square$

Letting $N = 2$, $\tau_1 = 0$, $\tau_2 = \tau$, $a_1 = 1 - a$, $a_2 = -b$, $c_1 = 0$, $c_2 = c$, we have

Corollary 2.6. *Suppose that $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ and there exists a constant $\alpha \in (0,1)$ such that for $n \in [n_0, \infty) \cap \mathbb{Z}$*

$$\begin{aligned} (2.12) \qquad & |c(n-1)| + \sum_{s=n-\tau(n)}^{n-1} |h_2(s)| \\ & + \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| (|h_1(s) - 1 + a(s)| + |h_2(s - \tau(s)) + b(s) - \varphi(s)|) \\ & + \sum_{s=n_0}^{n-1} |1 - H(s)| \left| \prod_{u=s+1}^{n-1} H(u) \right| \sum_{v=s-\tau(s)}^{s-1} |h_2(v)| \leqslant \alpha, \end{aligned}$$

where $H(n) = 1 - \sum\limits_{j=1}^{2} h_j(n)$ and $\varphi(n) = c(n) - c(n-1)H(n)$. Then the zero solution of (1.5) *is asymptotically stable if*

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \text{ as } n \to \infty.$$

Remark 2.7. When $h_1(s) = 1 - a(s)$ and $h_2(s) = 0$, Corollary 2.6 reduces to Theorem C.

For the special case $c_j = 0$, we can get

Corollary 2.8. *Suppose that $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ and there exists a constant $\alpha \in (0,1)$ such that for $n \in [n_0, \infty) \cap \mathbb{Z}$*

$$\begin{aligned} (2.13) \qquad & \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \left| \prod_{u=s+1}^{n-1} H(u) \right| |h_j(s - \tau_j(s)) - a_j(s)| \\ & + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1 - H(s)| \left| \prod_{u=s+1}^{n-1} H(u) \right| \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \leqslant \alpha. \end{aligned}$$

Then the zero solution of (1.3) *is asymptotically stable if*

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \text{ as } n \to \infty.$$

Remark 2.9. When $h_j(s) = a_j(g_j(s))$ for $j = 1, 2, \ldots, N$, Corollary 2.8 reduces to Theorem B.

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