ON $|A,\delta|_k$-SUMMABILITY OF ORTHOGONAL SERIES

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Dedicated to the memory of my Professor Muharrem Berisha

Abstract. In the paper, we prove two theorems on $|A,\delta|_k$ summability, $1 \leqslant k \leqslant 2$, of orthogonal series. Several known and new results are also deduced as corollaries of the main results.

Keywords: orthogonal series, matrix summability

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1. Introduction

Let $\sum\limits_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$ and let $A := (a_{nv})$ be a normal matrix, i.e. a lower triangular matrix with non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s := \{s_n\}$ to $As := \{A_n(s)\}$, where

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, 2, \ldots$$

In 1957, Flett [5] gave the following definition:

The infinite series $\sum\limits_{n=0}^{\infty} a_n$ is said to be absolutely $|A|_k$-summable, $k \geqslant 1$, if

$$\sum_{n=0}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k$$

converges, where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If this is the case, we write

$$\sum_{n=0}^{\infty} a_n \in |A|_k.$$

In [6], Flett considered a further extension of absolute summability in which he introduced a further parameter δ. The series $\sum\limits_{n=0}^{\infty} a_n$ is said to be $|A, \delta|_k$-summable, $k \geqslant 1$, $\delta \geqslant 0$, if

$$\sum_{n=0}^{\infty} n^{\delta k+k-1} |\bar{\Delta} A_n(s)|^k < \infty.$$

Let p denote the sequence $\{p_n\}$. For two given sequences p and q, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

The infinite series $\sum\limits_{n=0}^{\infty} a_n$ is absolutely (N, p, q)-summable if the series

$$\sum_{n=0}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

The notion of $|N, p, q|$ summability was introduced by Tanaka [3].

Let $\{\varphi_j\}$ be an orthonormal system defined in the interval (a, b). We assume that f belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{j=0}^{\infty} c_j \varphi_j(x), \tag{1.1}$$

where $c_j = \int_a^b f(x) \varphi_j(x)\, \mathrm{d}x \quad (j = 0, 1, 2, \ldots)$.

Following [4] we write

$$R_n := (p*q)_n, \quad R_n^j := \sum_{m=j}^{n} p_{n-m}q_m$$

where

$$R_n^{n+1} = 0, \quad R_n^0 = R_n.$$

We recall two results from [4].

Theorem 1.1 [4]**.** *If the series*

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|c_j|^2\right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j\varphi_j(x)$$

is $|N,p,q|$-summable almost everywhere.

Theorem 1.2 [4]**.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum\limits_{n=1}^{\infty}(n\Omega(n))^{-1}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum\limits_{n=1}^{\infty}|c_n|^2\Omega(n)w^{(1)}(n)$ converges, then the orthogonal series $\sum\limits_{j=0}^{\infty}c_j\varphi_j(x) \in |N,p,q|$ almost everywhere, where $w^{(1)}(n)$ is defined by $w^{(1)}(j) := j^{-1}\sum\limits_{n=j}^{\infty}n^2(R_n^j/R_n - R_{n-1}^j/R_{n-1})^2$.*

The main purpose of the present paper is to generalize Theorems 1.1 and 1.2 for $|A,\delta|_k$ summability of the orthogonal series (1.1), where $1 \leqslant k \leqslant 2$. Before stating the main results, we introduce some further notation.

With a normal matrix $A := (a_{nv})$ we associate two semi lower matrices $\bar{A} := (\bar{a}_{nv})$ and $\hat{A} := (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \quad n,i = 0,1,2,\ldots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00},\ \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,\ldots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively.

Throughout this paper we denote by K a constant that depends only on k and may be different in different relations.

2. Main results

We prove the following theorem.

Theorem 2.1. *If the series*

$$\sum_{n=0}^{\infty}\left\{n^{2(\delta+1-1/k)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right\}^{k/2}$$

converges for $1\leqslant k\leqslant 2$, *then the orthogonal series*

$$\sum_{j=0}^{\infty}c_j\varphi_j(x)$$

is $|A,\delta|_k$*-summable almost everywhere.*

P r o o f. Let

$$s_v(x)=\sum_{j=0}^{v}c_j\varphi_j(x)$$

be the partial sums of order v of the series (1.1). Then, for the matrix transform $A_n(s)(x)$ of the partial sums $s_v(x)$, we have

$$\begin{aligned}A_n(s)(x)&=\sum_{v=0}^{n}a_{nv}s_v(x)=\sum_{v=0}^{n}a_{nv}\sum_{j=0}^{v}c_j\varphi_j(x)\\&=\sum_{j=0}^{n}c_j\varphi_j(x)\sum_{v=j}^{n}a_{nv}=\sum_{j=0}^{n}\bar{a}_{nj}c_j\varphi_j(x).\end{aligned}$$

Hence

$$\begin{aligned}\bar{\Delta}A_n(s)(x)&=\sum_{j=0}^{n}\bar{a}_{nj}c_j\varphi_j(x)-\sum_{j=0}^{n-1}\bar{a}_{n-1,j}c_j\varphi_j(x)\\&=\bar{a}_{nn}c_n\varphi_n(x)+\sum_{j=0}^{n-1}\left(\bar{a}_{n,j}-\bar{a}_{n-1,j}\right)c_j\varphi_j(x)\\&=\hat{a}_{nn}c_n\varphi_n(x)+\sum_{j=0}^{n-1}\hat{a}_{n,j}c_j\varphi_j(x)=\sum_{j=0}^{n}\hat{a}_{n,j}c_j\varphi_j(x).\end{aligned}$$

Using Hölder's inequality and orthogonality, we have that

$$\int_a^b |\bar{\Delta} A_n(s)(x)|^k \,\mathrm{d}x \leqslant (b-a)^{1-k/2} \left(\int_a^b |A_n(s)(x) - A_{n-1}(s)(x)|^2 \,\mathrm{d}x \right)^{k/2}$$
$$= (b-a)^{1-k/2} \left(\int_a^b \left| \sum_{j=0}^{n} \hat{a}_{n,j} c_j \varphi_j(x) \right|^2 \mathrm{d}x \right)^{k/2}$$
$$= (b-a)^{1-k/2} \left[\sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2}.$$

Thus, the series

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \int_a^b |\bar{\Delta} A_n(s)(x)|^k \,\mathrm{d}x \leqslant K \sum_{n=1}^{\infty} \left[n^{2(\delta+1)-2/k} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2} \tag{2.1}$$

converges since the last one does by the assumption. Now, the Lemma of Beppo-Lévi implies the theorem. □

If we put

$$w^{(k)}(A,\delta;j) := \frac{1}{j^{2/k-1}} \sum_{n=j}^{\infty} n^{2(\delta+1/k)} |\hat{a}_{n,j}|^2 \tag{2.2}$$

then the following theorem holds.

Theorem 2.2. *Let $1 \leqslant k \leqslant 2$ and let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} (n\Omega(n))^{-1}$ converges. If the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{2/k-1}(n) w^{(k)}(A,\delta;n)$ converges, then the orthogonal series $\sum_{j=0}^{\infty} c_j \varphi_j(x)$ is $|A,\delta|_k$-summable almost everywhere, where $w^{(k)}(A,\delta;n)$ is defined by (2.2).*

P r o o f. Applying Hölder's inequality to the inequality (2.1) we get that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} \int_a^b |\bar{\Delta} A_n(s)(x)|^k \,\mathrm{d}x \leqslant K \sum_{n=1}^{\infty} n^{\delta k+k-1} \left[\sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2}$$
$$= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{1-k/2}} \left[n^{2\delta+1} \Omega^{2/k-1}(n) \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2}$$

$$\leqslant K\Big(\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))}\Big)^{1-k/2}\Big[\sum_{n=1}^{\infty}n^{2\delta+1}\Omega^{2/k-1}(n)\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\Big]^{k/2}$$

$$\leqslant K\Big\{\sum_{j=1}^{\infty}|c_j|^2\sum_{n=j}^{\infty}n^{2\delta+1}\Omega^{2/k-1}(n)|\hat{a}_{n,j}|^2\Big\}^{k/2}$$

$$\leqslant K\Big\{\sum_{j=1}^{\infty}|c_j|^2\Big(\frac{\Omega(j)}{j}\Big)^{2/k-1}\sum_{n=j}^{\infty}n^{2(\delta+1/k)}|\hat{a}_{n,j}|^2\Big\}^{k/2}$$

$$= K\Big\{\sum_{j=1}^{\infty}|c_j|^2\Omega^{2/k-1}(j)w^{(k)}(A,\delta;j)\Big\}^{k/2},$$

which is finite by virtue of the hypothesis of the theorem, and this completes the proof. □

The next section is devoted to applications of our main results.

3. Applications of the main results

We can specialize the matrix $A=(a_{nv})$ so that $|A,\delta|_k$ summability reduces to some known notions of absolute summability. This means that sufficient conditions obtained in the main results, under which the orthogonal series (1.1) is $|A,\delta|_k$-summable almost everywhere ($1\leqslant k\leqslant 2$), include sufficient conditions under which the orthogonal series (1.1) is absolute summable almost everywhere with different kinds of absolute summability notions. The most important particular cases of the $|A,\delta|_k$ summability notions are:

1. For $a_{n,v}=(n+1)^{-1}$ we obtain the Cesàro means $A_n(s)=(n+1)^{-1}\sum\limits_{v=0}^{n}s_v$, and $|A,\delta|_k\equiv|C,1,\delta|_k$ summability.
2. For $a_{n,v}=((n-v+1)\log n)^{-1}$ we obtain the harmonic means $A_n(s)=(\log n)^{-1}\sum\limits_{v=0}^{n}s_v/(n-v+1)$, and $|A,\delta|_k\equiv|H,1,\delta|_k$ summability.
3. For $a_{n,v}=\binom{n-v+\alpha+1}{\alpha-1}/\binom{n+\alpha}{\alpha}$, $0\leqslant\alpha\leqslant 1$, we obtain the Cesàro means (of order α) $A_n(s)=\binom{n+\alpha}{\alpha}^{-1}\sum\limits_{v=0}^{n}\binom{n-v+\alpha+1}{\alpha-1}s_v$, and $|A,\delta|_k\equiv|C,\alpha,\delta|_k$ summability.
4. For $a_{n,v}=p_{n-v}/P_n$ we obtain the Nörlund means $A_n(s)=P_n^{-1}\sum\limits_{v=0}^{n}p_{n-v}s_v$, and $|A,\delta|_k\equiv|N,p_n,\delta|_k$ summability.
5. For $a_{n,v}=q_v/Q_n$ we obtain the Riesz means $A_n(s)=Q_n^{-1}\sum\limits_{v=0}^{n}q_vs_v$, and $|A,\delta|_k\equiv|\overline{N},q_n,\delta|_k$ summability.

6. For $a_{n,v} = p_{n-v}q_v/R_n$, where $R_n = \sum\limits_{v=0}^{n} p_v q_{n-v}$, we obtain the generalized Nörlund means $A_n(s) = R_n^{-1} \sum\limits_{v=0}^{n} p_{n-v}q_v s_v$, and $|A, \delta|_k \equiv |N, p_n, q_n, \delta|_k$ summability.
7. For $a_{n,v} = (n+1)^{-1} P_v^{-1} \sum\limits_{k=0}^{v} p_{v-k}$, we obtain the t_n^{CN} means (see [7]) $A_n(s) = (n+1)^{-1} \sum\limits_{v=0}^{n} P_v^{-1} \sum\limits_{k=0}^{v} p_{v-k}s_k$, and $|A, \delta|_k \equiv |C^1 \cdot N_p, \delta|_k$ summability.

Now we shall discuss only some of the above cases for $\delta = 0$ (the other cases can be discussed in a similar way). For this purpose, first let us clarify that the results of [4] follow from the main results of this paper. Indeed, for $a_{n,v} = p_{n-v}q_v/R_n$ we have that

$$\begin{aligned}\hat{a}_{n,v} = \bar{a}_{n,v} - \bar{a}_{n-1,v} &= \sum_{j=v}^{n} a_{nj} - \sum_{j=v}^{n-1} a_{n-1,j} \\ &= \frac{1}{R_n} \sum_{j=v}^{n} p_{n-j}q_j - \frac{1}{R_{n-1}} \sum_{j=v}^{n-1} p_{n-1-j}q_j = \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}},\end{aligned}$$

whence

$$|\hat{a}_{n,v}|^2 = \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2.$$

Therefore, if we insert this equality, and take $\delta = 0$ and $k = 1$ in Theorems 2.1 and 2.2, then Theorems 1.1 and 1.2 follow immediately.

Also, some other known results are included in Theorem 2.1. Namely, for $a_{n,v} = p_{n-v}/P_n$ we get

$$\begin{aligned}\hat{a}_{n,j} &= \bar{a}_{n,j} - \bar{a}_{n-1,j} \\ &= \frac{1}{P_n} \sum_{i=j}^{n} p_{n-i} - \frac{1}{P_{n-1}} \sum_{i=j}^{n-1} p_{n-1-i} \\ &= \frac{1}{P_n P_{n-1}} \left(P_{n-1}P_{n-j} - P_n P_{n-1-j} \right) \\ &= \frac{1}{P_n P_{n-1}} \left((P_n - p_n)P_{n-j} - P_n(P_{n-j} - p_{n-j}) \right) \\ &= \frac{p_n}{P_n P_{n-1}} \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right) p_{n-j}.\end{aligned}$$

Hence, using Theorem 2.1 for $\delta = 0$ and $k = 1$, the following result holds.

Corollary 3.1 [1]. *If the series*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n P_{n-1}} \Big\{ \sum_{j=1}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 |c_j|^2 \Big\}^{1/2}$$

converges, then the orthogonal series $\sum_{j=0}^{\infty} c_j \varphi_j(x)$ *is* $|N, p|$*-summable almost everywhere.*

Also, for $a_{n,v} = q_v / Q_n$ one can find that

$$\hat{a}_{n,j} = \bar{a}_{n,j} - \bar{a}_{n-1,j} = -\frac{q_n Q_{j-1}}{Q_n Q_{n-1}}.$$

Therefore, using again Theorem 2.1 for $\delta = 0$ and $k = 1$, we obtain

Corollary 3.2 [2]. *If the series*

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \Big\{ \sum_{j=1}^{n} Q_{j-1}^2 |c_j|^2 \Big\}^{\frac{1}{2}}$$

converges, then the orthogonal series $\sum_{j=0}^{\infty} c_j \varphi_j(x)$ *is* $|\overline{N}, q|$*-summable almost everywhere.*

Some other interesting consequences are the corollaries formulated below.

Corollary 3.3. *If the series*

$$\sum_{n=0}^{\infty} \Big(\frac{n^{2(1-1/k)/k} p_n}{P_n P_{n-1}} \Big)^k \Big\{ \sum_{j=1}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 |c_j|^2 \Big\}^{k/2}$$

converges for $1 \leqslant k \leqslant 2$*, then the orthogonal series* $\sum_{j=0}^{\infty} c_j \varphi_j(x)$ *is* $|N, p|_k$*-summable almost everywhere.*

Remark 3.1. We note here that:

1. If $p_n = 1$ for all values of n then $|N, p|_k$ summability reduces to $|C, 1|_k$ summability
2. If $k = 1$ and $p_n = 1/(n+1)$ then $|N, p|_k$ is equivalent to $|R, \log n, 1|$ summability.

Corollary 3.4. *If the series*

$$\sum_{n=0}^{\infty}\left(\frac{n^{2(1-1/k)/k}q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{j=1}^{n}Q_{j-1}^2|c_j|^2\right\}^{k/2}$$

converges for $1\leqslant k\leqslant 2$, *then the orthogonal series* $\sum\limits_{j=0}^{\infty}c_j\varphi_j(x)$ *is* $|\overline{N},q|_k$*-summable almost everywhere.*

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