Maximal dissipation and well-posedness for the compressible Euler system

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

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Compressible Euler system

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Periodic (impermeable) boundary conditions

$$\Omega = \left[[-1,1]|_{\{-1,1\}} \right]^N, \ N = \boxed{2,3}$$

Initial data

$$\varrho(0,\cdot) = \varrho_0, \ (\varrho \mathbf{u}) = \varrho_0 \mathbf{u}_0, \ \varrho_0 > 0$$

Weak solutions

Regularity

$$\varrho \in L^\infty((0,T)\times\Omega)\cap C_{\rm weak}([0,T];L^1(\Omega)),\ \ \varrho > 0$$
$$\mathbf{u} \in L^\infty((0,T)\times\Omega;R^N),\ \ (\varrho\mathbf{u}) \in C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Equation of continuity and momentum balance

$$\int_\Omega \Big(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot) \Big)\ {\rm d}x = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \Big)\ {\rm d}x\ {\rm d}t$$

for any $\tau \in [0,T]$, and any $\varphi \in C^\infty([0,T]\times\Omega)$

$$\int_\Omega \Big((\varrho\mathbf{u})(\tau,\cdot)\cdot\varphi(\tau,\cdot) - \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot) \Big)\ {\rm d}x$$
$$= \int_0^\tau \int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho){\rm div}_x\varphi \Big)\ {\rm d}x\ {\rm d}t$$

for any $\tau \in [0,T]$, and any $\varphi \in C^\infty([0,T]\times\Omega;R^3)$

Energy and dissipation

Mechanical energy

$$\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) = \frac{1}{2}\frac{|\varrho\mathbf{u}|^2}{\varrho} + P(\varrho),\ \ P(\varrho) = \varrho\int_1^{\varrho}\frac{p(z)}{z^2}\ \mathrm{d}z$$

Energy balance - regular solutions

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\mathbf{u}\right] + \mathrm{div}_x\left(p(\varrho)\mathbf{u}\right) = 0$$

Energy dissipation - weak solutions

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\mathbf{u}\right] + \mathrm{div}_x\left(p(\varrho)\mathbf{u}\right) \le 0$$

Admissible weak solutions

Entropy inequality - weak formulation

$$\int_0^T \int_\Omega \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \partial_t \varphi + \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_\Omega p(\varrho) \mathbf{u} \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t + \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right) \varphi(0, \cdot) \ \mathrm{d}x \geq 0$$

for any $\varphi \in C_c^\infty([0, T) \times \Omega)$, $\varphi \geq 0$

Energy - weak form

$$\int_\Omega E(\tau+) \varphi \ \mathrm{d}x = \mathrm{ess} \lim_{t \to \tau+} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \varphi \ \mathrm{d}x, \ \ \tau \in [0, T)$$

$$\int_\Omega E(\tau-) \varphi \ \mathrm{d}x = \mathrm{ess} \lim_{t \to \tau-} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \varphi \ \mathrm{d}x, \ \ \tau \in (0, T]$$

Principle of maximal dissipation

Maximal dissipation [Dafermos 1974]

Let $\tau \in [0, T)$ and let $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ be another weak solution, defined in $[0, \tilde{T}]$, $\tau < \tilde{T} \leq T$, such that

$$\varrho = \tilde{\varrho}, \ \varrho \mathbf{u} = \tilde{\varrho} \tilde{\mathbf{u}} \text{ in } [0, \tau] \times \Omega.$$

Then there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$, $\tau_n > \tau$, $\tau_n \to \tau$ such that

$$\int_{\Omega} \tilde{E}(\tau_n +) \ \mathrm{d}x \geq \int_{\Omega} E(\tau_n +) \ \mathrm{d}x \text{ for all } n = 1, 2, \dots,$$

where E, $\tilde{E}$ is the mechanical energy associated with u, $\tilde{u}$

Incompressible Euler system

Equations

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

Initial conditions

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

DeLellis, Székelyhidi [2008]

There exist (infinite) set of initial data $\mathbf{v}_0$ in $L^2 \cap L^\infty(\Omega; R^3)$ such that the incompressible Euler system possesses infinitely many global in time solutions, with prescribed energy

$$\frac{1}{2}|\mathbf{v}|^2 = e, \ \ e(0, \cdot) = \frac{1}{2}|\mathbf{v}_0|^2 \text{ and the pressure } \Pi = -\frac{1}{3}|\mathbf{v}|^2$$

Infinitely many admissible solutions

Corollary of the result of DeLellis and Székelyhidi

$$\varrho(0,\cdot) = \overline{\varrho} > 0 \ \mathrm{const}, \ \ e = \mathrm{const}$$

There exists $\mathbf{u}_0 \in L^2 \cap L^\infty$ such that the *compressible* Euler system possesses infinitely many admissible weak solutions. The solutions satisfy $\varrho = \overline{\varrho}$ for all $t \geq 0$.

Improvement by Chiodaroli [2012]

$$\varrho(0,\cdot) = \varrho_0 > 0, \ \ \varrho_0 \in C^1(\Omega)$$

There exists $\mathbf{u}_0 \in L^2 \cap L^\infty$ and $T > 0$ such that the *compressible* Euler system possesses infinitely many admissible weak solutions in $[0, T]$. The solutions satisfy $\varrho = \varrho_0$ for all $t \in [0, T]$.

Method of convex integration, I

Set of subsolutions

$$X_{0,e}[0,T] = \left\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \ \Big| \right.$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbf{v}(T,\cdot) = \mathbf{v}_T, \ {\rm div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \in C^1((0,T) \times \Omega; R^3), \ \partial_t \mathbf{v} + {\rm div}_x \mathbb{U} = 0$$

$$\mbox{for a certain } \mathbb{U} \in C^1((0,T) \times \Omega; R^{3\times 3}_{{\rm sym},0}),$$

$$\left. \frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{\varrho} - \mathbb{U} \right] < e \mbox{ in } (0,T) \times \Omega \right\},$$

Distance function

$$\frac{3}{2} \lambda_{\max} \left[\frac{\mathbf{v} \otimes \mathbf{v}}{\varrho} - \mathbb{U} \right] \geq \frac{1}{2} \frac{|\mathbf{v}|^2}{\varrho},$$

Method of convex integration, II

Oscillatory lemma [DeLellis, Székelyhidi]

For any $\mathbf{v} \in X_{0,e}[0,T]$ there exist sequences

$$\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0,T)\times\Omega; R^3),\ \{\mathbb{U}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0,T)\times\Omega; R^3)$$

such that the functions $\mathbf{v} + \mathbf{w}_n$ belong to $X_{0,e}[0,T]$, with the associated tensor fields $\mathbb{U} + \mathbb{U}_n$,

$$\mathbf{w}_n \to 0 \text{ in } C_{\rm weak}([0,T]; L^2(\Omega; R^3)),$$

and

$$\liminf_{n\to\infty} \|\mathbf{w}_n\|^2_{L^2((0,T)\times\Omega)} \geq \Lambda \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(e - \frac{1}{2}\frac{|\mathbf{v}|^2}{\varrho} \right)^2 \ {\rm d}x\ {\rm d}t,\ \ \Lambda > 0,$$

where the constant Λ depends only on the norm of the quantities ϱ, ϱ^{-1}, e in $L^{\infty}((0,T)\times\Omega)$.

Problem with constant (in time) density

"Ansatz"

$$\varrho = \varrho_0 \in C^1(\Omega), \ \mathbf{v} = \varrho \mathbf{u}$$

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{\mathbf{v} \otimes \mathbf{v}}{\varrho} \right) + \nabla_x \Pi = 0$$

Solutions by convex integration

$$\frac{1}{2} \frac{|\mathbf{v}|^2}{\varrho} = \frac{1}{2} \varrho |\mathbf{u}|^2 = e$$

$$\Pi = -\frac{1}{3} \frac{|\mathbf{v}|^2}{\varrho} = -\frac{2}{3} e$$

Choice of kinetic energy

$$\Pi = p(\varrho) - \frac{2}{3}\chi(t) = -\frac{2}{3} e \ \Rightarrow e = \chi(t) - \frac{3}{2} p(\varrho)$$

Construction by convex integration

Step 1: Energy

Choose e (or rather χ) to ensure validity of the energy inequality

Step 2: Subsolution

Make sure that the space of subsolutions is non-empty

Step 3: Suitable subsolutions

Construct a new space of subsolutions (on a possibly shorter time interval) so that

$$\frac{1}{2}\frac{|\mathbf{v}_0|^2}{\varrho} = e(0, \cdot)$$

Modification via Helmholtz decomposition

Helmholtz decomposition

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \operatorname{div}_x \mathbf{v} = 0$$

Reformulation

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x \left(\partial_t \Psi + p(\varrho) \right) = 0$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Psi|^2}{\varrho} = e$$

Application of convex integration

Set of subsolutions

$$X_{0,e}[0,T] = \Big\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \ \Big|$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbf{v}(T,\cdot) = \mathbf{v}_T, \ {\rm div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \in C^1((0,T) \times \Omega; R^3), \ \partial_t \mathbf{v} + {\rm div}_x \mathbb{U} = 0$$

for a certain $\mathbb{U} \in C^1((0,T) \times \Omega; R^{3\times 3}_{{\rm sym},0})$,

$$\frac{3}{2} \lambda_{\rm max} \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] < e \text{ in } (0,T) \times \Omega \Big\},$$

Distance function

$$\frac{3}{2} \lambda_{\rm max} \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] \geq \frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Psi|^2}{\varrho},$$

Convex integration revisited

Step 1: Energy

Choose e (or rather χ) to ensure validity of the energy inequality

Step 2: Subsolution

Make sure that the space of subsolutions is non-empty

Step 3: Suitable subsolutions

Construct a new space of subsolutions so that

$$\frac{1}{2} \frac{|\mathbf{v}_0 + \nabla_x \Psi_0|^2}{\varrho_0} = e(0, \cdot)$$

Global admissible solutions

Global-in-time solutions

$$\varrho_0 \in C^1(\Omega),\ \varrho_0 > 0,\ \int_\Omega \varrho_0 \, \mathrm{d}x = M_0,\ |\nabla_x \varrho_0| < \varepsilon(M_0)$$

There exists $\mathbf{u}_0$ such that the compressible Euler system admits infinitely many global-in-time admissible solutions

Remark

The solutions coincide with the static state $[\overline{\varrho}, 0]$ for t large

Convex integration and maximal dissipation

Kinetic energy

$$\frac{1}{2}\varrho|\mathbf{u}|^2(t,\cdot) = e(t,\cdot),\ \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 = e(0,\cdot)$$

Construction

- Choose $\tilde{e}$,

$$\tilde{e}(0,\cdot) = e(0,\cdot),\ \tilde{e}(T,\cdot) = e(T,\cdot)$$

$$\tilde{e}(t,\cdot) < e(t,\cdot),\ t \in (0,T)$$

- Take

$$X_{0,\tilde{e}}[0,T] \subset X_{0,e}[0,T]$$

- Make sure

$$\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 = e(0,\cdot),\ X_{0,\tilde{e}}[0,T] \neq \emptyset$$