# On the weak solutions to the Euler-Fourier system

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

joint work with E.Chiodaroli (Zürich) and O.Kreml (Zürich, Praha)

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# Basic field equations

**Mass conservation**

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

**Momentum balance**

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

**Internal energy balance**

$$\partial_t (\varrho e(\varrho, \vartheta)) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \mathbf{q} = - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

# Constitutive relations

**Fourier's law**

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

**Gibbs' equation**

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left( \frac{1}{\varrho} \right)$$

**Thermodynamics stability**

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

# Boundary and initial conditions

**Periodic boundary conditions**

$$x \in \mathcal{T} = \left( [-1,1]|_{\{-1;1\}} \right)^N, \ \boxed{N = 3}$$

**Initial conditions**

$$\varrho(0,\cdot) = \varrho_0 > 0, \ \vartheta(0,\cdot) = \vartheta_0 > 0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

# Local well posedness

### Regularity of the data

$$\varrho_0, \ \vartheta_0, \ \mathbf{u}_0 \in W^{m,2}, \ m \geq 3$$

### Local in time existence

T. Alazard [2006] - local existence for large data
D. Serre [2008, 2012]

# Euler-Fourier system

**Mass conservation**

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

**Momentum balance**

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

**Internal energy balance**

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

# Existence of weak solutions

**Initial data**

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

**Global existence**

For any (smooth) initial data $\varrho_0$, $\vartheta_0$, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

**Regularity class**

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta \ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

# Incompressible Euler system

**Equations**

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

**Initial conditions**

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

**DeLellis, Székelyhidi [2008]**

There exist (infinite) set of initial data $\mathbf{v}_0$ in $L^2 \cap L^\infty(\Omega; R^3)$ such that the incompressible Euler system possesses infinitely many global in time solutions, with prescribed energy

$$\frac{1}{2}|\mathbf{v}|^2 = e, \ \ e(0, \cdot) = \frac{1}{2}|\mathbf{v}_0|^2 \text{ and the pressure } \Pi = -\frac{1}{3}|\mathbf{v}|^2$$

# Method of convex integration, I

## Set of subsolutions

$$X_{0,e}[0,T] = \Big\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3) \ \Big|$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbf{v}(T,\cdot) = \mathbf{v}_T, \ {\rm div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \in C^1((0,T) \times \Omega; R^3), \ \partial_t \mathbf{v} + {\rm div}_x \mathbb{U} = 0$$

$$\mbox{for a certain } \mathbb{U} \in C^1((0,T) \times \Omega; R^{3\times 3}_{{\rm sym},0}),$$

$$\frac{3}{2} \lambda_{\rm max} \left[ \mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < e \mbox{ in } (0,T) \times \Omega \Big\},$$

## Distance function

$$\frac{3}{2} \lambda_{\rm max} \left[ \mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] \geq \frac{1}{2} |\mathbf{v}|^2$$

$$e = \frac{1}{2} |\mathbf{v}|^2 \ \Rightarrow \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I}$$

# Method of convex integration, II

**Oscillatory lemma [DeLellis, Székelyhidi]**

For any $\mathbf{v} \in X_{0,e}[0, T]$ there exist sequences

$$\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0, T) \times \Omega; R^3), \ \{\mathbb{U}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0, T) \times \Omega; R^3)$$

such that the functions $\mathbf{v} + \mathbf{w}_n$ belong to $X_{0,e}[0, T]$, with the associated tensor fields $\mathbb{U} + \mathbb{U}_n$,

$$\mathbf{w}_n \to 0 \text{ in } C_{\rm weak}([0, T]; L^2(\Omega; R^3)),$$

and

$$\liminf_{n \to \infty} \|\mathbf{w}_n\|^2_{L^2((0,T)\times\Omega)} \geq \Lambda \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \ {\rm d}x \ {\rm d}t, \ \Lambda > 0,$$

where the constant $\Lambda$ depends only on the norm of $e$ in $L^{\infty}((0, T) \times \Omega)$.

# Application of convex integration, I

**Ansatz**

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \operatorname{div}_x \mathbf{v} = 0$$

**Equations**

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left( \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \Big( \partial_t (\varrho \vartheta) + \operatorname{div}_x \big( \vartheta (\mathbf{v} + \nabla_x \Psi) \big) \Big) - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \left( \frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

**"Energy"**

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

# Application of convex integration, II

**Fixing the density and the acoustic potential**

$$\varrho = \varrho(t,x),\ \Psi = \Psi(t,x),\ \Psi = -\Delta^{-1}[\partial_t \varrho]$$

**Computing the temperature as a function of v**

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x\Big(\vartheta(\mathbf{v} + \nabla_x \Psi)\Big)\right) - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x\left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho}\right)$$

$$\|\vartheta\|_{L^\infty((0,T)\times\Omega)} \leq c(\varrho, \Psi) \text{ independent of } \mathbf{v}$$

**Kinetic energy**

$$e = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}]$$

# Application of convex integration, III

**Set of subsolutions**

$$X_{0,e}[0,T] = \Big\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \ \Big|$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbf{v}(T,\cdot) = \mathbf{v}_T, \ {\rm div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \in C^1((0,T) \times \Omega; R^3), \ \partial_t \mathbf{v} + {\rm div}_x \mathbb{U} = 0$$

$$\text{for a certain } \mathbb{U} \in C^1((0,T) \times \Omega; R^{3\times 3}_{{\rm sym},0}),$$

$$\frac{3}{2} \lambda_{\max} \left[ \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] < e[\vartheta] \text{ in } (0,T) \times \Omega \Big\},$$

**Distance function**

$$\frac{3}{2} \lambda_{\max} \left[ \frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] \geq \frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Psi|^2}{\varrho},$$

# Energy solutions

**Energy conservation**

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x = 0$$

# Relative entropy (energy)

## Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\Big|\, r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

## Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho \Big( e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

## Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

# Dissipative solutions

**Relative entropy inequality**

$$\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)\right]_{t=0}^{\tau} - \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

# Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U}\Big)\cdot(\mathbf{U} - \mathbf{u})\ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r,\Theta) - p(\varrho,\vartheta)\Big)\mathrm{div}\mathbf{U} + \frac{\varrho}{r}(\mathbf{U}-\mathbf{u})\cdot\nabla_x p(r,\Theta)\right]\ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\mathbf{u}\cdot\nabla_x\Theta\right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\Theta\right)\ \mathrm{d}x$$

$$+ \int_\Omega \frac{r-\varrho}{r}\Big(\partial_t p(r,\Theta) + \mathbf{U}\cdot\nabla_x p(r,\Theta)\Big)\ \mathrm{d}x$$

# Dissipative solutions to the Euler-Fourier system

**Initial data**

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

**Infinitely many dissipative weak solutions**

For any regular initial data $\varrho_0$, $\vartheta_0$, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$