On the well-posedness problems for (certain) complete fluid systems

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Euler-Fourier system

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = \boxed{0}$$

Energy balance

$$\frac{3}{2}\left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\right] - \boxed{\Delta \vartheta} = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \ \vartheta(0,\cdot) = \vartheta_0,\ \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Well posedness

Local existence of classical solutions [Alazard , Serre (2008)]

For any given smooth initial data,

$$\varrho_0 > 0, \ \vartheta_0 > 0,$$

there exists $T_{\max}$ such that the system possesses a classical solution defined on the time interval $(0, T_{\max})$. The classical solution is unique in the class of smooth solutions.

Global existence of weak solutions [Chiodaroli, F., Kreml (2013)]

For any given smooth initial data,

$$\varrho_0 > 0, \ \vartheta_0 > 0,$$

and $\boxed{\textit{any}}$ $T > 0$, the system admits infinitely many (bounded) weak solutions in $(0, T)$.

Convex integration - DeLellis and Shékelyhidi

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \right) + \nabla_x \Pi = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbf{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I},\ \mathbb{U} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Convex integration continued

Implicit constitutive relation

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]$$

$$\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2$$

$$\boxed{\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}|\mathbf{v}|^2\mathbb{I}$$

Convex integration - subsolutions

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0,\ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e \text{ in } (0, T)$$

Oscillatory lemma

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0,\ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon,\ \mathbf{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(V)$$

$$\lambda_{\max}\left[(\mathbf{v}+\mathbf{w}_\varepsilon)\otimes(\mathbf{v}+\mathbf{w}_\varepsilon) - (\mathbb{U}+\mathbb{V}_\varepsilon)\right] < e$$

Energy

$$\liminf_{\varepsilon\to 0} \int_V \left(|\mathbf{v}+\mathbf{w}_\varepsilon|^2\right) \geq \int_V |\mathbf{v}|^2 + c\int_V \left(e - \frac{1}{2}|\mathbf{v}|^2\right)^\alpha$$

Infinitely many solutions

Necessary condition

There exists e such that the set of subsolutions is non-empty

Infinitely many solutions

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) - \boxed{\nabla_x \left(\frac{1}{3} |\mathbf{v}|^2 \right)} = 0$$

Pressure

$$\frac{1}{2} |\mathbf{v}|^2 = e,\ p = -\frac{1}{3} |\mathbf{v}|^2 = -\frac{2}{3} e \text{ in } (0, T)$$

Non-constant coefficients

Convex set

$$\frac{1}{2}\frac{1}{r(t,x)}|\mathbf{v}+\mathbf{q}(t,x)|^2$$

$$\leq \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{q}(t,x))\otimes(\mathbf{v}+\mathbf{q}(t,x))}{r(t,x)} - \mathbb{U}\right] < e \text{ in } (0,T)$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t (\mathbf{v}+\mathbf{q}) + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{q})\otimes(\mathbf{v}+\mathbf{q})}{r}\right) - \frac{1}{3}\nabla_x\left(\frac{|\mathbf{v}+\mathbf{q}|^2}{r}\right) = 0$$

Energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{q}(t,x)|^2}{r(t,x)} = e(t,x)$$

Applications to the Euler-Fourier system, I.

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \Big(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \Big) - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

Energy

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}] - \frac{3}{2} \partial_t \Psi$$

Application to the Euler-Fourier system, II

Construction of solutions

1. Fix ϱ and compute the acoustic potential Ψ

$$-\Delta \Psi = \partial_t \varrho$$

2. Compute $\vartheta = \vartheta[\mathbf{v}]$ for $\mathbf{v} \in L^\infty$

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x\Big(\vartheta(\mathbf{v} + \nabla_x \Psi)\Big)\right) - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho}\right)$$

3. Observe that $0 < \vartheta < \overline{\vartheta}$, $\overline{\vartheta}$ independent of $\mathbf{v}$
4. Take

$$e = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}]$$

and use the non-local variant of the results od DeLellis and Székelyhidi for the *incompressible* Euler system to find $\mathbf{v}$

Conservative solutions to the Euler-Fourier system

Total energy conservation

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{3}{2}\varrho\vartheta \right)(\tau,\cdot)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{3}{2}\varrho_0\vartheta_0 \right)\ \mathrm{d}x$$

Weak-strong uniqueness [Dafermos], [F. and Novotný]

A strong solution coincides with any conservative weak solution on their joint existence interval. Strong solutions are unique in the class of conservative weak solutions.

Existence of conservative solutions

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Conservative weak solutions [Chiodaroli, F., Kreml [2013]]

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many conservative weak solutions in $(0, T)$

Weak formulation revisited

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma \boxed{\geq} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta^2}$$

Total energy conservation

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{3}{2} \varrho \vartheta \right) (\tau, \cdot) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{3}{2} \varrho_0 \vartheta_0 \right) \ \mathrm{d}x$$

Maximal dissipation criterion?

Entropy production rate

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$s(\varrho, \vartheta) = -\log\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

$$\sigma \boxed{\geq} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta^2}$$

Maximal dissipation [Dafermos 1974]

- Maximize the entropy production rate σ
- Maximize the total entropy $\int_\Omega \varrho s(\varrho, \vartheta)\ \mathrm{d}x$
- Maximize the entropy $\varrho s(\varrho, \vartheta)$