Kreisel's Conjecture for L$\exists_1$

Matthias Baaz Pavel Pudlák

Introduction

The famous conjecture of Kreisel states:

> *If there exists k such that* $\mathbf{T}$ *proves* $\varphi(S^n(0))$ *in* $\leq k$ *steps for every* n*, then* $\mathbf{T}$ *proves* $\forall x \varphi(x)$.

Here $S^n(\ldots)$ denotes n-times iterated successor function. The conjecture was proposed for $\mathbf{T}$ = Peano Arithmetic in the formalization of Kleene (cf. [Parikh], [Friedman] problem N.34, [Kreisel]), but it is interesting for any $\mathbf{T}$ which contains some fragment of number theory. We shall prove Kreisel's conjecture for an axiomatic system with finitely many (basic) axioms and the schema of the least number principle for existential formulas $\mathbf{L\exists_1}$.

It should be stressed that the answer to the conjecture depends not so much on the sentences provable in $\mathbf{T}$ as it does depend on the presentation of $\mathbf{T}$ and the logical calculus used. For example, it is well known that $\mathbf{L\exists_1}$ is equivalent to $\mathbf{I\Sigma_1}$ [Kaye] and that $\mathbf{I\Sigma_1}$ is finitely axiomatizable. But our result does not imply anything about $\mathbf{I\Sigma_1}$ and it is not implied by the fact that for every sufficiently strong *finite* axiomatic system Kreisel's conjecture holds. Thus by writing $\mathbf{L\exists_1}$, $\mathbf{I\Sigma_1}$ etc. we shall always mean an axiomatic system, not just a deductively closed set of sentences. The reason is that the fact that $\mathbf{S}$ and $\mathbf{T}$ have the same logical consequences does not imply that we have the same bounds to the number of steps in the proofs. For instance one can prove the following statements:

Proposition 1.1 *(a) Let* $\mathbf{T}_0$ *be a finite axiomatization of a consistent number theory* $\mathbf{T}$ *axiomatized by quantifier-free induction scheme or identity scheme* $t = s \rightarrow \psi(t) = \psi(s)$ *(*$\psi(x)$ *arbitrary term). Then there is no* k *such that all axioms of* $\mathbf{T}$ *can be proved from* $\mathbf{T}_0$ *within* k *steps.*

(b) For each recursively enumerable consistent number theory $\mathbf{T}_0$ *there is a number theory* $\mathbf{T}$ *with exactly the same derivable formulas such that Kreisel's conjecture does not hold for* $\mathbf{T}$.

Proof. *(a)* By Theorem 6.1 of [Krajíček/Pudlák] and cut–elimination there exists a sequence of terms $\tau_1, \ldots, \tau_n$ for each formula $\varphi(x)$ and k such that

(*) $\mathbf{T}_0 \vdash \varphi(\tau_i)$, $1 \leq i \leq n$, and $\mathbf{T}_0 \vdash^k \varphi(\tau)$ implies that τ is substitution instance of some τ_i.

Let $d_0 := 0$, $d_{r+1} := d_r + 0$. Then

(**) there is a k such that for all r $\mathbf{T} \vdash^k d_r = 0$.

This was shown in [Yukami] using $\forall x(x = 0 \rightarrow \psi(x) = \psi(0))$ for an adequate φ. This principle is derivable using the same skeleton for all q by quantifier-free induction on $x = 0 \rightarrow \psi(x) = \psi(0)$. (*) contradicts (**) if there is a k such that all axioms of $\mathbf{T}$ are derivable from $\mathbf{T}_0$ within k steps.

(b) Consider a formula $\varphi(x)$ such that $\mathbf{T}_0 \vdash \varphi(S^n(0))$ for all n, but $\mathbf{T}_0 \not\vdash \forall x \varphi(x)$. Let $\mathbf{T} := \mathbf{T}_0 \cup \{\varphi(S^n(0)) | n \in \mathsf{N}\}$. $\square$

Clearly, it is also important that we talk about the number of steps (proof lines, sequents, formulas in the proof) but not about the total size of the proof (e.g. the length of a string encoding the proof). For the other concept the conjecture is trivial. The point is that there can be infinitely many proofs with $\leq k$ steps (for k sufficiently large) even if we consider the proofs of a fixed formula. The essential reason is that there can be arbitrarily large terms in short proofs.

Kreisel's conjecture does not have any immediate consequences relevant to other topics in logic and number theory. However it is very stimulating if we want to learn something about the role of terms in first order logic. This may have eventually quite practical consequences, since manipulation with terms is very important in computer science. In fact, our main tool — semiunification of terms — appeared in this area and in computer science almost at the same time (cf. [Baaz2], [Kfoury et al.]). The idea of using semiunification for Kreisel's conjecture is due to the first author, who used Hilbert's ε–calculus. The second author adapted it so that Herbrand's theorem is used instead.

1 Preliminaries

We shall consider a standard Hilbert style system of first order logic, say as in [Barwise]. *A proof of* φ *in* $\mathbf{T}$ *with* k *steps* is a sequence of formulas $< \varphi_1, \ldots, \varphi_k >$ such that each φ is either an axiom (logical or in $\mathbf{T}$) or follows from preceding ones by a logical rule; notation $\mathbf{T} \vdash^k \varphi$. The language of $\mathbf{T}$ should contain the constant 0 and the successor function $S(x)$ and a finite number of function and predicate symbols. For the general result from which we derive Kreisel's conjecture we even do not need 0 and S.

In the proof we shall need arrays of variables and terms. A d-dimensional array will always be marked with the left superscript d, e.g. ${}^d a$. By writing, say ^{2}a, we mean *the* *string* $a_{11}, a_{12}, \dots, a_{1n}$, $a_{21}, a_{22}, \dots, a_{2n}$, $a_{m1}, a_{m2}, \dots, a_{mn}$; ${}^2a_i$ is the string $a_{i1}, \dots, a_{in}$; ${}^2a_{ij}$ is just a_{ij} (thus we can omit the superscript here). A similar convention will be used for $d > 2$.

The schema $\mathbf{L\exists_1}$ are all formulas

$$\exists y_1 \forall z_1 (\exists z_2 \dots \exists_n \gamma({}^1p, {}^1z) \rightarrow \exists y_2 \dots \exists y_n \gamma({}^1p, {}^1y) \wedge y_1 \leq z_1), \tag{1}$$

where γ is an open formula. This schema is equivalent to

$$\exists^1 y \forall^1 z (\gamma({}^1p, {}^1z) \rightarrow \exists y_2 \dots \exists y_n \gamma({}^1p, {}^1y) \wedge {}^1y_1 \leq {}^1z_1), \tag{2}$$

which will be denoted by $\overline{\mathbf{L\exists_1}}$. The number of steps of the proof of this equivalence depends only on n. We shall see below that for a fixed k, we can bound n in proofs with k steps which use $\mathbf{L\exists_1}$. Thus for every k and r, there exists K such that if a formula of rank $\leq r$ (see below for the definition) can be derived from $\mathbf{L\exists_1}$ in k steps, then the formula can be derived from $\overline{\mathbf{L\exists_1}}$ in K steps.

Because of the deduction theorem, if Kreisel's conjecture holds for $\mathbf{T}$, then it holds for $\mathbf{T}$ augmented by any finite set of axioms. Thus we may assume that $\mathbf{T}$ consists only of the scheme $\overline{\mathbf{L\exists_1}}$.

We should point out that we use logic without equality which means that we allow the equality sign, but we treat it as a nonlogical predicate. As we assume a finite language, we need only finitely many equality axioms. It is an open problem what happens if we add infinitely many equality axioms, for instance the *schema of equality*

$$t = s \rightarrow \psi(t) \equiv \psi(s),$$

where $\psi(x)$, t, s are arbitrary terms. Kreisel's conjecture may still hold true, but our tool — Theorem 4.2 — fails (cf. Proposition 1.1).

2 Bounding the rank of formulas

Definition 2.1 *The* rank *of a formula φ is inductively defined as follows:*

(i) *rank*$(\varphi)=0$ *if φ is atomic;*

(ii) *rank*$(\varphi \wedge \psi) = 1 + \max(\mathit{rank}(\varphi), \mathit{rank}(\psi))$,
rank$(\neg\varphi) = 1 + \mathit{rank}(\varphi)$,
rank$(\exists x \varphi) = 1 + \mathit{rank}(\varphi)$,
etc. for other connectives and $\forall$.

The rank of a proof d is the maximal rank of a formula of d.

Our aim is to show that the rank of a proof with $\leq k$ steps can be bounded in the following sense.

Lemma 2.1 *For every k and r there exists R such that for every formula φ, if rank$(\varphi) \leq r$ and $\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi$, then there exists a proof d such that d: $\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi$ and rank$(d) \leq R$.*

Proof. The proof is essentially the same as in [Parikh]. But since we use a special scheme, we have to check that the same idea works also here.

Let $\varphi_1, \ldots, \varphi_k$ be a proof of φ (i.e. $\varphi = \varphi_k$) of rank $\leq r$. We construct a unification problem. We consider the language consisting of a constant c for atomic formulas, distinct variables v_β for each subformula β of the proof and function symbols for each connective and each quantifier: $f_\rightarrow$, $f_\neg$, $f_\exists$ etc.

The equations of the unification problem will be the following:

(1) For each application of a propositional rule in the proof, say

$$\frac{\alpha, \quad \alpha \rightarrow \beta}{\beta}$$

we add the equation

$$v_{\alpha\rightarrow\beta} = f_\rightarrow(v_\alpha, v_\beta).$$

(2) For each application of a quantifier rule, say

$$\frac{\alpha \rightarrow \beta}{\exists x\alpha \rightarrow \beta}$$

we add the equations

$$v_{\alpha\rightarrow\beta} = f_\rightarrow(v_\alpha, v_\beta).$$

$$v_{\exists\alpha\rightarrow\beta} = f_\rightarrow(f_\exists(v_\alpha), v_\beta).$$

If we have a quantifier axiom, say

$$\alpha(t) \rightarrow \exists x\alpha(x), \quad t \text{ a term},$$

we add the equation

$$v_\beta = f_\rightarrow(v_\gamma, f_\exists(v_\gamma)).$$

where β, γ are respectively

– the instance of the axiom above,

– $\alpha(x)$.

(3) For each instance of $\mathbf{L}\exists_\mathbf{1}$ which occurs in the proof, say

$$\exists y_1 \forall z_1(\exists z_2 \ldots \exists_n \gamma({}^1p, {}^1z) \rightarrow \exists y_2 \ldots \exists y_n \gamma({}^1p, {}^1y) \wedge y_1 \leq z_1),$$

we add an equation

$$v_\alpha = f_\exists f_\forall f_\rightarrow(v_\beta, f_\wedge(v_\beta, c_\delta)),$$

where α, β, δ are respectively

– the instance of $\mathbf{L\exists_1}$ above,

– $\exists z_2 \dots \exists_n \gamma({}^1p, {}^1z)$,

– $y_1 \leq z_1$.

(4) Finally we add

$$v_\varphi = \tau,$$

where τ is the term constructed from the constant corresponding to atomic formulas of φ and function symbols corresponding to the connectives and quantifiers in φ and which is isomorphic to φ.

Claim 1. *The size of the unification problem depends only on k and r.*

To prove the claim we just note that the size of each equation in (1) – (3) is fixed, and there are at most $2k$ such equations; the size of the equation (4) is bounded in terms of $rank(\varphi)$ which is $\leq r$.

Observe that the proof $\varphi_1, \dots, \varphi_k$ determines a solution to the unification problem, if we do not distinguish different atomic formulas and bounded variables. By the claim the most general unifier has size bounded in terms of k and r. Using such a unifier we construct a proof whose rank is bounded in terms of k and r. As the original proof is obtained by substitution from the most general unifier, we can think of it as a sequence $\varphi'_1, \dots, \varphi'_k$ obtained from $\varphi_1, \dots, \varphi_k$ by

(i) replacing some atomic formulas by the constant c,

(ii) replacing some subformulas by variables,

(iii) omitting the bounded variables from quantifiers.

Pick a suitable formula ξ so that it has small size (a constant independent of k and r) it is open, and does not contain logical variables occurring in the proof; say $0 = 0$. Construct $\varphi''_1 \dots \varphi''_k$ as follows. If c or a variable of φ'_i corresponds to some atomic formula α in φ_i, replace it by α. If a variable of φ'_i corresponds to a non-atomic formula, replace it by ξ. Finally, add the bounded variables to quantifiers according to $\varphi_1, \dots, \varphi_k$.

Claim 2. *$\varphi''_1, \dots, \varphi''_k$ is an $\mathbf{L\exists_1}$ proof of φ.*

We shall check that for every $i \leq k$, if φ_i is an instance of $\mathbf{L\exists_1}$ or follows from some previous formulas by a rule, then φ''_i is an instance of $\mathbf{L\exists_1}$ or etc. For propositional rules this is almost trivial. For quantifier rules we have to check that the conditions on occurrences of variables are satisfied. By our choice of ξ no clashes of variables can be introduced.

Now suppose φ_i is $\alpha(t) \to \exists x \alpha(x)$. Since φ_i is a substitution instance of φ'_i and by the construction of φ''_i from φ'_i, φ''_i must have the following form: it is φ_i with some pairs of subformulas $\psi(t)$ of $\alpha(t)$ and $\psi(x)$ of $\alpha(x)$ replaced both by ξ. Thus φ''_i is of the form $\alpha'(t) \to \exists x \alpha'(x)$ for some α. Exactly the same argument works for instances of $\mathbf{L\exists_1}$.

It remains to observe that $\varphi''_k = \varphi$ because of the equation (4). $\square$

Corollary 2.2 *For every k and r there exist K and R' such that for every formula φ, if rank$(\varphi) \leq r$ and $\mathbf{L\exists_1} \vdash^k \varphi$, then there exists a proof d' such that $\overline{\mathbf{L\exists_1}} \vdash^k \varphi$ and rank$(d') \leq R'$.*

Proof. Let d be the proof from Lemma 2.1. To obtain d' we insert a proof of the particular instance of $\mathbf{L\exists_1}$ from $\overline{\mathbf{L\exists_1}}$. The number of steps and the rank of the inserted proofs can be bounded using the ranks of the instances of $\mathbf{L\exists_1}$ which in turn are bounded in terms of k and r. $\square$

Lemma 2.3 *For every k and r there exist K and R such that for every formula φ, if rank$(\varphi) \leq r$ and $\mathbf{L\exists_1} \vdash^k \varphi$, then there exist a finite set A_0 of universal closures of instances of $\overline{\mathbf{L\exists_1}}$ and a proof d in pure predicate calculus of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$ such that d has $\leq K$ steps and rank$(d) \leq R$.*

Proof. Apply the deduction theorem. Clearly the increase of the number of steps and rank is bounded in the original parameters. $\square$

Now we can go on in the same manner and show that we can transform $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$ into a prenex normal form while keeping the number of steps and the rank bounded. In fact we can choose any of the possible prenex normal forms.

3 Herbrand's theorem

Let A be a formula in prenex normal form

$$\exists^2 x_1 \forall^2 y_1 \exists^2 x_2 \forall^2 y_2 \ldots \alpha(^1p, {}^2x_1, {}^2y_1, {}^2x_2, {}^2y_2, \ldots).$$

We define $He(A)$, the Herbrand form of A, to be

$$\exists^2 x_1 \exists^2 x_2 \ldots \alpha(^1p, {}^2x_1, {}^2f_1(^1p, {}^2x_1), {}^2x_2, {}^2f_2(^1p, {}^2x_1, {}^2x_2), \ldots).$$

where ${}^2f$ is an array of new function symbols.

Lemma 3.1 *A is provable in $\mathbf{T}$ iff $He(A)$ is provable in $\mathbf{T}$; $\mathbf{T}$ is any theory not containing the new function symbols of $He(A)$. Moreover the number of steps in the proof of $He(A)$ can be bounded using the number of steps in the proof of A and $rank(A)$ and vice versa.*

Proof. It is a well-known fact. $\Box$

Let us write $He(A)$ in the form

$$\exists^1 u \beta({}^1p, {}^1u).$$

Then we call a term instance of $He(A)$ any formula of the form

$$\beta({}^1p, {}^1t).$$

where ${}^1t$ is an array of terms.

Lemma 3.2 (Herbrand's Theorem) *(i) For every k and r, there exists a K such that for every formula in prenex normal form A, if A is provable in pure predicate calculus by a proof with $\leq k$ steps and rank $\leq r$, then there exist $\vartheta_1, \ldots, \vartheta_l$, $l \leq K$, term instances of $He(A)$ such that*

$$\vartheta_1 \vee \ldots \vee \vartheta_l$$

is a propositional tautology.

(ii) If $\vartheta_1 \vee \ldots \vee \vartheta_l$ is a propositional tautology where $\vartheta_1, \ldots, \vartheta_l$, are term instances of $He(A)$, A in a prenex normal form, then A is provable.

Proof. *(i)* Apply any of the standard constructive proofs of Herbrand's theorem. For instance let us consider cut-elimination in a Gentzen style calculus. There the increase can be bounded by the rank of cut formulas and their number.

(ii) This is a well-known fact. $\Box$

We shall consider Herbrand proofs of only a special class of formulas, namely $\bigwedge\!\!\!\bigwedge A_0 \rightarrow \varphi$ where A_0 are universal closures of instances of $\overline{\mathbf{L}\exists_\mathbf{1}}$ and φ is existential. They can be presented in a special form which we shall describe now.

Let φ be $\exists^1 x \psi({}^1x)$, and let the elements of A_0 be

$$\forall^1 p \exists^2 y_i \forall^2 z_i (\gamma_i({}^1p, {}^2z_i) \rightarrow \gamma_i({}^1p, {}^2y_i) \wedge {}^2y_{i1} \leq {}^2z_{i1}).$$

Then we take the following prenex normal form of $\bigwedge\!\!\!\bigwedge A_0 \rightarrow \varphi$:

(1) $$\exists^1 p \forall^2 y_i \exists^2 z_i \exists^1 x \bigvee\!\!\!\bigvee_i [(\gamma_i({}^1p, {}^2z_i) \rightarrow \gamma_i({}^1p, {}^2y_i) \wedge {}^2y_{i1} \leq {}^2z_{i1}) \rightarrow \psi({}^1x)].$$

We shall abbreviate the part in square brackets by

$$\Gamma_i({}^1p, {}^2z_i, {}^2y_i, {}^2y_{i1}, {}^2z_{i1}) \to \psi({}^1x).$$

Hence a Herbrand proof of the formula has the form

$$\bigvee\!\!\!\bigvee_{j.i}\Gamma_i({}^2s_j, {}^3q_{ji}, {}^2f_{i1}({}^2s_j), q_{ji1}) \to \psi({}^2u_j),$$

where ${}^2s$, ${}^3q$, ${}^2u$ are arrays of terms and ${}^2f$ is the array of new function symbols.

It is possible that for some $j \neq j'$ the strings ${}^2s_j$ and ${}^2s_{j'}$ are equal, while some $q_{jim} \neq q_{j'im}$ or $u_{jm} \neq u_{j'm}$. Therefore we group the disjuncts according to the strings ${}^2s_j$. After introducing new indices and renaming the arrays we get thus

$$\bigvee\!\!\!\bigvee_{j.i,l}\Gamma_i({}^2s_j, {}^4q_{jil}, {}^2f_i({}^2s_j), f_{i1}({}^2s_j), q_{ji1}) \to \psi({}^3u_{jl})).$$

In order to be able to prove that there exists a most general Herbrand proof with a given skeleton (see below for definition), we shall allow looser structur than the above one. Namely we allow Γ_i to depend also on j, hence also ${}^2f_i$ will depend on j. Thus we get

$$\bigvee\!\!\!\bigvee_{j.i,l}\Gamma_{ji}({}^2s_j, {}^4q_{jil}, {}^3f_{ji}({}^2s_j), f_{ji1}({}^2s_j), q_{jil1}) \to \psi({}^3u_{jl})).$$

This transformation is quite formal, we only added another index to Γ_i's and f_i's. However there may be propositional tautologies of this form where $\Gamma_{ji} \neq \Gamma_{j'i}$ and they can be interpreted as Herbrand proofs of φ from $\overline{\mathbf{L}\exists_1}$. Thus for the purpose of proving Kreisel's conjecture for $\mathbf{L}\exists_1$ we shall use the above shape of a formula as our notion of a *Herbrand proof.*

We shall analyze further this formula. It is possible (and actually happens, if we derive the formula as above) that $f_{jim} = f_{j'im}$ for $j \neq j'$. But note that, since ${}^2s_j \neq {}^2s_j'$ for $j \neq j'$, the terms

$$f_{jim}({}^2s_j) \quad \text{and} \quad f_{j'i'm'}({}^2s_{j'})$$

are always different if $i \neq i'$ or $j \neq j'$ or $m \neq m'$. Furthermore we can order the disjunction in such a way that

$$j < j' \Rightarrow \max_m |s_{jm}| \leq \max_m |s_{j'm}|,$$

where $|\dots|$ denotes the size of a term. We shall include these to conditions into our definition of a Herbrand proof.

Lemma 3.3 *(i) For every k and r, there exists K such that if $\mathbf{L}\exists_1 \vdash \exists^1 x\psi({}^1x)$ then, for some formulas γ_{ij} and terms ${}^2s$, ${}^4q$, ${}^3u$*

$$(2) \qquad \bigvee\!\!\!\bigvee_{j.i,l}[\gamma_{ji}({}^2s_j, {}^4q_{jil}) \to \gamma_{ji}({}^2s_j, {}^3f_{ji}({}^2s_j)) \wedge {}^3f_{ji1}({}^2s_j) \leq {}^4q_{jil1}) \to \psi({}^3u_{jl})]$$

is a Herbrand proof where the size of the arrays and the rank of each γ_{ji} is bounded by K.

(ii) If (1) is a Herbrand proof, then $\mathbf{L}\exists_1 \vdash \exists^1 x\psi({}^1x)$.

Proof. The first part has been proved above. The second part is proved in the usual way. Namely, we derive *(1)* from *(2)*, where we will have γ_{ji}'s instead of γ_i's. First apply the rule of introducing $\exists$ to terms ${}^4q$ and ${}^3u$. Thus, using the abbreviation above, we get

(3) $$\bigvee\!\!\!\!\bigvee_{j.i}\exists^3 z\exists^2 x(\Gamma_{ji}({}^2s_j, {}^3z_{ji}, f_{ji}({}^2s_j), f_{ji1}({}^2s_j), {}^3z_{ji1}) \to \psi({}^2x_j)).$$

Clearly, we can replace different strings x_j by a single one; formally it means to replace ${}^2x_j$ by ${}^1x$. Let the maximal j be J. We consider the disjuncts with $j = J$. By our definition of Herbrand proof

(i) all ${}^3f_{Jim}({}^2s_J)$ are different,

(ii) they do not occur elsewhere in *(3)*, since they contain s_{jm} of maximal size.

Replace all ${}^3f_{Jim}({}^2s_J)$ by different new variables ${}^3y_J$. By conditions (i), (ii) we can apply the generalization rule to ${}^3y_J$. Then apply the $\exists$-rule to ${}^2s_j$. Thus we get

$$[\exists^1 p\forall^3 y_J\exists^3 z\exists^1 x\bigvee\!\!\!\!\bigvee_{i}(\Gamma_{Ji}({}^1p, {}^3z_{Ji}, {}^3y_{Ji}, {}^3y_{Ji1}, {}^3z_{Ji1}) \to \psi({}^1x))] \vee \Phi,$$

where Φ is (3) with the restriction $j < J$. Now we do the same with Φ (note that ${}^3f_{jim}({}^2s_j)$ do not occur in the first disjunct) and so on. Eventually we obtain

$$\exists^1 p\forall^3 y_J\exists^3 z\exists^1 x\bigvee\!\!\!\!\bigvee_{j.i}(\Gamma_{ji}({}^1p, {}^3z_{ji}, {}^3y_{ji}, {}^3y_{ji1}, {}^3z_{ji1}) \to \psi({}^1x)),$$

which is essentially (1) except γ_{ij}'s are indexed by two indices j and i. $\square$

A formula will be called *term-free*, if it does not contain occurrences of function and constant symbols (variables are allowed). Let β be a term-free formula and ${}^2v$ an array of variables. A sequence $\beta_1({}^2v_1)\, ldots, \beta_l({}^2v_l)$ will be called a *Herbrand skeleton*, if

(4) $$\beta_1({}^2v_1) \vee \ldots \vee \beta_l({}^2v_l)$$

is a propositional tautology. Here we do not assume that all the variables in ${}^2v$ are distinct.

If $\beta_1({}^2t_1), \vee \ldots \vee \beta_l({}^2t_l)$ is a Herbrand proof and

$$t_{ij} = t_{i'j'} \Rightarrow v_{ij} = v_{i'j'},$$

then we say that *(4)* is the skeleton of the Herbrand proof.

We shall use the following two easy facts about Herbrand skeletons:

(i) For a given r and l there are only finitely many Herbrand skeletons with formulas of rank $\leq r$ and with l terms.

(ii) Any substitution instance of a Herbrand skeleton is also a propositional tautology.

Note that because of *(ii)*, when constructing a Herbrand proof from a Herbrand skeleton, we need only to substitute terms with a suitable shape.

4 Main theorem

Theorem 4.1 (Main Theorem) *Let A_0 be a finite set of the universal closures of instances of $\overline{\mathbf{L}\exists_\mathbf{1}}$, let $\varphi(t)$ be an existential formula which may contain function symbols not present in $\overline{\mathbf{L}\exists_\mathbf{1}}$. Suppose the prenex normal form of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi(\tau)$ has a Herbrand proof with skeleton S. Then there exists a term τ_0 such that*

(i) $\overline{\mathbf{L}\exists_\mathbf{1}} \to \varphi(\tau_0)$ *(in fact we have a Herbrand proof of $\bigwedge\!\!\!\bigwedge A_1 \to \varphi(\tau_0)$ with the skeleton S for some finite set A_1 of universal closures of instances of $\overline{\mathbf{L}\exists_\mathbf{1}}$);*

(ii) if τ' is a term such that $\bigwedge\!\!\!\bigwedge A_2 \to \varphi(\tau')$ has a Herbrand proof with skeleton S for some finite set A_2 of universal closures of instances of $\overline{\mathbf{L}\exists_\mathbf{1}}$, then τ' is a substitution instance of τ_0 (in fact the whole Herbrand proof of $\bigwedge\!\!\!\bigwedge A_2 \to \varphi(\tau')$ is a substitution instance of the Herbrand proof of $\bigwedge\!\!\!\bigwedge A_1 \to \varphi(\tau_0)$).

Before proving Kreisel's conjecture for $\mathbf{L}\exists_\mathbf{1}$ we shall derive another theorem from Theorem 4.1 from which the conjecture follows easily (cf. similar theorem for finitely axiomatized theories [Krajíček/Pudlák]). The proof of Theorem 4.1 is postponed to Section 6.

Theorem 4.2 *Let $\varphi(x)$ be an arbitrary formula, let k be a positive integer. Suppose that*

$$\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi(\tau).$$

Then there exist terms $\tau_0, \ldots, \tau_n$ such that

(i) $\mathbf{L}\exists_\mathbf{1} \vdash \varphi(\tau_0), \ldots, \varphi(\tau_n)$;

(ii) if $\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi(\tau')$, then τ' is a substitution instance of τ_i for some i.

The proof of Theorem 4.2 is based on Theorem 4.1 and the following lemma. Let us denote by $\forall\overline{\mathbf{L}\exists_\mathbf{1}}$ the universal closures of formulas of $\overline{\mathbf{L}\exists_\mathbf{1}}$.

Lemma 4.3 *For every k and r there exist skeletons $S_0, \ldots, S_n$ such that for every formula φ, if $\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi$ and rank$(\varphi) \le r$, then for some $A_0 \subseteq \forall\overline{\mathbf{L}\exists_\mathbf{1}}$, every prenex form of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$ has a Herbrand proof with a skeleton S_i for some i.*

Proof. By Lemma 2.3, if the number of steps in a proof of $\mathbf{L}\exists_\mathbf{1} \vdash \varphi$ and the rank of φ are bounded, we can find $A_0 \subseteq \forall\overline{\mathbf{L}\exists_\mathbf{1}}$ such that $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$ has a proof in pure predicate calculus whose rank is bounded. Moreover, we can take any prenex normal form of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$. By Lemma 3.2 we have a Herbrand proof of $\vartheta_1 \vee \ldots \vee \vartheta_l$ of the prenex form of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$, where l is bounded in terms of k and r. The rank of each ϑ_i is also bounded, since it is equal to the rank of (the prenex normal form of) $\bigwedge\!\!\!\bigwedge A_0 \to \varphi$. Let $\beta_1({}^2v_1) \vee \ldots \vee \beta({}^2v_l)$ be the skeleton of the Herbrand proof. Since $\beta_1, \ldots, \beta_l$ are term-free and of bounded rank, there are only finitely many such skeletons, up to renaming of variables ${}^2v$. Thus let $S_0, \ldots, S_n$ be these skeletons. $\Box$

Proof of Theorem 4.2: By Lemma 3.1 it suffices to consider only $\varphi(x)$ existential. Let k be given and suppose $\mathbf{L\exists_1} \vdash^k \varphi(\tau)$ for some term τ. Let $r = rank(\varphi(x))$; note that $rank(\varphi(\tau)) = r$ for every term τ. Hence by Lemma 4.3, $\mathbf{L\exists_1} \vdash^k \varphi(\tau)$ implies that the prenex form of $\bigwedge\!\!\!\bigwedge A_0 \to \varphi(\tau)$, for some $A_0 \subseteq \forall\overline{\mathbf{L\exists_1}}$, has a Herbrand proof whose skeleton is one of $S_0, \ldots, S_n$ (n depends only on k and r). By Theorem 4.1 we have "a most general" term τ_i for each S_i. $\Box$

Corollary 4.4 (Kreisel's conjecture for $\mathbf{L\exists_1}$) *Let $\mathbf{T}$ be the theory axiomatized by $\mathbf{Q}$ (Robinson Arithmetic) and $\mathbf{L\exists_1}$. Suppose k is an integer and φ is a formula such that for every m:*

$$\mathbf{T} \vdash^k \varphi(S^m(0)).$$

Then

$$\mathbf{T} \vdash \forall x \varphi(x)).$$

Proof. Apply Theorem 4.2 to $\bigwedge\!\!\!\bigwedge \mathbf{Q} \to \varphi(S^m(0))$ instead of φ. Let $\tau_0, \ldots, \tau_n$ be terms given by (ii). Take m bigger than the depth of all $\tau_0, \ldots, \tau_n$. Let τ_i be such that $S^m(0)$ is a substitution instance of τ_i. Then, clearly, τ_i is of the form $S^l(x)$ for some variable x.

By (i)

$$\mathbf{L\exists_1} \vdash \bigwedge\!\!\!\bigwedge \mathbf{Q} \to \varphi(S^l(x)),$$

whence

$$\mathbf{L\exists_1} + \mathbf{Q} \vdash \forall x \geq S^l(0) \quad \varphi(x).$$

Since

$$\mathbf{Q} \vdash x = 0 \vee x = S(0) \vee \ldots \vee x = S^{l-1}(0) \vee x \geq S^l(0),$$

and since all numeric instances of $\varphi(x)$ are provable, we have

$$\mathbf{L\exists_1} + \mathbf{Q} \vdash \forall x \varphi(x). \quad \Box$$

Note that the answer to Kreisel's conjecture depends heavily on the representation of the n-th number by $S^n(0)$: Let $\mathbf{T}$ be a sufficient strong number theory and let $\mathbf{T} \vdash \varphi(S^n(0))$ for all n. There is a sequence of terms $\tau_0, \tau_1, \ldots$ consisting of $0, S, +, \cdot, \dot{-}$ such that $\mathbf{T} \vdash \tau_n = S^n(0)$ for all n, and there is a k such that $\mathbf{T} \vdash^k \varphi(\tau_n)$ for all n ([Baaz1]). If $\mathbf{T}$ is r.e., we can find $\varphi(x)$ so that $\mathbf{T} \not\vdash \forall x \varphi(x)$ moreover.

On the other hand results similar to Kreisel's conjecture may be proved for other sequences of terms $0, F(0), F^2(0), \ldots$. For example take $F(0) := x + 0$. Then we get the following somewhat papradoxical statement.

Corollary 4.5 *Let* $\mathbf{T}$ *be finite and* $\mathbf{T}+\mathbf{L}\exists_\mathbf{1} \vdash \forall x(x+0=x)$. *Let* $d_0 := 0$, $d_{r+1} := d_r+0$. *Then for every* k *and* φ, *there exists an* n_0 *such that for all* $n \geq n_0$

$$\mathbf{T}+\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi(d_n) \Rightarrow \mathbf{T}+\mathbf{L}\exists_\mathbf{1} \vdash \forall x \varphi(x).$$

Proof. Derive $\mathbf{T}+\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi((\ldots((x+0)+0)+\ldots)+0)$ from $\mathbf{T}+\mathbf{L}\exists_\mathbf{1} \vdash^k \varphi(d_n)$ for sufficiently big n using the Main Theorem. $\square$

5 Semiunification

Our main tool to prove Theorem 4.1 will be semiunification.

Definition 5.1 *A semiunification problem is given by a set of pairs of terms* $(s_1,t_1),\ldots,(s_n,t_n)$. *A solution to the semiunification problem is a substitution* δ *such that there exist substitutions* $\sigma_1,\ldots,\sigma_n$ *such that*

$$s_1\delta = t_1\delta\sigma_1, \ldots, s_n\delta = t_n\delta\sigma_n;$$

a solution will be also called a semiunifier. A most general semiunifier *is a semiunifier* δ_0 *such that for every semiunifier* δ *there exists a substitution* δ' *such that* $\delta = \delta_0\delta'$ *(i.e.* $\delta(x) = \delta'(\delta_0(x))$*).*

Theorem 5.1 ([Baaz2]). *If a semiunification problem has a semiunifier then it has a most general one.*

We shall need a somewhat more general situation than the one of the definition.

Lemma 5.2 *Theorem 5.1 holds also if we impose additional conditions of the following kind:*

(i) for some pairs i,j: $\sigma_i = \sigma_j$;

(ii) for some variables v: $v\delta = v$;

(iii) for some pairs i,j: *if* ξ *is a variable such that* $\xi\sigma_i \neq \xi$, *then* ξ *does not occur in* $s_j\delta$;

(iv) for some pairs, a variable v *and a number* j: v *does not occur in* $s_j\delta$ *provided that* $v\delta = v$.

Proof. *(i)* Replace equations (s_i, t_i) and (s_j, t_j) by a single one $(f(s_i, s_j), f(t_i, t_j))$, f a binary function symbol.

(ii) Let $v_1, \dots, v_l$ be the variables for which we require $v\delta = v$. Choose $l+1$ different constants $c_0, \dots, c_l$ which do not occur elsewhere, and add equations $(v_1, c_0), \dots, (v_l, c_0)$ and $(v_1, c_l), \dots, (v_l, c_l)$. Let δ_0 be a most general semiunifier for the extended problem. The extra equations ensure that $u_1 = v_1\delta, \dots, u_l = v_l\delta$ are distinct variables. Let ρ be the partial mapping defined by

$$\rho(u_1) = v_1, \dots, \rho(u_l) = v_l.$$

Since ρ is one-to-one, it can be extended to $\bar{\rho}$ a total one-to-one mapping (permutation) defined on all variables. Then, clearly $\delta_0\bar{\rho}$ is a most general semiunifier satisfying

$$v_1\delta_0\bar{\rho} = v_1, \dots, v_l\delta_0\bar{\rho} = v_l.$$

(iii) For each such pair i, j add the equation (using *(i)* above)

$$s_j\delta = s_j\delta\sigma_i.$$

(iv) First add an equation (c, v) with the corresponding substitution σ, where c is a constant. Then, if

$$v = v\delta \quad \text{and} \quad c\delta = v\delta\sigma$$

then $v\sigma \neq v$. Now use *(iii)* to ensure that v does not occur in $s_j\delta$. $\square$

Let us also note that we may use also equations $s_i\delta = t_i\delta$ (i.e. without σ_i), because we can first unify these (using the existence of a most general unifier) and then use Theorem 5.1 for the rest.

6 Proof of the main theorem

In this section we shall prove Theorem 4.1.

Let $A_0 \subseteq \forall\overline{\mathbf{L}\exists_1}$ and $\varphi(\tau)$ be given. By definition a Herbrand proof of $\bigwedge\!\!\!\bigwedge A_0 \vdash \varphi(\tau)$ has the following form

$$(0) \qquad \bigvee\!\!\!\bigvee_{j.i,l} \Gamma_{ji}({}^2s_j, {}^4q_{jil}, {}^3f_{ji}({}^2s_j), f_{ji1}({}^2s_j), q_{jil1}) \to \psi({}^3u_{jl})).$$

where ${}^2s, {}^4q, {}^3u$ are arrays of terms and ${}^3f$ are new function symbols, $\Gamma_{ji}({}^1p, {}^2z, {}^2y, {}^2y_1, {}^2z_1)$ is an abbreviation of $\gamma_{ji}({}^1p, {}^2z) \to \gamma_{ji}({}^1p, {}^2y) \wedge {}^2y_1 \leq {}^2z_1$, and we have the following conditions:

$$(1) \qquad j > j' \Rightarrow \max|s_{jm}| \leq |s_{j'm}| \quad \text{and}$$

$${}^2s_j \neq {}^2s_j' \quad \text{(i.e. } s_{jm} \neq s_{j'm} \quad \text{for some } m\text{)},$$

(2) $$\text{all} \quad {}^3f_{ji}({}^2s_j) \quad \text{are distinct, i.e.}$$
$$f_{jir}({}^2s_j) = f_{j'i'r'}({}^2s_{j'}) \quad \text{iff} \quad i = i', r = r', j = j',$$

Because of condition (1) we have

(3) $${}^3f_{ji}({}^2s_j) \quad \text{do not occur in} \quad {}^2s_{j'}, \quad \text{for} \quad j' \leq j,$$

(because a term of the equal size or bigger cannot be a proper subterm of a given term).

Now replace all maximal subterms which have the outermost symbol some f_{jir} by distinct variables. Thus we obtain

(4) $$\bigvee\!\!\!\bigvee_{j.i,l}\Gamma_{ji}({}^2s^*_j, {}^4q^*_{jil}, {}^3v_{ji}, v_{ji1}, q^*_{jil1}) \to \psi({}^3u^*_{jl})).$$

where ${}^3v$ is an array of variables and ${}^2s^*$, ${}^4q^*$, ${}^3u^*$ are arrays of terms. The conditions (2), (3) are now transformed to

(5) $$\text{all variables in } {}^3v \text{ are different,}$$

(6) $$\text{the variables of } {}^3v_{ji} \text{ do not occur in } {}^2s^*_{ji} \text{ for } \; j' \leq j,$$

Claim 1. *If a formula of the form* (4) *satisfies* (5) *and* (6), *one can substitute terms in it in such a way that a formula of the form* (0) *is obtained.*

Proof. For $j = 1, 2, \ldots$, gradually replace v_{jir} by f_{ir} applied to the array of terms obtained from ${}^2s^*_j$ at the previous stage. Thus first we replace v_{1ir} by $f_{ir}({}^2s^*_1)$; in this way ${}^2s^*_2$ may be changed to some ${}^2s^{**}_2$, hence in the second stage we replace v_{2ir} by $f_{ir}({}^2s^{**}_2$ and so on. Note that ${}^2s^*_1$ is never changed; after the first stage ${}^2s^{**}_2$ is never changed 2 etc.. Thus the formula gradually acquires the form (0).

Next we shall rewrite (4) using terms and term-free formulas. We get

(7) $$\bigvee\!\!\!\bigvee_{j.i,l}((\delta_{ji}({}^2t_i({}^2s^*_j, {}^4q^*_{jil})) \to \delta_{ij}({}^2t_i({}^2s^*_j, {}^3v_{ji})) \wedge v_{ji1} \leq q^*_{jil1} \to \beta({}^1w(\tau, {}^3u^*_{jl}))),$$

where δ_i and β are term-free. Let us abbreviate this formula by

(8) $$\bigvee\!\!\!\bigvee_{j.i,l}\Delta_{ji}({}^2t_i({}^2s^*_j, {}^4q^*_{jil}), {}^2t_i({}^2s^*_j, {}^3v_{ji}), v_{ji1}, q^*_{jil1}, {}^1w(\tau, {}^3u^*_{jl})).$$

Put
$${}^3t^*_{ji}({}^2V_j, {}^3v_{ji}) := {}^2t_i({}^2s^*_j, {}^3v_{ji}),$$

where ${}^2V_j$ are the variables of ${}^2s^*_j$. Hence the condition (6) can be expressed by

(9) $${}^3v_{ji} \text{ do not occur in } {}^2V_{j'} \text{ for } j' \leq j,$$

and (8) becomes

$$\bigvee\!\!\!\bigvee_{j.i,l}\Delta_{ji}({}^3t^*_{ji}({}^2V_j, {}^4q^*_{jil}), {}^3t^*_{ji}({}^2V_j, {}^3v_{ji}), v_{ji1}, q^*_{jil1}, {}^1w(\tau, {}^3u^*_{jl})). \tag{10}$$

Claim 2. *If a formula of the form* (10) *satisfies* (5) *and* (9)*, then it can be put into the form* (0) *by a suitable substitution.*

Proof. Just note that (10) is a special case of (7), where we have variables instead of terms ${}^2s^*_j$. As (7) is only a more explicit form of (4) we get Claim 2 from Claim 1. $\square$

The skeleton of the above proof is

$$\bigvee\!\!\!\bigvee_{j.i,l}\Delta_{ji}({}^4a_{jil}, {}^3b_{ji}, c_{ji}, d_{ji}, {}^3e_{jl}) \tag{11}$$

where ${}^4a, {}^3b, {}^2c, {}^2d, {}^3e$ are arrays of variables which are not supposed to be mutually distinct.

We have shown that any Herbrand proof with the above skeleton determines a formula of the form (10) with conditions (5) and (9) and vice versa such a formula determines a Herbrand proof of some $\bigwedge\!\!\!\bigwedge A_0 \rightarrow \varphi(\tau)$. Now we shall show that the structure of terms in (10) with conditions (5) and (9) can be determined by a semi unification problem. Hence a most general semiunifier will determine a most general formula of the form (10) (plus (5),(9)), thus also in a sense a most general Herbrand proof with the above skeleton. In particular we obtain a most general term τ_0 on the place of τ.

The semiunification problem is defined as follows:

variables: ${}^4a, {}^3b, {}^2c, {}^2d, {}^3e, h, {}^3x$,
substitutions: δ, σ_{jil},
equations:

(i) ${}^1w(h, {}^3x_{jl})\delta = {}^3e_{jl}\delta$,
(ii) ${}^4a_{jil}\delta = {}^3b_{ji}\delta\sigma_{jil}$,
(iii) $d_{jil}\delta = c_{ji}\delta\sigma_{jil}$,

restrictions:

(iv) $c_{ji}\delta = c_{ji}$,
(v) if $\sigma_{jil}(\xi) \neq \xi$, then ξ does not occur among the variables of ${}^3b_{j'i'}\delta$ for $j' < j$ or ($j' = j$ and $i' \neq i$),
(vi) if $\sigma_{jil}(\xi) \neq \xi$, then ξ is different from every $c_{j'i'}$ for $j' \neq j$ or $i' \neq i$ (or both),

(vii) c_{ji} does not occur among the variables of $b_{j'i'}\delta$ for $j' < j$ or ($j' = j$ and $i' \neq i$).

Furthermore, we identify c_{ji} with v_{ji1}.

We must stress that the variables ${}^4a, {}^3b, {}^2c, {}^2d, {}^3e$ are not distinct. They come from the Herbrand skeleton. (Otherwise the equations (i)–(iii) had a trivial solution.)

First we shall show that (10) with (5) and (9) determines a solution to the semiunification problem. Put

$$\begin{aligned}
{}^4a_{jil}\delta &= {}^3t^*_{ji}({}^2V_j, {}^3q^*_{jil}),\\
{}^3b_{ji}\delta &= {}^3t^*_{ji}({}^2V_j, {}^3v_{ji}),\\
c_{ji}\delta &= v_{ji1},\\
d_{jil}\delta &= q^*_{jil1},\\
{}^3e_{jl}\delta &= {}^1w(\tau, {}^3u^*_{jl}),\\
h\delta &= \tau,\\
{}^3x_{jl}\delta &= u^*_{jl},\\
\sigma_{jil}({}^3v_{ji}) &= {}^3q^*_{ji},\\
\sigma_{jil}({}^2V_j) &= {}^2V_j
\end{aligned}$$

This definition is possible, since we assume that (10) has the Herbrand skeleton (11). It remains to check the restrictions. (iv) is satisfied by definition; if $\sigma_{ji}(\xi) \neq \xi$, then ξ must be some v_{jim}, hence we get (v) from (5) and (9); to see (vi), recall that $c_{ji} = v_{ji1}$, and that (5) says that the variables in ${}^3v$ are distinct; finally (vii) means that v_{ji1} does not occur among ${}^2V_{j'}$ for $j' \leq j$ and among ${}^3v_{ji}$ for $j' < j$ or ($j' = j$ and $i' \neq i$), which follows from (9) and (5) respectively in the same way as above.

The last thing to do is to show that any solution to the semiunification problem determines a formula of the form (10) satisfying conditions (5) and (9). Trivially the terms $v_{ji1}(= c_{ji})$, $q^*_{jil}(= d_{jil}\delta)$ and ${}^1w(\tau, {}^3u^*_{jl})(= e_{jl}\delta)$ have the required form. We get some terms ${}^4a_{jil}\delta$ and ${}^3b_{ji}\delta$ and we must write them in the form ${}^3t^*_{ji}({}^2V_j, {}^3q^*_{jil})$ and ${}^3t^*_{ji}({}^2V_j, {}^3v_{ji})$, respectively. This is done as follows: ${}^2V_j$ are the variables that occur in ${}^3b_{ji}\delta$ for some i and such that σ_{jil} are identity on them for *all* i and l, i.e. $\sigma_{jil}({}^2V_j) ={}^2 V_j$. The variables ${}^3v_{ji}$ are the remaining variables of ${}^3b_{ji}\delta$ and the variables $c_{ji}(= v_{ji1})$. We shall prove the conditions (5) and (9).

Proof of (5): Consider v_{jim}, $m \neq 1$. This means that some $\sigma_{ji'l}$ is not identity on v_{jim} (i.e. $\sigma_{ji'l}(v_{jim}) \neq v_{jim}$). By (v) it is only possible that $i' = i$. Also by (v) v_{jim} must be different from all $v_{j'i'm}$ with $j' < j$ or ($j' = j$ and $i' \neq i$), and $m' \neq 1$ while (vi) gives this also for $m' = 1$. For $m = 1$ we get the same from (vii). Thus we have for all v_{jim}: v_{jim} is different from $v_{j'i'm}$ with $j' < j$ or ($j' = j$ and $i' \neq i$), but this simply means that they are all different.

Proof of (9): Let v_{jim} be given. It cannot occur among ${}^2V_j$ by definition. For $j' < j$, v_{jim} cannot occur among ${}^2V_{j'}$ by (v) (for $m \neq 1$) and (vii) (for $m = 1$).

At last we should comment on the fact that $\varphi(x)$ may use some function symbols which are not present in $\overline{\mathbf{L}\exists_1}$. It is easily seen that these function symbols cannot occur in the terms t^*_{ji} obtained from a most general semiunifier, since the original t_{ji}'s do not contain them and they are substitution instances of t_{ji}'s. $\square$

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Matthias Baaz
Institut für Algebra und Diskrete Mathematik
Technische Universität Wien
Wiedner Hauptstraße 8-10
A–1040 Wien
Austria

Pavel Pudlák
Mathematical Institute
ČSAV
Žitná 25
11567 Praha 1
Czechoslovakia