A posteriori error estimates

Part II – Complementary estimates

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Outline

- Toy problem
- Derivation of complementary estimates
- Two options
  - (A) Error majorant
  - (B) Dual finite elements
- Energy minimization
- Method of hypercircle
- Numerical examples
- Conclusions

Toy problem

Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad a(u, v) = \mathcal{F}(v) \quad \forall v \in V$$

Notation:

- $a(u, v) = (\nabla u, \nabla v)$
- $\mathcal{F}(v) = (f, v)$
- $(\varphi, \psi) = \int_\Omega \varphi\psi \, \mathrm{d}x$
- Error: $e = u - u_h$
- Energy norm: $\|e\|^2 = a(e, e) = (\nabla e, \nabla e) = \|\nabla e\|_0^2$

Derivation

Divergence thm.: $(\operatorname{div} \mathbf{y}, v) + (\mathbf{y}, \nabla v) = 0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega), v \in V$

Friedrichs' inequality: $\|v\|_0 \leq C_{\mathrm{F}} |||v||| \quad \forall v \in V$

Theorem: Let $u_h \in V$ be arbitrary then

$$|||u - u_h||| \leq \eta(u_h, \mathbf{y})$$

$$\eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof: $v \in V$

$$\begin{aligned} a(u - u_h, v) &= (f, v) - (\nabla u_h, \nabla v) = (f + \operatorname{div} \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\ &\leq \|f + \operatorname{div} \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\ &\leq \left(C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0\right) |||v||| \end{aligned}$$

Set $v = u - u_h$. $\square$

Derivation

Divergence thm.: $(\operatorname{div}\mathbf{y}, v) + (\mathbf{y}, \nabla v) = 0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega), v \in V$

Friedrichs' inequality: $\|v\|_0 \leq C_{\mathrm{F}} |\!|\!| v |\!|\!| \quad \forall v \in V$

Theorem: Let $u_h \in V$ be arbitrary then

$$|\!|\!| u - u_h |\!|\!| \leq \eta(u_h, \mathbf{y})$$

$$\eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div}\mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Lemma: Let $u \in V$ be the exact solution.
Then $|\!|\!| u - u_h |\!|\!| = \eta(u_h, \nabla u)$.

Two options

(A) Error majorant

$$\|\!|u - u_h|\!\| \leq \eta(u_h, \mathbf{y})$$
$$\eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

[S. Repin et al., 2000–]

(B) Dual finite elements

$$\|\!|u - u_h|\!\| \leq \widetilde{\eta}(u_h, \mathbf{y})$$
$$\widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$
$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : f + \operatorname{div} \mathbf{y} = 0\}$$

[J. Haslinger, I. Hlaváček, M. Křížek, 1970s–80s]

(A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \leq \widehat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_{\mathrm{F}}^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1+\beta) \|\mathbf{y} - \nabla u_h\|_0^2 \quad \forall \beta > 0$$

Proof: $(A+B)^2 \leq \left(1 + \frac{1}{\beta}\right) A^2 + (1+\beta) B^2 \quad \forall \beta > 0$

Equality for $\beta = A/B$. □

(A) Error majorant

$$\|u - u_h\| \le \eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \le \widehat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_{\mathrm{F}}^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1+\beta) \|\mathbf{y} - \nabla u_h\|_0^2 \quad \forall \beta > 0$$

Notation: $\mathbf{W} = \mathbf{H}(\operatorname{div}, \Omega)$

Complementary problem (equivalent formulations):

(i) Find $\mathbf{y} \in \mathbf{W} : \eta(u_h, \mathbf{y}) \le \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$

(ii) Find $\mathbf{y} \in \mathbf{W}$ and $\beta > 0 : \widehat{\eta}^2(u_h, \mathbf{y}, \beta) \le \widehat{\eta}^2(u_h, \mathbf{w}, \widehat{\beta}) \quad \forall \mathbf{w} \in \mathbf{W},\ \widehat{\beta} > 0$

If $\beta > 0$ fixed:

(iii) Find $\mathbf{y} \in \mathbf{W} : \widehat{\eta}^2(u_h, \mathbf{y}, \beta) \le \widehat{\eta}^2(u_h, \mathbf{w}, \beta) \quad \forall \mathbf{w} \in \mathbf{W}$

(iv) Find $\mathbf{y} \in \mathbf{W} : (\operatorname{div} \mathbf{y}, \operatorname{div} \mathbf{w}) + \frac{\beta}{C_{\mathrm{F}}^2}(\mathbf{y}, \mathbf{w}) = \frac{\beta}{C_{\mathrm{F}}^2}(\nabla u_h, \mathbf{w}) - (f, \operatorname{div} \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$

(A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y}) = C_{\mathrm{F}} \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \leq \widehat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_{\mathrm{F}}^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1+\beta) \|\mathbf{y} - \nabla u_h\|_0^2 \quad \forall \beta > 0$$

Notation: $\mathbf{W} = \mathbf{H}(\operatorname{div}, \Omega)$

Practical implementation:

- $\mathbf{W}_h \subset \mathbf{W}, \quad \dim \mathbf{W}_h < \infty$
  e.g. Raviart-Thomas elements of degree p:
  $$\mathbf{W}_h^p = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \mathbf{w}_h|_K \in P^p(K) \ \forall K \in \mathcal{T}_h\}$$
- Set values for β and C_{F}
- Find $\mathbf{y}_h \in \mathbf{W}_h$:
  $$(\operatorname{div} \mathbf{y}_h, \operatorname{div} \mathbf{w}_h) + \frac{\beta}{C_{\mathrm{F}}^2}(\mathbf{y}_h, \mathbf{w}_h) = \frac{\beta}{C_{\mathrm{F}}^2}(\nabla u_h, \mathbf{w}_h) - (f, \operatorname{div} \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h$$
- Compute $\eta(u_h, \mathbf{y}_h)$

Friedrichs' constant C_{F}

$$\|v\|_0 \le C_{\mathrm{F}} \|\!|v|\!\| \quad \forall v \in V$$

(a) Analytical estimate (Mikhlin, 1986): $V = H_0^1(\Omega)$

$$C_{\mathrm{F}} \le \frac{1}{\pi}\left(\frac{1}{|a_1|} + \cdots + \frac{1}{|a_d|}\right)^{-1/2}, \quad \Omega \subset a_1 \times \cdots \times a_d,$$

Friedrichs' constant C_{F}

$$\|v\|_0 \le C_{\mathrm{F}} \|\!|v|\!\| \quad \forall v \in V$$

(b) Numerical upper bound:

$$C_{\mathrm{F}} = \sup_{v \in V} \frac{\|v\|_0}{\|\!|v|\!\|} \quad \Leftrightarrow \quad \lambda_1 = \inf_{v \in V} \frac{\|\!|v|\!\|^2}{\|v\|_0^2}, \quad C_{\mathrm{F}}^2 = 1/\lambda_1$$

Eigenvalue problem: $u_i \in V: \quad a(u_i, v) = \lambda_i (u_i, v) \quad \forall v \in V$

Galerkin approxim.: $u_i^h \in V_h: \ a(u_i^h, v^h) = \lambda_i^h (u_i^h, v^h) \quad \forall v^h \in V_h$

$V_h \subset V$

$$\Rightarrow \lambda_1^h = \inf_{v_h \in V_h} \frac{\|\!|v_h|\!\|^2}{\|v_h\|_0^2} \quad \Rightarrow \quad \lambda_1 \le \lambda_1^h \quad \Rightarrow \quad 1/\lambda_1^h \le C_{\mathrm{F}}^2$$

Sigillito, Kuttler (1970s): $\overline{\lambda}_1^h \le \lambda_1 \quad \Rightarrow \quad C_{\mathrm{F}}^2 \le 1/\overline{\lambda}_1^h$

(B) Dual finite elements

$$|||u - u_h||| \leq \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : f + \mathrm{div}\,\mathbf{y} = 0\}$$

Complementary problem:

(i) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\widetilde{\eta}(u_h, \mathbf{y}) \leq \widetilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\frac{1}{2}\|\mathbf{y}\|_0^2 \leq \frac{1}{2}\|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

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Lemma 1: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Proof: (i) $\Leftrightarrow$ (ii)

$$\|\mathbf{y} - \nabla u_h\|_0^2 \leq \|\mathbf{w} - \nabla u_h\|_0^2$$

$$\|\mathbf{y}\|_0^2 - 2(\mathbf{y}, \nabla u_h) + \|\nabla u_h\|_0^2 \leq \|\mathbf{w}\|_0^2 - 2(\mathbf{w}, \nabla u_h) + \|\nabla u_h\|_0^2$$

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$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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Proof: (i) $\Leftrightarrow$ (ii)

$$\|\mathbf{y} - \nabla u_h\|_0^2 \le \|\mathbf{w} - \nabla u_h\|_0^2$$

$$\|\mathbf{y}\|_0^2 - 2(\mathbf{y}, \nabla u_h) \le \|\mathbf{w}\|_0^2 - 2(\mathbf{w}, \nabla u_h)$$

(B) Dual finite elements

$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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$$\|\mathbf{y}\|_0^2 - 2(f, u_h) \le \|\mathbf{w}\|_0^2 - 2(f, u_h)$$

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$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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Proof: (i) $\Leftrightarrow$ (ii)

$$\|\mathbf{y} - \nabla u_h\|_0^2 \le \|\mathbf{w} - \nabla u_h\|_0^2$$

$$\|\mathbf{y}\|_0^2 \qquad \le \|\mathbf{w}\|_0^2$$

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$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 1: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Proof: (ii) $\Rightarrow$ (iii)

$J(t) = \left\|\mathbf{y} + t\mathbf{w}^0\right\|_0^2$, $\qquad J(t)$ has minimum at $t = 0$

$$0 = J'(0) = \lim_{t \to 0} \frac{\left\|\mathbf{y} + t\mathbf{w}^0\right\|_0^2 - \|\mathbf{y}\|_0^2}{t}$$

(B) Dual finite elements

$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : f + \mathrm{div}\,\mathbf{y} = 0\}$$

Complementary problem:

(i) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\widetilde{\eta}(u_h, \mathbf{y}) \le \widetilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\frac{1}{2}\|\mathbf{y}\|_0^2 \le \frac{1}{2}\|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 1: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Proof: (ii) $\Rightarrow$ (iii)

$J(t) = \left\|\mathbf{y} + t\mathbf{w}^0\right\|_0^2, \qquad J(t)$ has minimum at $t = 0$

$$0 = J'(0) = \lim_{t \to 0} \frac{\|\mathbf{y}\|_0^2 + 2t(\mathbf{y}, \mathbf{w}^0) + t^2\left\|\mathbf{w}^0\right\|_0^2 - \|\mathbf{y}\|_0^2}{t} = 2(\mathbf{y}, \mathbf{w}^0)$$

(B) Dual finite elements

$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : f + \mathrm{div}\,\mathbf{y} = 0\}$$

Complementary problem:

(i) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\widetilde{\eta}(u_h, \mathbf{y}) \le \widetilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\frac{1}{2}\|\mathbf{y}\|_0^2 \le \frac{1}{2}\|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 1: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

Proof: (iii) $\Rightarrow$ (ii)

$$\mathbf{w} \in \mathbf{Q}(f), \quad \exists \mathbf{w}^0 \in \mathbf{Q}(0) : \mathbf{w} = \mathbf{y} + \mathbf{w}^0, \qquad (\mathbf{y}, \mathbf{w}) = \|\mathbf{y}\|_0^2$$

$$0 \le \|\mathbf{w} - \mathbf{y}\|_0^2 = \|\mathbf{w}\|_0^2 - 2(\mathbf{y}, \mathbf{w}) + \|\mathbf{y}\|_0^2 = \|\mathbf{w}\|_0^2 - \|\mathbf{y}\|_0^2 \qquad \square$$

(B) Dual finite elements

$$|||u - u_h||| \le \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : f + \mathrm{div}\, \mathbf{y} = 0\}$$

Complementary problem:

(i) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\widetilde{\eta}(u_h, \mathbf{y}) \le \widetilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\frac{1}{2}\|\mathbf{y}\|_0^2 \le \frac{1}{2}\|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 2: $\mathbf{y} = \nabla u \in \mathbf{Q}(f)$ is the unique solution of (i)–(iii)

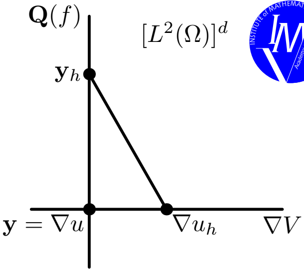
Proof:

If $\mathbf{y}_1 \in \mathbf{Q}(f)$ and $\mathbf{y}_2 \in \mathbf{Q}(f)$ satisfy (iii):

$\Rightarrow \mathbf{y}_2 - \mathbf{y}_1 \in \mathbf{Q}(0)$ and $(\mathbf{y}_2 - \mathbf{y}_1, \mathbf{w}^0) = 0$

$\Rightarrow \|\mathbf{y}_2 - \mathbf{y}_1\|_0^2 = 0 \quad \Rightarrow \quad \mathbf{y}_2 = \mathbf{y}_1$

(B) Dual finite elements



$$|\!|\!| u - u_h |\!|\!| \leq \widetilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : f + \mathrm{div}\, \mathbf{y} = 0\}$$

Complementary problem:

(i) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\widetilde{\eta}(u_h, \mathbf{y}) \leq \widetilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $\frac{1}{2} \|\mathbf{y}\|_0^2 \leq \frac{1}{2} \|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find $\mathbf{y} \in \mathbf{Q}(f)$: $(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 3: $\widetilde{\eta}^2(u, \mathbf{y}_h) + \widetilde{\eta}^2(u_h, \mathbf{y}) = \widetilde{\eta}^2(u_h, \mathbf{y}_h) \qquad \forall u_h \in V,\ \mathbf{y}_h \in \mathbf{Q}(f)$

$$\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u_h\|_0^2$$

Proof:

$$\|\mathbf{y}_h - \nabla u + \nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u\|_0^2 + \|\nabla u - \nabla u_h\|_0^2$$

$$(\mathbf{y}_h - \nabla u, \nabla u - \nabla u_h) = 0$$

(B) Dual finite elements

Practical implementation: $d = 2$, Ω simply connected

- $\overline{\mathbf{q}}(x_1, x_2) = -\left(\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0\right)^\top \quad \Rightarrow \quad -\operatorname{div} \overline{\mathbf{q}} = f$
- $\mathbf{Q}(f) = \overline{\mathbf{q}} + \mathbf{Q}(0) = \overline{\mathbf{q}} + \mathbf{curl}\, H^1(\Omega) \qquad \mathbf{curl} = (\partial_2, -\partial_1)^\top$
- Complementary problem:

  (iii) $\mathbf{y} = \overline{\mathbf{q}} + \mathbf{curl}\, z \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

  (iv) $z \in H^1(\Omega) : \qquad (\mathbf{curl}\, z, \mathbf{curl}\, v) = -(\overline{\mathbf{q}}, \mathbf{curl}\, v) \quad \forall v \in H^1(\Omega)$

  (v) $z \in H^1(\Omega) : \qquad (\nabla z, \nabla v) = -(\overline{\mathbf{q}}, \mathbf{curl}\, v) \quad \forall v \in H^1(\Omega)$
- Galerkin approximation:

  $$z_h \in Z_h \subset H^1(\Omega) : \quad (\nabla z_h, \nabla v_h) = -(\overline{\mathbf{q}}, \mathbf{curl}\, v_h) \quad \forall v_h \in Z_h$$
- $\mathbf{y}_h = \overline{\mathbf{q}} + \mathbf{curl}\, z_h$
- Compute $\widetilde{\eta}(u_h, \mathbf{y}_h)$

Energy minimization

$$a(u,v) = (\nabla u, \nabla v), \qquad \mathcal{F}(v) = (f, v), \qquad a^*(\mathbf{y}, \mathbf{w}) = (\mathbf{y}, \mathbf{w})$$

Primal problem:

$$u \in V: \quad a(u,v) = \mathcal{F}(v) \quad \forall v \in V$$

$$u \in V: \quad J(u) = \min_{v \in V} J(v), \quad J(v) = \frac{1}{2} a(v,v) - \mathcal{F}(v)$$

Complementary problem:

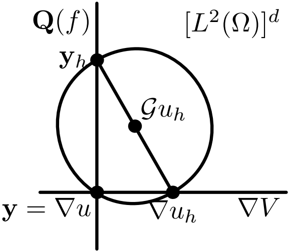
$$\mathbf{y} \in \mathbf{Q}(f): \quad a^*(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$$

$$\mathbf{y} \in \mathbf{Q}(f): \quad J^*(\mathbf{y}) = \min_{\mathbf{w} \in \mathbf{Q}(f)} J^*(\mathbf{w}), \quad J^*(\mathbf{w}) = \frac{1}{2} a^*(\mathbf{w}, \mathbf{w})$$

Complementarity of energies:

$$J(u) + J^*(\mathbf{y}) = -\frac{1}{2} a(u,u) + \frac{1}{2} a^*(\nabla u, \nabla u) = 0$$

Method of hypercircle



Theorem: If

- $u \in V$ is primal solution
- $u_h \in V$, $\mathbf{y}_h \in \mathbf{Q}(f)$ arbitrary
- $\mathcal{G} u_h = (\mathbf{y}_h + \nabla u_h)/2$

Then

$$\|\nabla u - \mathcal{G} u_h\|_0 = \frac{1}{2}\widetilde{\eta}(u_h, \mathbf{y}_h).$$

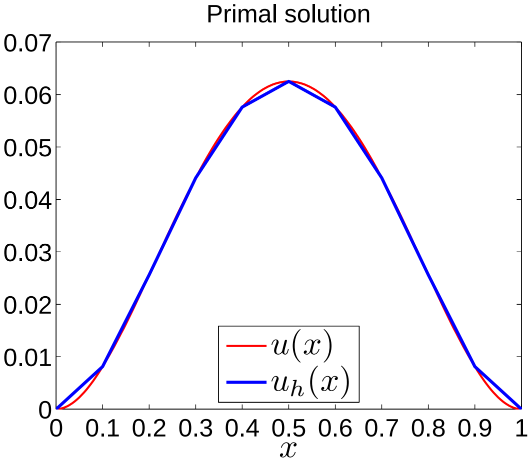
Proof:

$$\begin{aligned} 4\,\|\nabla u - \mathcal{G} u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$

Numerical illustration of hypercircle

$$-u''(x) = 2 - 12x + 12x^2 \text{ in } (0,1), \quad u(0) = u(1) = 0$$

$$u(x) = x^2(1-x)^2$$

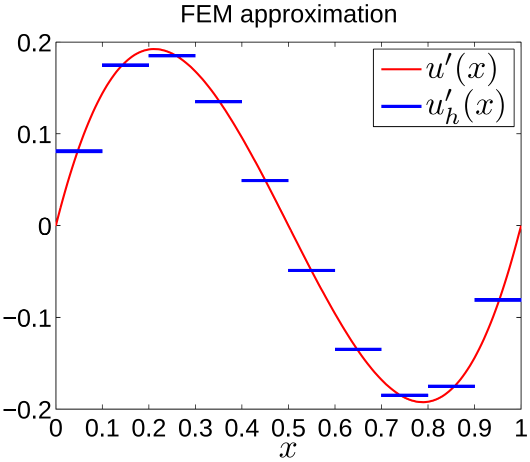


Numerical illustration of hypercircle

$$-u''(x) = 2 - 12x + 12x^2 \text{ in } (0,1), \quad u(0) = u(1) = 0$$

$$\mathbf{y}_h(x) = u'(x) = 2x(2x-1)(x-1)$$

$$\mathcal{G}u_h(x) = (\mathbf{y}_h(x) + u_h')/2$$



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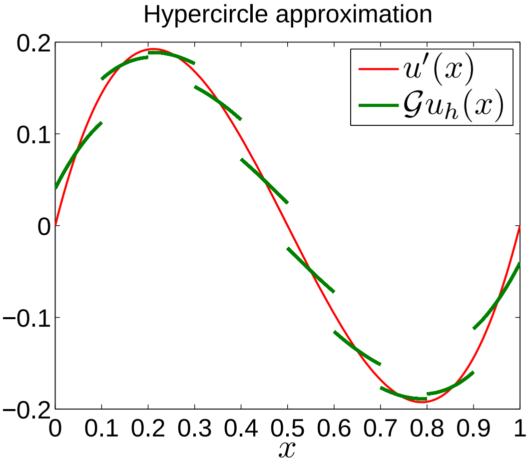
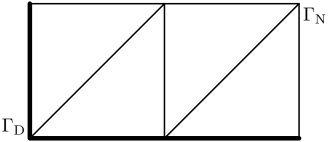
$$\frac{\|\nabla u - \mathcal{G}u_h\|_0}{\|u_h\|_0} \doteq 9.3\,\%$$



Example 1

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
$$u = 0 \text{ on } \Gamma_{\mathrm{D}}$$
$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_{\mathrm{N}}$$



$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

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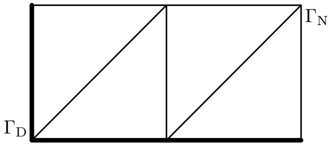
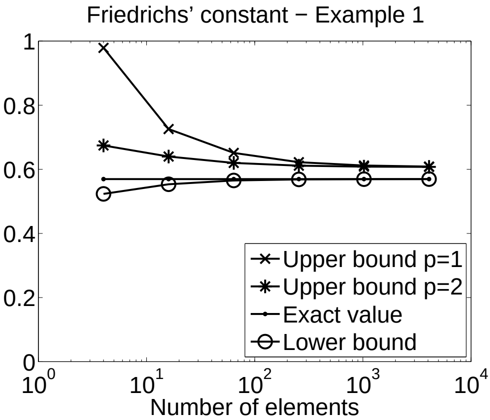


$$f = \frac{5\pi^2}{16} u$$
$$u = \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

$$C_{\mathrm{F}} = \frac{4}{\sqrt{5}\pi} \doteq 0.5694$$
$$C_{\mathrm{F}}^{\mathrm{low}} = 0.5693$$
$$C_{\mathrm{F}}^{\mathrm{up}} = 0.6004$$



Example 1

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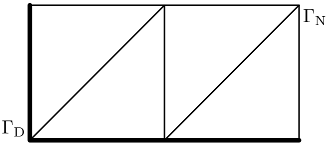
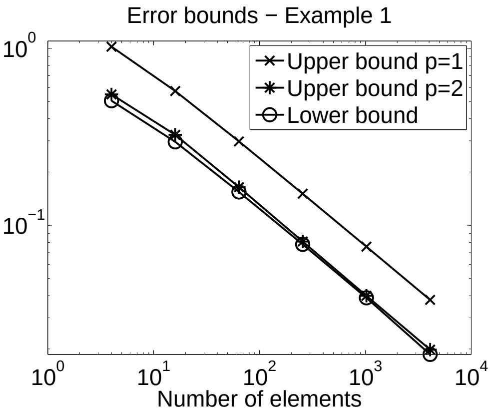


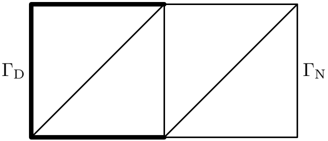
Lower bound:
reference solution

Upper bound:
error majorant



Example 2

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
$$u = 0 \text{ on } \Gamma_{\mathrm{D}}$$
$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_{\mathrm{N}}$$



$$f = \frac{5\pi^2}{16}\sin\frac{\pi x_1}{4}\sin\frac{\pi x_2}{2}$$

Example 2

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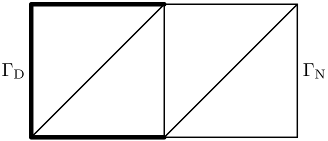
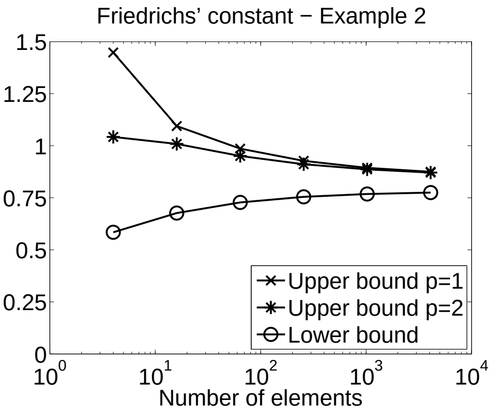


$$f = \frac{5\pi^2}{16} \sin\frac{\pi x_1}{4} \sin\frac{\pi x_2}{2}$$

$C_{\mathrm{F}} = ?$
$C_{\mathrm{F}}^{\mathrm{low}} = 0.7750$
$C_{\mathrm{F}}^{\mathrm{up}} = 0.8712$



Example 2

$$-\Delta u = f \text{ in } (0,2)\times(0,1)$$
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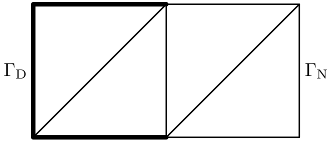

Lower bound:
reference solution

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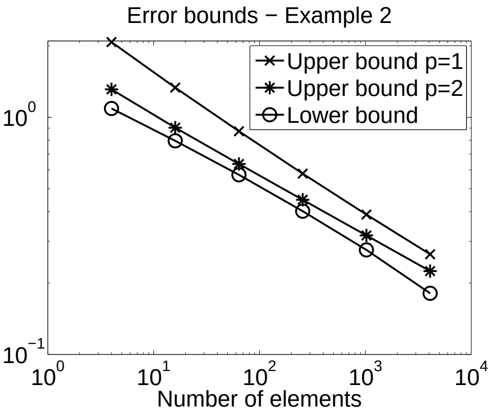

Conclusions

- Guaranteed upper bounds
  - if exact arithmetic
  - if exact quadrature
- Arbitrary numerical method for $u_h \in V$
  - solving complementary problem is expensive
- Particular numerical method for $u_h \in V$
  - postprocessing of $\nabla u_h \quad \Rightarrow \quad$ fast $\mathbf{y}_h$
- Total error
  - including quadrature and algebraic errors
- Technical difficulties
  - Friedrichs' constant
  - handling of $\mathbf{Q}(f)$

History

2000– S. Repin (S. Korotov, J. Valdman, S. Sauter, M. Frolov, . . .)

M. Vohralík (R. Fučík, I. Cheddadi, M.I. Prieto, . . .)

1976– I. Hlaváček (J. Haslinger, M. Křížek, J. Vacek, J. Weisz, . . .)

1971 J.P. Aubin and H.G. Burchard

1957 J.L. Synge

Books:

- P. Neittaanmäki, S. Repin, Reliable methods for computer simulation, error control and a posteriori estimates, Elsevier, Amsterdam, 2004.
- S. Repin, A posteriori estimates for partial differential equations, de Gruyter, Berlin, 2008.

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- T. Vejchodský: Complementarity - the way towards guaranteed error estimates, in: Programs and Algorithms of Numerical Mathematics 15, Institute of Mathematics, Prague, 2010, pp. 205–220. (Preprint 231.)
- M. Ainsworth, T. Vejchodský: Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems, Numer. Math. 119 (2011) 219–243. (Preprint 208.)
- T. Vejchodský: Complementarity based a posteriori error estimates and their properties, in press Math. Comput. Simulation, 2011. (Preprint 190.)

Thank you for your attention

Part II – Complementary estimates

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