Mathematics of fluids in motion

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(all pictures in the text thanks to *wikipedia*)

Motto

Johann von Neumann [1903-1957]

In mathematics you don't understand things. You just get used to them.

Fluids in the real world

- wheather prediction
- ships, planes, cars, trains
- astrophysics, gaseous stars
- rivers, floods, oceans, tsunami waves
- human body, blood motion

MATHEMATICAL ISSUES

- Modeling
- Analysis of models, well-posedness, stability, determinism (?)
- Numerical analysis and implementations, computations

Millennium problems (?)

Clay Mathematics Institute, Providence, RI

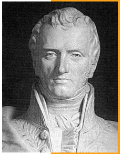
- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture

- Navier-Stokes Equation

- P vs NP Problem
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills and Mass Gap

Navier-Stokes system

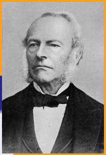
- $\mathbf{u} = \mathbf{u}(t, x)$ .... velocity of an incompressible viscous fluid
- $\Pi = \Pi(t, x)$ .... pressure



Claude Louis Marie Henri Navier [1785-1836]

Incompressibility constraint

$$\mathrm{div}_x \mathbf{u} = 0$$



George Gabriel Stokes [1819-1903]

Momentum balance

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \Delta \mathbf{u}$$

Mathematical modeling of fluids in motion

Molecular dynamics

Fluids understood as huge families of individual particles (atoms, molecules)

Kinetic models

Large ensembles of particles in *random* motion, description in terms of averages

Continuum fluid mechanics

Phenomenological theory based on observable quantities - mass density, temperature, velocity field

Models of turbulence

Essentially based on classical continuum mechanics but description in terms of averaged quantities

Good models?

Stephen William Hawking [*1942]

A model is a good model if it:

- Is elegant
- Contains few arbitrary or adjustable elements
- Agrees with and explains all existing observation
- Makes detailed predictions about future observations that disprove or falsify the model if they are not borne out

Linear vs. nonlinear models

Linear equations

- Solutions built up from elementary functions - modes
- Solvability by means of the symbolic calculus - Laplace and Fourier transform
- Limited applicability

Nonlinear equations

- Explicit solutions known only exceptionally: solitons, simple shock waves
- Possible singularities created by nonlinearity - blow up and/or shocks
- Almost all genuine models are nonlinear

Solvability - classical sense

Jacques Hadamard, [1865 - 1963]

- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Solvability - modern way

Jacques-Louis Lions,
[1928 - 2001]

- **Approximations.** Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically
- **Uniform bounds.** Approximate solutions possesses uniform bounds depending solely on the data
- **Stability.** The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

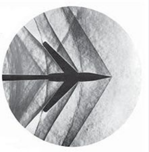
Singularities in nonlinear models

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time

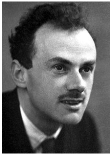


Weak vs. strong

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*
- Derivatives in the equations replaced by integrals:

$$\frac{\partial u}{\partial x} \approx \varphi \mapsto -\int u \partial_x \varphi, \ \varphi \text{ a smooth } \textit{test} \text{ function}$$

Dirac distribution: $\delta_0 : \ \varphi \mapsto \varphi(0)$



Paul Adrien Maurice Dirac [1902-1984]

Field equations - classical vs. weak formulation

- $\mathbf{u} = \mathbf{u}(t, x)$ .... velocity field
- $\varrho = \varrho(t, x)$ ....mass density

Mass conservation

$$\int_B \varrho(t_2, \cdot)\ \mathrm{d}x - \int_B \varrho(t_1, \cdot)\ \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \varrho \mathbf{u} \cdot \mathbf{n}\ \mathrm{dS}_x$$

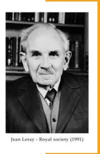
Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Weak formulation

$$\int \int \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi\ \mathrm{d}x\mathrm{d}t = 0 \text{ for } \textit{any} \text{ smooth } \varphi$$

State of the art



Jean Leray [1906-1998] Global existence of weak solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

What may go wrong...

What is not (?) in classical models

- the fluid velocity may become large or even infinite
- infinite speed of propagation
- "incompressibility" and the non-local character of the pressure in the incompressible models

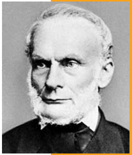
Mathematical problems

- Gap between the existence and uniqueness theory - weak solutions exist globally in time but are not (known to be) unique; strong (classical) solutions (are known to) exist only locally in time
- Possibility of blow-up or concentrations of solutions at some points
- Possibility of fast oscillations, shock waves (?)

Way out?

- Better (more accurate) models
- Better mathematics
- Both?

Do some solutions lose energy?



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Kinetic energy balance for a viscous incompressible fluid

classical: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} |\mathbf{u}|^2 \ \mathrm{d}x = -\nu \int |\nabla_x \mathbf{u}|^2$$

weak: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} |\mathbf{u}|^2 \ \mathrm{d}x \boxed{\leq} - \nu \int |\nabla_x \mathbf{u}|^2$$

Complete fluid systems

State variables

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

Thermodynamic functions

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Transport

Viscous stress

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Heat flux

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$$

Field equations

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x=0$$

Mass conservation

$$\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})=0$$

Momentum balance

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho,\vartheta)=\mathrm{div}_x\mathbb{S}(\vartheta,\nabla_x\mathbf{u})$$

Entropy production

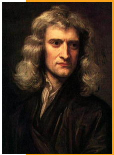
$$\partial_t(\varrho s)+\mathrm{div}_x(\varrho s\mathbf{u})+\mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\right)\boxed{\geq}\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u}-\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Second law

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$

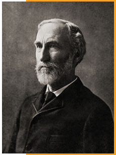


Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Gibbs' relation



Willard Gibbs
[1839-1903]

Gibbs' relation:

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Navier's slip

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan} + \beta[\mathbf{u}]_{\tan} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Mathematics of complete system

- Weak solutions exist globally in time for any physically admissible data
- Strong solutions exist locally in time
- **Weak-strong uniqueness.** A weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists. Strong solutions are unique in the class of weak solutions
- **Long-time stability.** Any weak solution stabilizes to an equilibrium state for large time
- **Conditional regularity.** Any weak solution with a bounded velocity gradient is regular (strong)

However...

Sir Winston Churchill, [1874–1965]

However beautiful the strategy, you should occasionally look at the results

Open questions

Despite the well know fact that the Navier-Stokes equations and related models have been successfully used many times as a platform for modeling and numerical implementations for many real world problems, we still don't know if:

- Are the *weak* solutions to the incompressible/compressible models uniquely determined by the data?
- Does the *density* in the compressible models remain bounded if it was initially?
- Does the *density* in the compressible models remain bounded *below* away from zero if it was initially?
- Does the velocity gradient remain bounded?