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Chérif Amrouche
Mohamed Meslameni
Šárka Nečasová

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The Stationary Oseen Equations in an Exterior Domain: An Approach in Weighted Sobolev Spaces

Chérif Amrouche Mohamed Meslameni Šárka Nečasová *

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Abstract

In this work, we study the linearized Navier-Stokes equations in an exterior domain of $\mathbb{R}^3$ at the steady state, that is, the Oseen equations. We are interested in the existence and the uniqueness of weak, strong and very weak solutions in L^p-theory which makes our work more difficult. Our analysis is based on the principle that linear exterior problems can be solved by combining their properties in the whole space $\mathbb{R}^3$ and the properties in bounded domains. Our approach rests on the use of weighted Sobolev spaces.

Keywords: Oseen equations, weighted Sobolev spaces, exterior domain.
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1 Introduction

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ of class $C^{1,1}$ representing an obstacle and let Ω be the exterior region occupied by the fluid, *i.e.* $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. We consider here the Oseen equations in Ω obtained formally by linearising the Navier-Stokes equations: For a given vector field $\boldsymbol{f}$, a function h and a boundary value $\boldsymbol{g}$, we are looking for a velocity field $\boldsymbol{u}$ of the fluid and a pressure π which fulfil:

$$-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma, \tag{1.1}$$

where, $\boldsymbol{v}$ is a given velocity field belonging to $\boldsymbol{L}^3(\Omega)$ satisfying the divergence free condition. In fact, the Oseen approximation is typical for a flow occurring in an exterior region because it describes the physical properties of a system constituted by an object moving with a small, constant velocity in a viscous liquid, at least at large distances from the object where the viscous effects become less important. But, in bounded region, the Oseen approximation loses its physical meaning, while, from the mathematical point of view, it presents no difficulties and can be handled as a corollary to the theory developed for the Stokes system. It should be observed, however, that the Oseen problem has different structures, one of them is given by the following equations:

$$-\Delta\, \boldsymbol{u} + k\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma, \tag{1.2}$$

with $k > 0$. Problem (1.2) has been studied by many authors, from different points of view and it would be too long to list them all here so we give some examples. One of the first complete work on (1.2) is due to Faxén [21] who generalized the method introduced by Odqvist [33] for the Stokes problem. More recently, Finn [22] used Galerkin's method to establish existence of solutions of (1.2) including weighted estimates. For the $\Omega = \mathbb{R}^3$, Babenko [11] used the Lizorkin's Multiplier Theorem in the investigation of (1.2). The results proved by Finn and Babenko were generalized and improved by Galdi in [23] and very expanded in Chapter VII of his book [24]. Galdi approach is based in the functional framework, homogeneous Sobolev spaces,which is one of possible tools, how to describe the behaviour of solution in the large distance. In [13], Farwig used anisotropic weighted L^2 spaces for the investigation of the exterior problem. Spaces with the weight function η_0^α are also used in [14] and [15], but the weighted estimates are only obtained for the derivatives of second order of functions. We also mention [30] or [34] where the convolution with the fundamental solution of the Oseen problem is studied in L^p space

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with the anisotropic weight function η_β^α. Also see work of Kračmar, Penel for generalized Oseen problem [31]. Recently, problem (1.2), has been studied by Amrouche and Razafison [9] and more recently by Amrouche and Nguyen [8]. Note that, in [9] and in [8], problem (1.2) was setted in weighted Sobolev spaces in order to provide an explicit description of the behavior of the functions and all its derivatives at infinity.

When Ω is a bounded domain, the existence, uniqueness and regularity properties of the solutions for the Oseen problem (1.1) and (1.2) are well known in the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$, see [10] for example for the problem (1.1). It is well known that it is not possible to extend this result to the case of unbounded domains, for example the whole space $\mathbb{R}^3$ or the exterior domain, here the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$ are not adequate. Therefore, a specific functional framework is necessary which also has to take into account the behaviour of the functions at infinity. Our approach is similar to that [9] and [8], which is the use of the weighted Sobolev spaces $\boldsymbol{W}_\alpha^{m,p}(\Omega)$ introduced by Hanouzet [29], Cantor [12], Kudrjavcev [32] (see Section 2 for the notations and details). Moreover we are interested also in the very weak solutions. The concept of very weak solutions for Stokes equations was introduced by Giga in 1981, see [26], by Amrouche and Girault in 1994 in a domain class $C^{1,1}$, see [4]. More recently this concept was extended by Amrouche, Rodríguez - Bellido, see [10].

Finally, very weak solution in different spirit was intensively studied by Galdi, Simader, Farwig, Kozono and Sohr, see [16, 17, 18, 19, 20, 25] to a setting in classical L^q-spaces. For non-steady case we can refer to the work of Amann see [2, 3] in the setting of Besov spaces.

2 Basic Concepts on Weighted Sobolev Spaces

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denotes its distance to the origin. In order to control the behaviour at infinity of our functions and distributions we use for basic weights the quantity $\rho(\mathbf{x}) = (1 + r^2)^{1/2}$ which is equivalent to r at infinity, and to one on any bounded subset of $\mathbb{R}^3$ and the quantity $\ln(2 + r^2)$. We define $\mathcal{D}(\Omega)$ as the linear space of infinite differentiable functions with compact support on Ω. Now, let $\mathcal{D}'(\Omega)$ denotes the dual space of $\mathcal{D}(\Omega)$, often called the space of distributions on Ω. We denote by $< .,. >$ the duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for any nonnegative integers m and real numbers $p > 1$ and α, setting

$$k = k(m,p,\alpha) = \begin{cases} -1, & \text{if } \ \frac{3}{p} + \alpha \notin \{1, ..., m\}, \\ m - \frac{3}{p} - \alpha, & \text{if } \ \frac{3}{p} + \alpha \in \{1, ..., m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W_\alpha^{m,p}(\Omega) = \{ & u \in \mathcal{D}'(\Omega); \\ & \forall\, \lambda \in \mathbb{N}^3 : \ 0 \le |\lambda| \le k,\ \rho^{\alpha - m + |\lambda|}(\ln(2 + r^2))^{-1} D^\lambda u \in L^p(\Omega); \\ & \forall\, \lambda \in \mathbb{N}^3 : \ k + 1 \le |\lambda| \le m, \rho^{\alpha - m + |\lambda|} D^\lambda u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$||u||_{W_\alpha^{m,p}(\Omega)} = \Bigg(\sum_{0 \le |\lambda| \le k} ||\rho^{\alpha - m + |\lambda|}(\ln(2 + r^2))^{-1} D^\lambda u||^p_{L^p(\Omega)} + \sum_{k+1 \le |\lambda| \le m} ||\rho^{\alpha - m + |\lambda|} D^\lambda u||^p_{L^p(\Omega)} \Bigg)^{1/p}.$$

For $m = 0$, we set

$$W_\alpha^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega); \ \ \rho^\alpha\, u \in L^p(\Omega)\}.$$

We note that the logarithmic weight only appears if $\dfrac{3}{p} + \alpha \in \{1, ..., m\}$ and all the local properties of $W_\alpha^{m,p}(\Omega)$ coincide with those of the corresponding classical Sobolev spaces $W^{m,p}(\Omega)$. We set $\mathring{W}_\alpha^{m,p}(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ for the norm $\| \, . \, \|_{W_\alpha^{m,p}(\Omega)}$. Then, the dual space of $\mathring{W}_\alpha^{m,p}(\Omega)$, denoting by $W_{-\alpha}^{-m,p'}(\Omega)$, is a space of distributions. When $\Omega = \mathbb{R}^3$, we have $W^{m,p}_\alpha(\mathbb{R}^3) = \mathring{W}^{m,p}_\alpha(\mathbb{R}^3)$. If Ω is a Lipschitz exterior domain, then we have

$$\mathring{W}^{1,p}_\alpha(\Omega) = \left\{ v \in W^{1,p}_\alpha(\Omega),\ v = 0 \quad \text{on} \quad \partial\Omega \right\},$$

and

$$\mathring{W}^{2,p}_\alpha(\Omega) = \left\{ v \in W^{2,p}_\alpha(\Omega),\ v = \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega \right\},$$

where $\dfrac{\partial v}{\partial \boldsymbol{n}}$ is the normal derivate of v. The spaces $W^{1,p}_\alpha(\Omega)$ or $W^{2,p}_\alpha(\Omega)$ contain some polynomial functions. In the cases $m = 1$ or $m = 2$:

$$\mathcal{P}_j \subset W^{m,p}_\alpha(\Omega) \qquad \text{with} \quad \begin{cases} j = [m - (3/p + \alpha)] & \text{if} \quad 3/p + \alpha \notin \mathbb{Z}^-, \\ j = m - (3/p + \alpha) - 1 & \text{if} \quad 3/p + \alpha \in \mathbb{Z}^-, \end{cases} \tag{2.1}$$

where $[s]$ denotes the integer part of the real number s and $\mathcal{P}_j$ is the space of polynomials of degree less then j. We recall the following Sobolev embeddings for any real values α and $1 < p < 3$,

$$W^{1,p}_\alpha(\Omega) \hookrightarrow W^{0,p*}_\alpha(\Omega) \quad \text{where} \quad p* = \frac{3p}{3-p}. \tag{2.2}$$

and, by duality, we have

$$W^{0,q}_{-\alpha}(\Omega) \hookrightarrow W^{-1,p'}_{-\alpha}(\Omega) \quad \text{where} \quad q = \frac{3p'}{3+p'}.$$

On the other hand, if $\frac{3}{p} + \alpha \notin \{1, ..., m\}$, we have the following continuous embedding:

$$W^{m,p}_\alpha(\Omega) \hookrightarrow W^{m-1,p}_{\alpha-1}(\Omega) \hookrightarrow ... \hookrightarrow W^{0,p}_{\alpha-m}(\Omega). \tag{2.3}$$

Moreover, the Hardy inequality holds, for $1 < p < \infty$,

$$\forall\ u \in \mathring{W}^{m,p}_\alpha(\Omega), \quad ||u||_{W^{m,p}_\alpha(\Omega)} \leq C||\nabla\, u||_{\boldsymbol{W}^{m-1,p}_\alpha(\Omega)},$$

where $C = C(p, \alpha, \Omega) > 0$ and when $\Omega = \mathbb{R}^3$, we have

$$\forall u \in W^{1,p}_\alpha(\mathbb{R}^3), \quad \begin{cases} ||u||_{W^{1,p}_\alpha(\mathbb{R}^3)} \leq ||\nabla\, u||_{\boldsymbol{W}^{0,p}_\alpha(\mathbb{R}^3)}, & \text{if} \quad 3/p + \alpha > 1, \\ ||u||_{W^{1,p}_\alpha(\mathbb{R}^3)/\mathcal{P}_0} \leq ||\nabla\, u||_{\boldsymbol{W}^{0,p}_\alpha(\mathbb{R}^3)}, & \text{otherwise}, \end{cases}$$

where $\mathcal{P}_0$ stands for the space of constant functions in $W^{1,p}_\alpha(\mathbb{R}^3)$ when $3/p + \alpha \leq 1$ and C satisfies $C = C(p, \alpha) > 0$.
We consider the following spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \text{div}\, \boldsymbol{v} = 0\} \quad \text{and} \quad \boldsymbol{V}_p(\Omega) = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,p}_0(\Omega);\ \text{div}\boldsymbol{v} = 0\}.$$

In addition, we introduce the space

$$\boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \text{div}\, \boldsymbol{v} = 0\}.$$

Note that these definition will be also used with Ω replaced by $\mathbb{R}^3$ and for $k \in \mathbb{N}^*$:

$$A_k = \left\{ \mathbf{x} \in \mathbb{R}^3;\ k < |\mathbf{x}| < 2k \right\}.$$

Finally, we use bold type characters to denote vector distributions or spaces of vector distributions with 3 components and $C > 0$ usually denotes a generic constant the value of which may change from line to line.

3 Generalized solutions for the Oseen problem

We will prove in this section the existence and the uniqueness of weak solutions for the Oseen problem in an exterior domain Ω. We start by proving the existence and the uniqueness of solution in the Hilbert case *i.e* in $\boldsymbol{W}^{1,2}_0(\Omega) \times L^2(\Omega)$.

3.1 Generalized solutions in $\boldsymbol{W}_0^{1,2}(\Omega)$

Theorem 3.1. *Let*

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega), \quad h \in L^2(\Omega), \quad \boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{1/2,2}(\Gamma).$$

Then, Problem (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$*. Moreover, there exist some constants* $C_1 > 0$ *and* $C_2 > 0$ *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C_1 \big(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| h \|_{L^2(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{1/2,2}(\Gamma)}) \big), \tag{3.1}$$

$$\| \pi \|_{L^2(\Omega)} \leq C_2 \big(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + (1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})(\| h \|_{L^2(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{1/2,2}(\Gamma)}) \big), \tag{3.2}$$

where $C_1 = C(\Omega)$ *and* $C_2 = C_1 \big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}\big)$.

Proof. In order to prove the existence of solution, first we lift the boundary and the divergence data using Lemma 3.3 of [28]. Then, there exists $\boldsymbol{u}_0 \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that $\operatorname{div} \boldsymbol{u}_0 = h$ in Ω, $\boldsymbol{u}_0 = \boldsymbol{g}$ on Γ and:

$$\| \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{1,2}(\Omega)} \leq C(\| h \|_{L^2(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{1/2,2}(\Gamma)}). \tag{3.3}$$

Therefore, it remains to find $(\boldsymbol{z}, \pi) = (\boldsymbol{u} - \boldsymbol{u}_0, \pi) \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that:

$$-\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \Omega, \quad \boldsymbol{z} = 0 \quad \text{on} \quad \Gamma,$$

being $\widetilde{\boldsymbol{f}} = \boldsymbol{f} + \Delta \boldsymbol{u}_0 + \boldsymbol{v} \cdot \nabla \boldsymbol{u}_0$. Observe that $\boldsymbol{v} \cdot \nabla \boldsymbol{u}_0 = \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}_0)$ and $\boldsymbol{u}_0 \in \boldsymbol{W}_0^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ then we have $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,2}(\Omega)$. Using the density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $\boldsymbol{V}_2(\Omega)$ (see Theorem 2.6 of [27]), we deduce that the previous problem is equivalent to: Find $\boldsymbol{z} \in \boldsymbol{V}_2(\Omega)$ such that:

$$\int_\Omega \nabla \boldsymbol{z} \cdot \nabla \boldsymbol{\varphi} \, dx - b(\boldsymbol{v}, \boldsymbol{z}, \boldsymbol{\varphi}) = \big\langle \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \big\rangle_{\boldsymbol{W}_0^{-1,2}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} \qquad \forall \quad \boldsymbol{\varphi} \in \boldsymbol{V}_2(\Omega), \tag{3.4}$$

where $b = \langle \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{z}), \boldsymbol{\varphi} \rangle_{\boldsymbol{W}_0^{-1,2}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)}$ is a trilinear antisymmetric form with respect to the last two variables, well-defined for $\boldsymbol{v} \in \boldsymbol{L}^3(\Omega)$, $\boldsymbol{z}$, $\boldsymbol{\varphi} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)$. By Lax-Milgram theorem we can deduce the existence of unique solution $\boldsymbol{z} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)$ and using that $\boldsymbol{W}_0^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have

$$\begin{aligned}
\| \boldsymbol{z} \|_{\boldsymbol{W}_0^{1,2}(\Omega)} &\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \Delta \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \operatorname{div} (\boldsymbol{v} \otimes \boldsymbol{u}_0) \|_{\boldsymbol{W}_0^{-1,2}(\Omega)}) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \| \boldsymbol{v} \otimes \boldsymbol{u}_0 \|_{\boldsymbol{L}^2(\Omega)}) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}\big) \| \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{1,2}(\Omega)} \big) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}\big) \Big(\| h \|_{L^2(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{1/2,2}(\Gamma)} \Big),
\end{aligned}$$

which added to estimate (3.3) makes (3.1). Now, $-\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}_2(\Omega), \quad \Big\langle -\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \Big\rangle_{\boldsymbol{W}_0^{-1,2}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} = 0.$$

As a consequence to Corollary 3.2 in [28], there exists a unique $\pi \in L^2(\Omega)$ such that:

$$-\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \pi = \widetilde{\boldsymbol{f}} \quad \text{in} \quad \Omega$$

with $\| \pi \|_{L^2(\Omega)} \leq C \| \nabla \pi \|_{\boldsymbol{W}_0^{-1,2}(\Omega)}$. Finally, estimate (3.2) follows from the previous equation and estimate for $\boldsymbol{z}$. □

3.2 Generalized solutions in $\boldsymbol{W}_0^{1,p}(\Omega)$

We are interested in this subsection in the following class of regularity for data:

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \boldsymbol{v} \in \boldsymbol{H}_3(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma),$$

where

$$\boldsymbol{H}_3(\Omega) = \{ \boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega); \ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{W^{-1/3,3}(\Gamma) \times W^{1/3,3/2}(\Gamma)} = 0 \}.$$

Throughout the rest of this work, if we do not say otherwise, we assume that $\boldsymbol{v} \in \boldsymbol{H}_3(\Omega)$. Firstly, we recall the definition of the kernel $\mathcal{S}^p_\alpha(\Omega)$ of the Stokes operator for any real value α and $1 < p < \infty$:

$$\mathcal{S}^p_\alpha(\Omega) = \left\{(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_\alpha(\Omega) \times W^{0,p}_\alpha(\Omega);\ -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{0} \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\right\}.$$

$\mathcal{S}^p_\alpha(\Omega)$ is characterized, see [1] for more details. With the same method as in [1], we want to characterize the kernel $\mathcal{N}^p_0(\Omega)$ of the Oseen operator with Dirichlet boundary conditions:

$$\mathcal{N}^p_0(\Omega) = \Big\{(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_0(\Omega) \times L^p(\Omega);\ -\Delta\,\boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\,\pi = \boldsymbol{0} \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\Big\}.$$

We will start by $p > 2$ and the characterization of the kernel $\mathcal{N}^p_0(\Omega)$ when $p \leqslant 2$ will be done in the end of this section. We introduce the space of polynomials for each integer k:

$$\mathbf{N}_k = \{(\boldsymbol{\lambda}, \mu) \in \mathcal{P}_k \times \mathcal{P}_{k-1},\ \operatorname{div} \boldsymbol{\lambda} = 0,\ -\Delta\,\boldsymbol{\lambda} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\lambda}) + \nabla\,\mu = \boldsymbol{0}\}.$$

In particular, recall that $\mathbf{N}_k = \{(\boldsymbol{0}, 0)\}$ whenever $k < 0$ and that $\mathbf{N}_0 = \mathbb{R}^3 \times \{0\}$.

Our analysis is based on the principle that linear exterior problems can be solved by combining their properties in the whole space $\mathbb{R}^3$ and the properties in bounded domains. Let us begin by recalling some results in $\mathbb{R}^3$:

Theorem 3.2. *(Amrouche, Meslameni and Nečasová [7]). Let* $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}_0(\mathbb{R}^3)$ *satisfying the compatibility condition:*

$$\langle f_i, 1\rangle_{W^{-1,p}_0(\mathbb{R}^3)\times W^{1,p'}_0(\mathbb{R}^3)} = 0 \quad \text{for any } i = 1, 2, 3 \quad \text{if} \quad p \leq 3/2 \tag{3.5}$$

and let $h \in L^p(\mathbb{R}^3)$ *and* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. *Then the Oseen problem* (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *if* $p < 3$ *and if* $p \geq 3$, $\boldsymbol{u}$ *is unique up to an additive constant vector. In addition, we have*

$$||\boldsymbol{u}||_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}^{-1,p}_0(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right). \tag{3.6}$$

The second result:

Lemma 3.3. *(Amrouche, Meslameni and Nečasová [7]). Supposing that* $1 < r \leq 2 < p$. *Let* $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}_0(\mathbb{R}^3) \cap \boldsymbol{W}^{-1,r}_0(\mathbb{R}^3)$ *satisfying the compatibility condition* (3.5) *if* $r \leq 3/2$ *and* $h \in L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ *and* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. *Then the Oseen problem* (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}^{1,p}_0(\mathbb{R}^3) \cap \boldsymbol{W}^{1,r}_0(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ *such that*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}^{1,r}_0(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \times \\ \left(||\boldsymbol{f}||_{\boldsymbol{W}^{-1,p}_0(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}^{-1,r}_0(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)} + ||h||_{L^r(\mathbb{R}^3)}\right). \end{aligned} \tag{3.7}$$

Finally, we recall:

Theorem 3.4. *(Amrouche, Meslameni and Nečasová [7]). Suppose that* $1 < p < 3$ *and* $p \neq 3/2$. *Let* $h \in W^{1,p}_1(\mathbb{R}^3)$ *and* $\boldsymbol{f} \in \boldsymbol{W}^{0,p}_1(\mathbb{R}^3)$ *such that*

$$\int_{\mathbb{R}^3} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} = 0 \quad \text{if} \quad p < 3/2, \tag{3.8}$$

and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. *Then the Oseen problem* (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}_1(\mathbb{R}^3) \times W^{1,p}_1(\mathbb{R}^3)$ *satisfying the following estimate:*

$$||\boldsymbol{u}||_{\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)} + ||\pi||_{W^{1,p}_1(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^6(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)} + \|h\|_{W^{1,p}_1(\mathbb{R}^3)}). \tag{3.9}$$

Now we can prove the following result:

Theorem 3.5. *Suppose that* $p > 2$.

i) If $p < 3$, *then* $\mathcal{N}^p_0(\Omega) = \{(\boldsymbol{0}, 0)\}$.

ii) If $p \geq 3$, then

$$\mathcal{N}_0^p(\Omega) = \left\{(\boldsymbol{z}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu), \quad (\boldsymbol{\lambda}, \mu) \in \boldsymbol{N}_{[1-3/p]}\right\}, \tag{3.10}$$

where $(\boldsymbol{z}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ denotes the unique solution in $\bigcap_{3/2<r\leq p} \boldsymbol{W}_0^{1,r}(\Omega) \times \boldsymbol{L}^r(\Omega)$ of the following equations

$$-\Delta\, \boldsymbol{z} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{z} + \nabla\, \eta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega,\ \boldsymbol{z} = \boldsymbol{\lambda} \quad \text{on} \quad \Gamma. \tag{3.11}$$

Proof. The proof follows the idea of [5]. Let $(\boldsymbol{u}, \pi)$ be an element of $\mathcal{N}_0^p(\Omega)$ and let $\boldsymbol{u}$ and π be extended by zero in Ω'. The extended functions, denoted by $\widetilde{\boldsymbol{u}}$ and $\widetilde{\pi}$ respectively belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$. Now, we extend $\boldsymbol{v}$ in $\mathbb{R}^3$ in the following way: We solve the following Neumann problem in Ω':

$$\Delta\, \theta = 0 \quad \text{in} \quad \Omega' \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma.$$

Owing to the boundary condition, this problem has a solution $\theta \in W^{1,3}(\Omega')$. Let us take

$$\boldsymbol{w} = \nabla\, \theta \quad \text{in} \quad \Omega' \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{v} \quad \text{in} \quad \Omega.$$

Then $\boldsymbol{w}$ belongs to $\boldsymbol{L}^3(\mathbb{R}^3)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ then we have

$$\begin{aligned} < \operatorname{div} \boldsymbol{w}, \varphi >_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= -\int_{\mathbb{R}^3} \boldsymbol{w} \cdot \nabla\, \varphi \, dx \\ &= -\int_{\Omega} \boldsymbol{v} \cdot \nabla\, \varphi \, dx - \int_{\Omega'} \nabla\, \theta \cdot \nabla\, \varphi \, dx \\ &= \ < \boldsymbol{v} \cdot \boldsymbol{n}, \varphi >_\Gamma - < \frac{\partial \theta}{\partial \boldsymbol{n}}, \varphi >_\Gamma = 0, \end{aligned}$$

where $< .,. >_\Gamma = < .,. >_{W^{-1/3,3}(\Gamma)\times W^{1/3,3/2}(\Gamma)}$. Then $\operatorname{div} \boldsymbol{w} = 0$ in $\mathbb{R}^3$ and thus $\boldsymbol{w}$ belongs to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Set

$$-\Delta\, \widetilde{\boldsymbol{u}} + \boldsymbol{w} \cdot \nabla\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi} := \mathbf{F} \quad \text{and} \quad \operatorname{div} \widetilde{\boldsymbol{u}} := e \quad \text{in} \quad \mathbb{R}^3. \tag{3.12}$$

Then $(\mathbf{F}, e)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and obviously they have a compact support. Since $p > 2$, we deduce that $(\mathbf{F}, e)$ belongs to $\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. It follows from Lemma 3.3 that there exists a solution $(\boldsymbol{z}, \eta)$ in $\boldsymbol{W}_0^{1,r}(\mathbb{R}^3) \times L^r(\mathbb{R}^3)$ for any $r \in]3/2, p]$ such that

$$-\Delta(\widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w} \cdot \nabla(\widetilde{\boldsymbol{u}} - \boldsymbol{z})) + \nabla(\widetilde{\pi} - \eta) = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(\widetilde{\boldsymbol{u}} - \boldsymbol{z}) = 0 \quad \text{in} \quad \mathbb{R}^3.$$

If $p < 3$, we deduce from the argument of uniqueness in Theorem 3.2 that $(\widetilde{\boldsymbol{u}} - \boldsymbol{z}, \widetilde{\pi} - \eta) = (\boldsymbol{0}, 0)$ and thus $\widetilde{\boldsymbol{u}}$ and $\widetilde{\pi}$ belongs respectively to $\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)$ and $L^r(\mathbb{R}^3)$ for any $3/2 < r \leq p$, which implies that $(\boldsymbol{u}, \pi)$ belongs to $\mathcal{N}_0^2(\Omega)$ and so $(\boldsymbol{u}, \pi) = \{(\boldsymbol{0}, 0)\}$. If $p \geq 3$, using again Theorem 3.2, we necessarily have $(\widetilde{\boldsymbol{u}} - \boldsymbol{z}, \widetilde{\pi} - \eta) = (\boldsymbol{\lambda}, \mu) \in \mathbf{N}_{[1-3/p]}$ and since $\boldsymbol{u} = \boldsymbol{0}$ on Γ, the restriction of $(\boldsymbol{z}, \eta)$ to Ω is nothing else but $(\boldsymbol{z}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ which verifies (3.11). Observe that in this case, $\boldsymbol{\lambda}$ is a vector constant of $\mathbb{R}^3$ and $\mu = 0$. ☐

Remark 3.6.

1. We shall see at the end of this section that in fact $\mathcal{N}_0^p(\Omega) = \{(\boldsymbol{0}, 0)\}$ for $1 < p < 3$.
2. Of course, we have seen at the beginning of this section that $\mathcal{N}_0^2(\Omega) = \{(\boldsymbol{0}, 0)\}$.

The next lemma solves Problem (1.1) with homogeneous boundary conditions and a right-hand side $\boldsymbol{f}$ and h with bounded support.

Lemma 3.7. *Assume that $p > 2$ and $\boldsymbol{g} = \boldsymbol{0}$ on Γ. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$ such that $\boldsymbol{f}$ and h have a compact support. Then, the Oseen Problem (1.1) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$ and $\pi \in L^2(\Omega) \cap L^p(\Omega)$.*

Proof. By virtue of Lemma 2.1 of [5], the right-hand side $\boldsymbol{f}$ belongs also to $\boldsymbol{W}_0^{-1,2}(\Omega)$. Since $p > 2$ and support of h is compact, we have $h \in L^2(\Omega)$. Due to Theorem 3.1, Problem (1.1) has exactly one solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. The remainder of the proof is devoted to establish that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. Take R_0 sufficiently large so that both the supports of $(\boldsymbol{f}, h)$ are contained in B_{R_0} and $\overline{\Omega'} \subset B_{R_0}$. Let λ and μ be two scalar, nonnegative functions in $C^\infty(\mathbb{R}^3)$ that satisfy

$$\forall\ x \in B_{R_0}, \quad \lambda(x) = 1, \quad \operatorname{supp} \lambda \subset B_{R_0+1}, \quad \forall\ x \in \mathbb{R}^3, \quad \lambda(x) + \mu(x) = 1.$$

Let Ω_{R_0+1} denotes the intersection $\Omega \cap B_{R_0+1}$ and let C_{R_0} denote the exterior (*i.e.* the complement) of B_{R_0}. Then, we can write

$$\boldsymbol{u} = \lambda\, \boldsymbol{u} + \mu\, \boldsymbol{u}, \qquad \pi = \lambda\, \pi + \mu\, \pi.$$

As μ is very smooth and vanishes on B_{R_0}, then $\mu\, \boldsymbol{f} = \boldsymbol{0}$ and $\mu\, h = 0$. Let us extend $(\boldsymbol{u}, \pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and let $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ such as in Theorem 3.5. After an easy calculation, we obtain that the pair $(\mu\widetilde{\boldsymbol{u}}, \mu\widetilde{\pi})$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\mu\, \widetilde{\boldsymbol{u}}) + \boldsymbol{w} \cdot \nabla(\mu\, \widetilde{\boldsymbol{u}}) + \nabla\, (\mu\, \widetilde{\pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\mu\, \widetilde{\boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = (\Delta\, \lambda)\widetilde{\boldsymbol{u}} - (\nabla\, \lambda)\widetilde{\pi} + 2\nabla\, \lambda \cdot \nabla\, \widetilde{\boldsymbol{u}} - (\boldsymbol{w} \cdot \nabla\, \lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = -\nabla\, \lambda \cdot \widetilde{\boldsymbol{u}}\ .$$

Moreover, owing to the supports of μ and λ, $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{L}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. In addition, if $\mathcal{O}$ is a Lipschitzian bounded domain, we have $\boldsymbol{L}^2(\mathcal{O}) \hookrightarrow \boldsymbol{W}^{-1,q}(\mathcal{O})$ and $H^1(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$ for any $2 \leqslant q \leqslant 6$. Hence, we shall assume for the time being that $2 < p \leqslant 6$ and afterward, we shall use a bootstrap argument. Then $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. It follows from Lemma 3.3 that there exists $(\boldsymbol{z}, \theta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{z} + \boldsymbol{w} \cdot \nabla\, \boldsymbol{z} + \nabla\, \theta = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = e_1 \quad \text{in} \quad \mathbb{R}^3.$$

And thus,

$$-\Delta(\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w} \cdot \nabla(\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \nabla(\mu\widetilde{\pi} - \theta) = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z}) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

with $\mu\widetilde{\pi} - \theta \in L^2(\mathbb{R}^3)$ and $\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)$. Then

$$\langle -\Delta(\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w} \cdot \nabla(\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \nabla(\mu\widetilde{\pi} - \theta), \mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z} \rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} = 0,$$

and so

$$\|\nabla\, (\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z})\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0.$$

Thus $\mu\, \widetilde{\boldsymbol{u}} - \boldsymbol{z} = \boldsymbol{0}$ and so $\mu\, \widetilde{\pi} - \theta = 0$. Consequently, $(\mu\, \widetilde{\boldsymbol{u}}, \mu\, \widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$.

In particular, we have $\mu\, \widetilde{\boldsymbol{u}} = \boldsymbol{u}$ and $\mu\, \widetilde{\pi} = \pi$ outside B_{R_0+1}, so the restriction of $\boldsymbol{u}$ to ∂B_{R_0+1} belongs to $\boldsymbol{W}^{1/p',p}(\partial\, B_{R_0+1})$. Therefore, $(\boldsymbol{u}, \pi)$ satisfies:

$$-\Delta\, \boldsymbol{u} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega_{R_0+1}, \quad \boldsymbol{u}|_{\partial B_{R_0+1}} = \widetilde{\boldsymbol{u}} \quad \text{and} \quad \boldsymbol{u}|_\Gamma = \boldsymbol{0}.$$

Observe that for any $\varphi \in W^{1,2}(\Omega_{R_0+1})$ we have

$$\int_{\Omega_{R_0+1}} \boldsymbol{u} \cdot \nabla\, \varphi\, dx = -\int_{\Omega_{R_0+1}} \varphi \operatorname{div} \boldsymbol{u}\, dx + \int_{\partial\, \Omega_{R_0+1}} \varphi \boldsymbol{u} \cdot \boldsymbol{n}\, dx.$$

In particular, for $\varphi = 1$, we have

$$\int_{\Omega_{R_0+1}} h(x)\, dx = \int_{\partial\, \Omega_{R_0+1}} \boldsymbol{u} \cdot \boldsymbol{n}\, dx$$

and thus, according to Theorem 15 see [10], this problem has a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega_{R_0+1}) \times L^p(\Omega_{R_0+1})$. This implies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ if $2 < p \leq 6$. Now, suppose that $p > 6$. The above argument shows that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,6}(\Omega) \times L^6(\Omega)$ and we can repeat the same argument with $p = 6$ instead of $p = 2$ using the fact that if $\mathcal{O}$ is a Lipschitzian bounded domain, we have $\boldsymbol{L}^6(\mathcal{O}) \hookrightarrow \boldsymbol{W}^{-1,t}(\mathcal{O})$ for any real number $t > 1$. This establishes the existence of solution $(\boldsymbol{u}, \pi)$ in $\boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ of Problem (1.1) when $p > 2$. Uniqueness follows from the fact that $\boldsymbol{W}_0^{1,2}(\Omega)$ does not contain the vector constant functions. □

The next lemma solves Problem (1.1) with non homogeneous boundary conditions and a right-hand side $\boldsymbol{f}$ and h with bounded support.

Lemma 3.8. *Under the assumptions of Lemma 3.7, for each* $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$*, Problem* (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \times L^p(\Omega) \cap L^2(\Omega)$.

Proof. Let $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$ and take R sufficiently large so that $\overline{\Omega'} \subset B_R$. Set $\Omega_R = \Omega \cap B_R$, then there exists $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega_R)$ solution of the problem $-\Delta \boldsymbol{z} = \boldsymbol{0}$ in Ω_R, $\boldsymbol{z} = \boldsymbol{g}$ on Γ and $\boldsymbol{z} = \boldsymbol{0}$ on ∂B_R. We extend $\boldsymbol{z}$ by zero out of B_R. The extended function denoted by $\widetilde{\boldsymbol{z}}$ has a compact support in $\overline{\Omega'}$ and belongs to $\boldsymbol{W}_0^{1,p}(\Omega)$ and once we set $\boldsymbol{u}' = \boldsymbol{u} - \widetilde{\boldsymbol{z}}$. Then Problem (1.1) is equivalent to the following problem: Find $(\boldsymbol{u}', \pi)$ such that

$$\begin{cases} -\Delta \boldsymbol{u}' + \boldsymbol{v} \cdot \nabla \boldsymbol{u}' + \nabla \pi = \boldsymbol{f} + \boldsymbol{v} \cdot \nabla \widetilde{\boldsymbol{z}} + \Delta \widetilde{\boldsymbol{z}}, \\ \text{div } \boldsymbol{u}' = h + \text{div } \widetilde{\boldsymbol{z}} \text{ in } \Omega, \boldsymbol{u}'|_{\partial\Omega} = \boldsymbol{0}, \end{cases} \tag{3.13}$$

where data belonging to the space $\boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ with the compact support in $\bar{\Omega}$. Then we will apply Lemma 3.7 . □

Corollary 3.9. *Assume that $p > 2$ and let $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$. Then there exists $(\boldsymbol{u}, \pi) \in \Big(\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)\Big) \times \big(L^p(\Omega) \cap L^2(\Omega)\big)$ such that*

$$-\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, \quad \text{div } \boldsymbol{u} = 0 \quad \textit{in} \quad \Omega \quad \text{and} \quad \boldsymbol{u}|_\Gamma = \boldsymbol{g}.$$

Proof. Let $R_0 > 0$ such that $\overline{\Omega'} \subset B_{R_0}$. Take $\psi \in \mathcal{D}(\mathbb{R}^3)$ with support in Ω_{R_0} and such that

$$\int_{\Omega_{R_0}} \psi(x) dx + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} ds = 0.$$

According to Theorem 12 [10], there exists $(\boldsymbol{z}, \eta) \in \boldsymbol{W}^{1,p}(\Omega_{R_0}) \times L^p(\Omega_{R_0})$ such that

$$-\Delta \boldsymbol{z} + \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla \eta = 0, \text{ div } \boldsymbol{z} = \psi \text{ in } \Omega_{R_0}, \ \boldsymbol{z}|_{\partial B_{R_0}} = \boldsymbol{0}, \boldsymbol{z}|_\Gamma = \boldsymbol{g}.$$

If we denote the extension by $(\boldsymbol{0}, 0)$ of $(\boldsymbol{z}, \eta)$ outside B_{R_0} by $(\tilde{\boldsymbol{z}}, \tilde{\eta})$ then $(\tilde{\boldsymbol{z}}, \tilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\begin{aligned} -\Delta \tilde{\boldsymbol{z}} + \boldsymbol{v} \cdot \nabla \tilde{\boldsymbol{z}} + \nabla \tilde{\eta} \quad &:= \quad \boldsymbol{\xi} \qquad \text{in} \quad \Omega, \\ \text{div } \tilde{\boldsymbol{z}} \quad &= \quad \psi \qquad \text{in} \quad \Omega, \\ \tilde{\boldsymbol{z}} \quad &= \quad \boldsymbol{g} \qquad \text{on} \quad \Gamma. \end{aligned}$$

Observe that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$ with compact support in $\overline{\Omega'}$. From Theorem 3.7 we have a solution $(\boldsymbol{w}, \tau) \in \big(\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega)\big) \times \big(L^p(\Omega) \cap L^2(\Omega)\big)$ to the problem

$$-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \tau = -\boldsymbol{\xi}, \text{ div } \boldsymbol{w} = -\psi \text{ in } \Omega \quad \text{and} \quad \boldsymbol{w}|_\Gamma = \boldsymbol{0}.$$

Then the pair $(\boldsymbol{u}, \pi) = (\tilde{\boldsymbol{z}} + \boldsymbol{w}, \tilde{\eta} + \tau)$ has the required properties. □

As consequence, we prove the following result:

Theorem 3.10. *Assume that $p > 2$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$. Then Problem (1.1) has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ unique up to an element of $\mathcal{N}_0^p(\Omega)$.*

Proof. **i) First case**: $\boldsymbol{g} = \boldsymbol{0}$.
We would like to extend data $(\boldsymbol{f}, h) \in \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ to the whole space. According to Corollary 1.3 of [1] there exists a second-tensor $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$ such that div $\boldsymbol{F} = \boldsymbol{f}$. Then we extend $\boldsymbol{F}$ (resp. h) by zero into the whole space and we denote this extension by $\tilde{\boldsymbol{F}}$ (resp. $\tilde{h}$). Set $\tilde{\boldsymbol{f}} = \text{div } \tilde{\boldsymbol{F}}$. It is clear that $(\tilde{\boldsymbol{f}}, \tilde{h}) \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. Now, we consider the following equation:

$$-\Delta \tilde{\boldsymbol{z}} + \boldsymbol{w} \cdot \nabla \tilde{\boldsymbol{z}} + \nabla \tilde{\eta} = \tilde{\boldsymbol{f}} \quad \text{and} \quad \text{div } \tilde{\boldsymbol{z}} = \tilde{h} \quad \text{in} \quad \mathbb{R}^3,$$

with $\boldsymbol{w} \in \boldsymbol{L}_\sigma^3(\mathbb{R}^3)$ introduced in the proof of Theorem 3.5. Applying the theory of Oseen problem in $\mathbb{R}^3$, we deduce that this problem has a unique solution $(\tilde{\boldsymbol{z}}, \tilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $p < 3$ and if $p \geq 3$, $\tilde{\boldsymbol{z}}$ is unique solution up to a constant vector. In addition, we have:

$$\|\tilde{\boldsymbol{z}}\|_{\boldsymbol{W}^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + \|\tilde{\eta}\|_{L^p(\mathbb{R}^3)} \leq C \left(\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3} + \|\tilde{h}\|_{L^p(\mathbb{R}^3)} \right). \tag{3.14}$$

Denoting the restriction to Ω by $(\boldsymbol{z}, \eta)$ and by $\gamma \boldsymbol{z} \in \boldsymbol{W}^{1/p',p}(\Gamma)$ the trace of $\boldsymbol{z}$ on Γ. According to Corollary 3.9, we have the existence of $(\boldsymbol{\xi}, \nu) \in \big(\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \times L^p(\Omega) \cap L^2(\Omega)\big)$ such that

$$-\Delta \boldsymbol{\xi} + \boldsymbol{v} \cdot \nabla \boldsymbol{\xi} + \nabla \nu = 0, \text{ div } \boldsymbol{\xi} = 0 \text{ in } \Omega, \boldsymbol{\xi}|_\Gamma = -\gamma \boldsymbol{z}.$$

Hence, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \boldsymbol{\xi}, \eta + \nu)$ belongs to $\boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and satisfies Problem (1.1) with $\boldsymbol{g} = \boldsymbol{0}$.

ii) Second case: Nonhomogeneous boundary data. Each $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$ has a lifting $\boldsymbol{\chi} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that

$$\|\boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}.$$

Setting $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{\chi}$, then Problem (1.1) is equivalent to the following problem: Find $(\boldsymbol{u}', \pi)$ such that

$$\begin{aligned} -\Delta \boldsymbol{u}' + \boldsymbol{v} \cdot \nabla \boldsymbol{u}' + \nabla \pi &= \boldsymbol{f} + \Delta \boldsymbol{\chi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\chi} && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u}' &= h - \operatorname{div} \boldsymbol{\chi} && \text{in} \quad \Omega, \\ \boldsymbol{u}' &= 0 && \text{on} \quad \Gamma. \end{aligned}$$

Set $\boldsymbol{f}_{\boldsymbol{\chi}} = \boldsymbol{f} + \Delta \boldsymbol{\chi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\chi}$ and $h_{\boldsymbol{\chi}} = h - \operatorname{div} \boldsymbol{\chi}$. As $p > 2$, $\boldsymbol{v} \cdot \nabla \boldsymbol{\chi} \in \boldsymbol{L}^r(\Omega)$, with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$ and $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$. Hence, $\boldsymbol{f}_{\boldsymbol{\chi}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$. From previous step we know that this problem has a solution in $\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$. Uniqueness follows from the definition of the kernel $\mathcal{N}_0^p(\Omega)$. □

In particular, it follows from Theorem 3.10 that, for any $p \geq 2$, the Oseen operator

$$\mathcal{O} : \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathcal{N}_0^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$$

defined by : $\mathcal{O}(\boldsymbol{u}, \pi) = \big(-\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi, \operatorname{div} \boldsymbol{u}\big)$ is obviously continuous and since both spaces are Banach spaces, it is an isomorphism. Thus there exists a constant $C(\boldsymbol{v})$ depending on $\boldsymbol{v} \in \boldsymbol{L}_\sigma^3(\Omega)$, Ω and p such that

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}_0^p(\Omega)} ||\boldsymbol{u} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi + \mu||_{L^p(\Omega)} \leq C(\boldsymbol{v})\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)}\big). \tag{3.15}$$

The following existence result can be stated via a dual argument.

Theorem 3.11. *Suppose that* $1 < p < 2$ *and* $\boldsymbol{g} = \boldsymbol{0}$. *Let* $(\boldsymbol{f}, h) \in \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ *such that for any* $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$, *we have*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \mu \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = 0. \tag{3.16}$$

Then the Oseen problem (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$.

Proof. On one hand, Green formula yields, for all $\boldsymbol{w} \in \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)$ and $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$

$$\langle -\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} =$$

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)}.$$

Taking into account that $p' > 2$, we have $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) = \boldsymbol{v} \cdot \nabla \boldsymbol{w} \in \boldsymbol{L}^r(\Omega)$ with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p'}$ and $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega)$. On the other hand, for all $\eta \in L^{p'}(\Omega)$,

$$\langle \boldsymbol{u}, \nabla \eta \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} = -\langle \operatorname{div} \boldsymbol{u}, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)}.$$

Then problem (1.1) with $\boldsymbol{g} = \boldsymbol{0}$ has the following equivalent variational formulation: find $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that for all $(\boldsymbol{w}, \eta) \in \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)$,

$$\begin{aligned} &\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla \eta \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = \\ &= \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)}. \end{aligned} \tag{3.17}$$

According to Theorem 3.10, for each $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{w}, \eta) \in (\mathring{\boldsymbol{W}}_0^{1,p'}(\Omega) \times L^{p'}(\Omega))/\mathcal{N}_0^{p'}(\Omega)$ such that

$$-\Delta \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla \eta = \boldsymbol{f}', \quad \operatorname{div} \boldsymbol{w} = h' \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma,$$

with the following estimate

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}_0^{p'}(\Omega)} ||\boldsymbol{w} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p'}(\Omega)} + ||\eta + \mu||_{L^{p'}(\Omega)} \leq C(\boldsymbol{v})\big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} + ||h'||_{L^{p'}(\Omega)}\big). \tag{3.18}$$

Let T be a linear form defined from $\mathbf{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ onto $\mathbb{R}$ by:

$$T: \ (\boldsymbol{f}', h') \mapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} .$$

Observe that for any pair $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ and for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$ we have

$$\begin{aligned} |T(\boldsymbol{f}', h')| &= \left| \langle \boldsymbol{f}, \boldsymbol{w} + \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta + \mu \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} \right| \\ &\leq ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} ||\boldsymbol{w} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p'}(\Omega)} + ||h||_{L^p(\Omega)} ||\eta + \mu||_{L^{p'}(\Omega)}. \end{aligned}$$

Using (3.18), we prove that

$$|T(\boldsymbol{f}', h')| \leq C(\boldsymbol{v}) \big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)} \big) \big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\Omega)} + ||h'||_{L^{p'}(\Omega)} \big).$$

Thus the linear form T is continuous on the following space $\mathbf{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$T(\boldsymbol{f}', h') = \langle \boldsymbol{u}, \boldsymbol{f}' \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, h' \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} ,$$

with

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leq C(\boldsymbol{v}) \big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)} \big). \tag{3.19}$$

By definition of T, it follows

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = \langle \boldsymbol{u}, \boldsymbol{f}' \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, h' \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} ,$$

which is the variational formulation (3.17). □

Now, let us prove the appropriate estimate for the Oseen problem (1.1) and we start by the case $1 < p < 2$:

Theorem 3.12. *Suppose that $1 < p < 2$ and let $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ be the unique solution of the Oseen problem* (1.1) *given by Theorem 3.11 with the following data:*

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \text{and} \quad \boldsymbol{g} = \boldsymbol{0}$$

and for any $(\boldsymbol{\xi}, \eta) \in \mathcal{N}_0^{p'}(\Omega)$, we have

$$\langle \boldsymbol{f}, \boldsymbol{\xi} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = 0.$$

Then $(\boldsymbol{u}, \pi)$ satisfies the following estimate :

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C \big(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)} \big)^2 \big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}) ||h||_{L^p(\Omega)} \big). \tag{3.20}$$

Proof. Since $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, then it follows from Corollary 1.3 of [1] that $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \boldsymbol{L}^p(\Omega)$ and $||\,\mathbf{F}\,||_{\boldsymbol{L}^p(\Omega)} \leq C ||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)}$. Let extend $\mathbf{F}$ and h by zero in Ω'. The extended functions, denoted by $\widetilde{\mathbf{F}}$ and $\widetilde{h}$ belong to $\boldsymbol{L}^p(\mathbb{R}^3)$ and set $\widetilde{\boldsymbol{f}} = \operatorname{div} \widetilde{\mathbf{F}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Let ϕ be a truncation function: $\phi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \phi(t) \leqslant 1$ for any $t \in \mathbb{R}^3$ and

$$\phi(t) = \begin{cases} 1 & \text{if} \quad |\, t \,| \leqslant 1, \\ 0 & \text{if} \quad |\, t \,| \geqslant 2. \end{cases}$$

Let λ be a cut off function, defined on $\mathbb{R}^3$ by $\lambda(\mathbf{x}) = \phi(\dfrac{\mathbf{x}}{R})$ for any R sufficiently large so that $\overline{\Omega'} \subset B_R$. Set $\mu = 1 - \lambda$. Let Ω_R denote the intersection $\Omega \cap B_R$. Now, let $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ be the unique solution of the Oseen problem (1.1) given by Theorem 3.11 and let us extend $(\boldsymbol{u}, \pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and let $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ such as in Theorem 3.5. Then

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leq ||\mu \, \widetilde{\boldsymbol{u}}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\lambda \, \boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\mu \, \widetilde{\pi}||_{L^p(\mathbb{R}^3)} + ||\lambda \, \pi||_{L^p(\Omega_R)}. \tag{3.21}$$

After an easy calculation, we obtain that the pair $(\mu\widetilde{\boldsymbol{u}}, \mu\widetilde{\pi})$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\mu\,\widetilde{\boldsymbol{u}}) + \operatorname{div}(\boldsymbol{w}\otimes(\mu\,\widetilde{\boldsymbol{u}})) + \nabla(\mu\,\widetilde{\pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\mu\,\widetilde{\boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = \mu\widetilde{\boldsymbol{f}} + (\Delta\,\lambda)\widetilde{\boldsymbol{u}} - (\nabla\,\lambda)\widetilde{\pi} + 2\nabla\,\lambda\cdot\nabla\,\widetilde{\boldsymbol{u}} - (\boldsymbol{w}\cdot\nabla\,\lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = \mu\,\widetilde{h} - \nabla\,\lambda\cdot\widetilde{\boldsymbol{u}}\ .$$

From Theorem 3.2, we have

$$\begin{aligned} ||\mu\,\widetilde{\boldsymbol{u}}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\mu\,\widetilde{\pi}||_{L^p(\mathbb{R}^3)} &\leq C(1+||\boldsymbol{w}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\Big(||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{w}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||e_1||_{L^p(\mathbb{R}^3)}\Big) \\ &\leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})\Big(||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_1||_{L^p(\mathbb{R}^3)}\Big) \quad (3.22) \end{aligned}$$

Now, let $\boldsymbol{\varphi}\in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$, then we have

$$\begin{aligned} |\,\langle(\Delta\,\lambda)\widetilde{\boldsymbol{u}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| &\leq C\int_{A_R}|\frac{1}{R^2}\boldsymbol{\varphi}(\Delta\,\phi)\widetilde{\boldsymbol{u}}|\,d\mathbf{x} \\ &\leq C_R||\frac{\widetilde{\boldsymbol{u}}}{\rho}||_{\boldsymbol{L}^p(A_R)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}, \end{aligned}$$

where $C_R := C\dfrac{(1+4R^2)^{1/2}}{R}$, $C>0$.

$$|\,\langle(\nabla\,\lambda)\widetilde{\pi}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| \leq C_R||\widetilde{\pi}||_{L^p(A_R)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}.$$

$$|\,\langle 2\nabla\,\lambda\cdot\nabla\,\widetilde{\boldsymbol{u}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| \leq C_R||\nabla\,\widetilde{\boldsymbol{u}}||_{\boldsymbol{L}^p(A_R)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}.$$

$$|\,\langle(\boldsymbol{w}\cdot\nabla\,\lambda)\widetilde{\boldsymbol{u}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| \leq C_R||\boldsymbol{w}\otimes\widetilde{\boldsymbol{u}}||_{\boldsymbol{L}^p(A_R)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}.$$

$$|\left\langle \mu\widetilde{\boldsymbol{f}}, \boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| = |\int_\Omega \mathbf{F}\nabla(\mu\boldsymbol{\varphi})\,d\mathbf{x}| \leq C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}.$$

Now, let $\varphi\in L^{p'}(\mathbb{R}^3)$, then we have

$$\begin{aligned} |\int_{\mathbb{R}^3}(\mu\,\widetilde{h} - \nabla\,\lambda\cdot\widetilde{\boldsymbol{u}})\varphi\,d\mathbf{x}| &\leq C\left[||h||_{L^p(\Omega)}||\mu\,\varphi||_{L^{p'}(\Omega)} + C_R||\frac{\widetilde{\boldsymbol{u}}}{\rho}||_{\boldsymbol{L}^p(A_R)}||\varphi||_{L^{p'}(\mathbb{R}^3)}\right] \\ &\leq C\left[||h||_{L^p(\Omega)} + C_R||\frac{\widetilde{\boldsymbol{u}}}{\rho}||_{\boldsymbol{L}^p(A_R)}\right]||\varphi||_{L^{p'}(\mathbb{R}^3)}. \end{aligned}$$

Then, we deduce from the previous inequalities that

$$\begin{aligned} &||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_1||_{L^p(\mathbb{R}^3)} \leq \\ &\leq C\Big[||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + C_R||\nabla\,\widetilde{\boldsymbol{u}}||_{\boldsymbol{L}^p(A_R)} + C_R||\frac{\widetilde{\boldsymbol{u}}}{\rho}||_{\boldsymbol{L}^p(A_R)} \\ &+ C_R\left\{||\widetilde{\pi}||_{L^p(A_R)} + ||\boldsymbol{w}\otimes\widetilde{\boldsymbol{u}}||_{\boldsymbol{L}^p(A_R)}\right\} \\ &+ \big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)\left\{C_R||\frac{\widetilde{\boldsymbol{u}}}{\rho}||_{\boldsymbol{L}^p(A_R)} + ||h||_{L^p(\Omega)}\right\}\Big]. \end{aligned} \quad (3.23)$$

Similarly, the pair $(\lambda\,\boldsymbol{u}, \lambda\,\pi)$ satisfies the following equations in Ω_R:

$$-\Delta(\lambda\,\boldsymbol{u}) + \operatorname{div}(\boldsymbol{v}\otimes(\lambda\,\boldsymbol{u})) + \nabla(\lambda\,\pi) := \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div}(\lambda\,\boldsymbol{u}) := e_2 \quad \text{in} \quad \Omega_R, \quad (\lambda\,\boldsymbol{u})|_{\partial B_{2R}} = \mathbf{0} \quad \text{and} \quad (\lambda\,\boldsymbol{u})|_\Gamma = \mathbf{0},$$

with

$$\boldsymbol{f}_2 = \lambda\,\boldsymbol{f} + (\Delta\,\mu)\boldsymbol{u} - (\nabla\,\mu)\pi + 2\nabla\,\mu\cdot\nabla\,\boldsymbol{u} - (\boldsymbol{v}\cdot\nabla\,\mu)\boldsymbol{u} \quad \text{and} \quad e_2 = \lambda\,h - \nabla\,\mu\cdot\boldsymbol{u}\ .$$

Using Theorem 15 of [10], we prove that

$$||\lambda\,\boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\lambda\,\pi||_{L^p(\Omega_R)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})^2\left(||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_2||_{L^p(\Omega_R)}\right). \quad (3.24)$$

As in the beginning of the proof, we show that

$$||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} \leq C\left[(1+\frac{1}{R})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + C_R||\frac{\boldsymbol{u}}{\rho}||_{\boldsymbol{L}^p(A_R)} + \frac{1}{R}\left\{||\pi||_{L^p(A_R)} + ||\nabla\,\boldsymbol{u}|_{\boldsymbol{L}^p(A_R)} + ||\boldsymbol{v}\otimes\boldsymbol{u}||_{\boldsymbol{L}^p(A_R)}\right\}\right] \quad (3.25)$$

and that

$$||e_2||_{L^p(\Omega_R)} \leq C\left[C_R||\frac{\boldsymbol{u}}{\rho}||_{\boldsymbol{L}^p(A_R)} + ||h||_{L^p(\Omega)}\right]. \quad (3.26)$$

Using (3.21)-(3.26) and tending R to ∞, we prove thanks to dominated convergence theorem the estimate (3.20). □

Remark 3.13. Under the assumptions of Theorem 3.12 and supposing that $6/5 \leq p < 2$, the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate :

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C\big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||h||_{L^p(\Omega)}\big).$$

Indeed, we shall use in the proof of Theorem 3.12 the following estimate

$$||\lambda\,\boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\lambda\,\pi||_{L^p(\Omega_R)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})\left(||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_2||_{L^p(\Omega_R)}\right),$$

instead of (3.24), see Proposition 3 of [10].

Now, we study the nonhomogeneous boundary data *i.e* $\boldsymbol{g} \neq \boldsymbol{0}$ on Γ.

Corollary 3.14. *Suppose that* $1 < p < 2$. *Let* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$ *such that for any* $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$, *we have*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda}\rangle_{\boldsymbol{W}_0^{-1,p}(\Omega)\times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \mu\rangle_{L^p(\Omega)\times L^{p'}(\Omega)} + \langle \boldsymbol{g}, (\mu I - \nabla\,\boldsymbol{\lambda})\cdot\boldsymbol{n}\rangle_\Gamma = 0. \quad (3.27)$$

Then the Oseen problem (1.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ *such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^2\Big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)\big(||h||_{L^p(\Omega)} + ||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big)\Big) \quad (3.28)$$

Proof. Let $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$, then there exists $\boldsymbol{\chi} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{\chi} = \boldsymbol{g}$ on Γ and

$$\|\boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}. \quad (3.29)$$

Setting $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{\chi}$, then Problem (1.1) is equivalent to the following problem: Find $(\boldsymbol{u}', q)$ such that

$$\begin{array}{rcll} -\Delta\boldsymbol{u}' + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}') + \nabla\pi &=& \boldsymbol{f} + \Delta\boldsymbol{\chi} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\chi}) & \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u}' &=& h - \operatorname{div}\boldsymbol{\chi} & \text{in} \quad \Omega, \\ \boldsymbol{u}'|_{\partial\Omega} &=& 0. & \end{array}$$

Set $\boldsymbol{f}_{\boldsymbol{\chi}} = \boldsymbol{f} + \Delta\boldsymbol{\chi} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\chi})$ and $h_{\boldsymbol{\chi}} = h - \operatorname{div}\boldsymbol{\chi}$. As $1 < p < 2$ then $\boldsymbol{\chi} \in \boldsymbol{L}^{p*}(\Omega)$ and $\boldsymbol{v}\otimes\boldsymbol{\chi} \in \boldsymbol{L}^p(\Omega)$. Thus $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\chi}) \in \boldsymbol{W}_0^{-1,p}(\Omega)$. Hence, $\boldsymbol{f}_{\boldsymbol{\chi}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$. It is clear that $(\boldsymbol{f}_{\boldsymbol{\chi}}, h_{\boldsymbol{\chi}})$ satisfies the compatibility condition (3.16). Then from Theorem 3.11 we know that this problem has a solution in $\mathring{\boldsymbol{W}}_0^{1,p}(\Omega)\times L^p(\Omega)$. In addition, using Theorem 3.12 we deduce that

$$||\boldsymbol{u}'||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C\big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)^2\big(||\boldsymbol{f}_{\boldsymbol{\chi}}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)||h_{\boldsymbol{\chi}}||_{L^p(\Omega)}\big).$$

It follows from (3.29) that

$$||\boldsymbol{f}_{\boldsymbol{\chi}}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)||h_{\boldsymbol{\chi}}||_{L^p(\Omega)} \leq$$

$$C\left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)(||h||_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\right). \quad (3.30)$$

Then (3.28) is a trivial consequence of the previous inequality. □

Remark 3.15. We suppose now that $p > 2$. As in Theorem 3.11, using a dual argument with the estimate (3.20) of Theorem 3.12, we prove that if $\boldsymbol{g} = \boldsymbol{0}$,

$$\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\xi}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\eta||_{L^p(\Omega)} \leqslant C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||\,h\,||_{L^p(\Omega)}\big).$$

As in Corollary 3.14, when $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$, we prove that

$$\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\xi}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\eta||_{L^p(\Omega)} \leqslant C\big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)^3 \times$$
$$\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)} + (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big).$$

The next theorem summarizes the result of existence and uniqueness of generalized solutions of Problem (1.1) when $1 < p < \infty$:

Theorem 3.16. *Let Ω be an exterior domain with $C^{1,1}$ boundary. If $p \geq 2$, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$, Problem (1.1) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathcal{N}_0^p(\Omega)$ and there exists a constant C, independent of $\boldsymbol{u}$, π, $\boldsymbol{f}$, h, $\boldsymbol{g}$ and $\boldsymbol{v}$, such that*

$$\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\xi}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\eta||_{L^p(\Omega)} \leq C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3$$
$$\times\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||\,h\,||_{L^p(\Omega)} + \big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big). \tag{3.31}$$

If $1 < p < 2$, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$ which satisfy the necessary compatibility condition (3.27), Problem (1.1) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and there exists a constant C, independent of $\boldsymbol{u}$, π, $\boldsymbol{f}$, h, $\boldsymbol{g}$ and $\boldsymbol{v}$, such that

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leq C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^2\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)(||\,h\,||_{L^p(\Omega)} + ||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\big). \tag{3.32}$$

4 Strong solutions for the Oseen problem

4.1 Strong solutions in $\boldsymbol{W}_0^{2,p}(\Omega)$

We begin by proving the existence and uniqueness of strong solution in $\boldsymbol{W}_0^{2,p}(\Omega)$ for $1 < p < 3$ in the following sense.

Theorem 4.1. *For $1 < p < 3$, let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$. Then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)/\mathcal{N}_0^{p*}(\Omega)$ such that*

$$\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^{p*}(\Omega)} \|\boldsymbol{u}+\boldsymbol{\xi}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi+\eta\|_{W_0^{1,p}(\Omega)} \leq$$

$$C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^4\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\Big). \tag{4.1}$$

Proof. For all $1 < p < 3$, Sobolev embedding holds *i.e* $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p*}(\Omega)$. Observe that $h \in W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}^{1-1/p*,p*}(\Gamma)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p*}(\Omega)$. Since $p* > 3/2$ *i.e* $(p*)' < 3$, we deduce that $\mathcal{N}_0^{(p*)'}(\Omega) = \{\boldsymbol{0}, 0\}$. Using Theorem 3.16 (there is no compatibility condition), we prove that there exists a solution

$$(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p*}(\Omega) \times L^{p*}(\Omega)$$

for the Oseen problem (1.1) with the following estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p*}(\Omega)} + ||\pi||_{L^{p*}(\Omega)} \leq C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3 \times$$
$$\Big(\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p*}(\Omega)} + ||\,h\,||_{L^{p*}(\Omega)}\big) + \big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1-1/p*,p*}(\partial\Omega)}\Big). \tag{4.2}$$

Since $(\boldsymbol{v}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, we can apply the Stokes regularity theory see [1] to deduce the existence of $(\boldsymbol{z},\eta) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ verifying:

$$-\Delta\boldsymbol{z} + \nabla\eta = \boldsymbol{f} - \boldsymbol{v}\cdot\nabla\,\boldsymbol{u} \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{z} = \boldsymbol{g} \quad \text{on } \Gamma.$$

Moreover, the estimate holds

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\eta\|_{W_0^{1,p}(\Omega)} \leq$$

$$C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\,\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right), \tag{4.3}$$

with C denoting a constant only depending on p and Ω. Let $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $\theta = \eta - \pi$, then we have $(\boldsymbol{w}, \theta)$ belongs to $\mathcal{S}_0^{p*}(\Omega)$. Therefore, if $1 < p < 3/2$ *i.e* $3/2 < p* < 3$, we deduce from [1] that $\mathcal{S}_0^{p*} = (\mathbf{0}, 0)$ and thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and we deduce from (4.2) and (4.3) that:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi\|_{W_0^{1,p}(\Omega)} \leq$$

$$C\big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)^4\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right). \tag{4.4}$$

If $p \geq 3/2$ *i.e* $p* \geq 3$, we deduce from the Stokes regularity theory see [1], that $(\boldsymbol{w}, \theta)$ belongs to $\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega) \subset \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$. Now, using the following embeddings $\boldsymbol{W}_0^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p*}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$ and using (4.2), (4.3) and (4.4), we deduce that

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p*}(\Omega)} + \|\theta\|_{L^{p*}(\Omega)} \leq$$

$$C\big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)^4\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right).$$

Observe that in the finite dimensional case, all norms are equivalent so we have

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\theta\|_{W_0^{1,p}(\Omega)} \leq$$

$$C\big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)^4\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \big(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right)$$

and thus we obtain (4.4). The uniqueness of the solution $(\boldsymbol{u}, \pi)$ follows from $\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p*}(\Omega) \times L^{p*}(\Omega)$ and also in $\boldsymbol{W}_0^{1,p*}(\Omega) \times L^{p*}(\Omega)$ the solution is unique up to an element of $\mathcal{N}_0^{p*}(\Omega)$. □

4.2 Strong solutions in $\boldsymbol{W}_1^{2,p}(\Omega)$

In this subsection, we take $\boldsymbol{f}$ in weighted space $\boldsymbol{L}^p(\Omega)$, more precisely $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, and the data h in the corresponding weighted Sobolev space $W_1^{1,p}(\Omega)$.

Theorem 4.2. *Suppose that $1 < p < 3$ and $p \neq 3/2$. Let $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\Omega)$, $h \in W_1^{1,p}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$ that satisfy the compatibility condition (3.27) if $p < 2$. Then the Oseen problem (1.1) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega)/\mathcal{N}_0^p(\Omega)$ such that*

$$\begin{aligned}\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u} + \boldsymbol{\xi}||_{\boldsymbol{W}_1^{2,p}(\Omega)} + ||\pi + \eta||_{W_1^{1,p}(\Omega)} &\leq C\big(1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^6 \\ \times\left(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\Omega)} + \big(1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)\big(||\,h\,||_{W_1^{1,p}(\Omega)} + ||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big)\right)&.\end{aligned} \tag{4.5}$$

Proof. ***i) Regularity***:
Since the following embeddings hold $W_1^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, $\boldsymbol{W}^{2-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}^{1/p',p}(\Gamma)$, resp. for $p \neq 3/2$ we have $\boldsymbol{W}_1^{0,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$, according to Theorem 3.16 it follows the existence of a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ to the Oseen problem (1.1) if $p < 2$ and if $p \geq 2$ it is unique up to an element of $\mathcal{N}_0^p(\Omega)$. Moreover the following estimate is satisfied

$$\begin{aligned}\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u} + \boldsymbol{\xi}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi + \eta||_{L^p(\Omega)} &\leq C\big(1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3 \\ \times\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_1^{0,p}(\Omega)} + ||\,h\,||_{W_1^{1,p}(\Omega)} + \big(1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{2-1/p,p}(\partial\Omega)}\big)&.\end{aligned} \tag{4.6}$$

The rest of the proof is similar to that of Lemma 3.7, we introduce the same partition of unity as in Lemma 3.7. With the same notation, we can write

$$\boldsymbol{u} = \lambda\, \boldsymbol{u} + \mu\, \boldsymbol{u}, \qquad \pi = \lambda\, \pi + \mu\, \pi.$$

Let us extend $(\mu\, \boldsymbol{u}, \mu\, \pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\mu\, \boldsymbol{u}}, \widetilde{\mu\, \pi})$ belongs to $\boldsymbol{W}^{1,p}_0(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and let $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ such as in Theorem 3.5. A quick computation in $\mathcal{D}'(\mathbb{R}^3)$, shows that the pair $(\widetilde{\mu\, \boldsymbol{u}}, \widetilde{\mu\, \pi})$ satisfies the following equations:

$$-\Delta(\widetilde{\mu\, \boldsymbol{u}}) + \boldsymbol{w} \cdot \nabla(\widetilde{\mu\, \boldsymbol{u}}) + \nabla\,(\widetilde{\mu\, \pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\widetilde{\mu\, \boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = \mu \widetilde{\boldsymbol{f}} + (\Delta\, \lambda)\widetilde{\boldsymbol{u}} - (\nabla\, \lambda)\widetilde{\pi} + 2\nabla\, \lambda \cdot \nabla\, \widetilde{\boldsymbol{u}} - (\boldsymbol{w} \cdot \nabla\, \lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = \widetilde{\mu\, h} - \nabla\, \lambda \cdot \widetilde{\boldsymbol{u}}\ .$$

Moreover, owing to the supports of μ and λ, $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{W}^{0,p}_1(\mathbb{R}^3) \times W^{1,p}_1(\mathbb{R}^3)$. It is clear that $\boldsymbol{f}_1$ satisfies (3.8) and thus it follows from Theorem 3.4, that there exists a unique $(\boldsymbol{z}, \theta) \in \boldsymbol{W}^{2,p}_1(\mathbb{R}^3) \times W^{1,p}_1(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{z} + \boldsymbol{w} \cdot \nabla\, \boldsymbol{z} + \nabla\, \theta = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = e_1 \quad \text{in} \quad \mathbb{R}^3.$$

and thus,

$$-\Delta(\widetilde{\mu\, \boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w} \cdot \nabla(\widetilde{\mu\, \boldsymbol{u}} - \boldsymbol{z}) + \nabla(\widetilde{\mu\, \pi} - \theta) = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(\widetilde{\mu\, \boldsymbol{u}} - \boldsymbol{z}) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

with $(\widetilde{\mu\, \pi} - \theta) \in L^p(\mathbb{R}^3)$ and $(\widetilde{\mu\, \boldsymbol{u}} - \boldsymbol{z}) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3)$. Then, using the argument of uniqueness in section 4, we deduce that $\widetilde{\mu\, \boldsymbol{u}} - \boldsymbol{z} = \boldsymbol{0}$ and $\widetilde{\mu\, \pi} - \theta = 0$. Consequently, $(\widetilde{\mu\, \boldsymbol{u}}, \widetilde{\mu\, \pi})$ belongs to $\boldsymbol{W}^{2,p}_1(\mathbb{R}^3) \times W^{1,p}_1(\mathbb{R}^3)$. In particular, we have $\widetilde{\mu\, \boldsymbol{u}} = \boldsymbol{u}$ and $\widetilde{\mu\, \pi} = \pi$ outside B_{R_0+1}, so the restriction of $\boldsymbol{u}$ to ∂B_{R_0+1} belongs to $\boldsymbol{W}^{2-1/p,p}(\partial\, B_{R_0+1})$. Therefore, $(\boldsymbol{u}, \pi)$ satisfies:

$$-\Delta\, \boldsymbol{u} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega_{R_0+1}, \quad \boldsymbol{u}|_{\partial B_{R_0+1}} = \widetilde{\mu\, \boldsymbol{u}} \quad \text{and} \quad \boldsymbol{u}|_\Gamma = \boldsymbol{g}.$$

Observe that for any $\varphi \in W^{1,p'}(\Omega_{R_0+1})$ we have

$$\int_{\Omega_{R_0+1}} \boldsymbol{u} \cdot \nabla\, \varphi\, dx = -\int_{\Omega_{R_0+1}} \varphi \operatorname{div} \boldsymbol{u}\, dx + \int_{\partial\, \Omega_{R_0+1}} \varphi \boldsymbol{u} \cdot \boldsymbol{n}\, dx.$$

In particular, for $\varphi = 1$, we have

$$\int_{\Omega_{R_0+1}} h(x)\, dx = \int_{\partial\, \Omega_{R_0+1}} \boldsymbol{u} \cdot \boldsymbol{n}\, dx = \int_{\partial\, B_{R_0+1}} \boldsymbol{u} \cdot \boldsymbol{n}\, dx + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, dx. \tag{4.7}$$

and thus, according to Theorem 14 and Corollary 7 of [10], this problem has a unique $(\boldsymbol{u}, \pi)$ in $\boldsymbol{W}^{2,p}(\Omega_{R_0+1}) \times W^{1,p}(\Omega_{R_0+1})$. This implies that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega)$. The uniqueness of the solution $(\boldsymbol{u}, \pi)$ follows from this inclusion $\boldsymbol{W}^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega) \subset \boldsymbol{W}^{1,p}_0(\Omega) \times L^p(\Omega)$ which holds for $p \neq 3/2$.

ii) A priori Estimate:

First observe that each solution $(\boldsymbol{\xi}, \eta) \in \boldsymbol{W}^{1,p}_0(\Omega) \times L^p(\Omega)$ to the Oseen problem (1.1) with null data obviously belongs to $\boldsymbol{W}^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega)$. In fact the proof is very similar to that of Lemma 3.11 of [6]. Conversly, we have $\boldsymbol{W}^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega) \subset \boldsymbol{W}^{1,p}_0(\Omega) \times L^p(\Omega)$. Now, considering the first step of regularity, it follows that the continuous operator

$$\mathcal{O}' : \boldsymbol{W}^{2,p}_1(\Omega) \times W^{1,p}_1(\Omega)/\mathcal{N}^p_0(\Omega) \longrightarrow \boldsymbol{W}^{0,p}_1(\Omega) \times W^{1,p}_1(\Omega) \times \boldsymbol{W}^{2-1/p,p}(\Gamma)$$

defined by : $\mathcal{O}'(\boldsymbol{u}, \pi) = \big(-\Delta\, \boldsymbol{u} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} + \nabla\, \pi, \operatorname{div} \boldsymbol{u}, \boldsymbol{u}|_\Gamma\big)$ is an isomorphism. Thus there exists a constant $C(\boldsymbol{v})$ depending on $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$, Ω and p such that

$$\inf_{(\boldsymbol{\xi},\eta) \in \mathcal{N}^p_0(\Omega)} ||\boldsymbol{u} + \boldsymbol{\xi}||_{\boldsymbol{W}^{2,p}_1(\Omega)} + ||\pi + \eta||_{W^{1,p}_1(\Omega)} \leq C(\boldsymbol{v})\big(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\Omega)} + ||h||_{W^{1,p}_1(\Omega)} + ||\boldsymbol{g}||_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\big). \tag{4.8}$$

Proceeding then as in Theorem 3.12 and Corollary 3.14, we can characterize the constant $C(\boldsymbol{v})$ and we obtain (4.5). □

Remark 4.3. As in the case of the Oseen problem in $\mathbb{R}^3$, for $p \geq 3$ and $\alpha = 0$ or $\alpha = 1$, the hypothesis of $\boldsymbol{f} \in \boldsymbol{W}^{0,p}_\alpha(\Omega)$, $h \in W^{1,p}_\alpha(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$ and $\boldsymbol{v} \in \boldsymbol{H}_3(\Omega)$ is not sufficient to ensure the existence of strong solutions for problem (1.1) in $\boldsymbol{W}^{2,p}_\alpha(\Omega) \times W^{1,p}_\alpha(\Omega)$.

5 Very weak solutions for the Oseen problem

In this section, we are interested in the existence and the uniqueness of very weak solutions for the Oseen problem (1.1).

5.1 Preliminary results

We recall some density results and Green formulas proved in [6]. Let us introduce the following space:

$$\boldsymbol{X}^{\ell}_{r,p}(\Omega) = \left\{ \boldsymbol{\varphi} \in \mathring{\boldsymbol{W}}^{1,r}_{\ell}(\Omega);\ \operatorname{div} \boldsymbol{\varphi} \in \mathring{W}^{1,p}_{\ell}(\Omega) \right\}.$$

According to Poincaré-type inequality (see [5]), this space can be equipped with the following norm:

$$\| \boldsymbol{\varphi} \|_{\boldsymbol{X}^{\ell}_{r,p}(\Omega)} = \sum_{1 \leq i,j \leq 3} \| \frac{\partial \varphi_i}{\partial x_j} \|_{W^{0,r}_{\ell}(\Omega)} + \| \operatorname{div} \boldsymbol{\varphi} \|_{W^{1,p}_{\ell}(\Omega)} .$$

Note that if $\boldsymbol{f} \in (\boldsymbol{X}^{\ell}_{r,p}(\Omega))'$ with $\ell = 1$ or $\ell = 0$ then there exist $\mathbb{F}_0 = (f_{ij})_{1 \leq i,j \leq 3} \in \boldsymbol{W}^{0,r'}_{-\ell}(\Omega)$ and $f_1 \in W^{-1,p'}_{-\ell}(\Omega)$ such that:

$$\boldsymbol{f} = \operatorname{div} \mathbb{F}_0 + \nabla f_1. \tag{5.1}$$

Moreover, we can define

$$\| \boldsymbol{f} \|_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]'} = \max \left\{ \| f_{ij} \|_{W^{0,r'}_{-\ell}(\Omega)}, 1 \leq i,j \leq 3, \| f_1 \|_{W^{-1,p'}_{-\ell}(\Omega)} \right\}.$$

The first result is given by the following lemma:

Lemma 5.1. *(Amrouche and Meslameni [6]). Suppose that* $0 \leq \frac{1}{r} - \frac{1}{p} \leq \frac{1}{3}$*, then*

i) For all $q \in W^{-1,p}_{-1}(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{X}^{1}_{r',p'}(\Omega)$*, we have*

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^{1}_{r',p'}(\Omega)]' \times \boldsymbol{X}^{1}_{r',p'}(\Omega)} = - \langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}^{1,p'}_{1}(\Omega)} . \tag{5.2}$$

ii) If in addition $p' \neq 3$*, then for all* $q \in W^{-1,p}_{0}(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{X}^{0}_{r',p'}(\Omega)$*, we have*

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^{0}_{r',p'}(\Omega)]' \times \boldsymbol{X}^{0}_{r',p'}(\Omega)} = - \langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{0}(\Omega) \times \mathring{W}^{1,p'}_{0}(\Omega)} . \tag{5.3}$$

Giving a meaning to the trace of a very weak solution of the Oseen problem is not trivial task. We need to introduce appropriate spaces. First, we consider the space:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega),\ \boldsymbol{\psi}|_{\Gamma} = 0,\ \operatorname{div} \boldsymbol{\psi}|_{\Gamma} = 0 \right\},$$

that can also be described (see [6]) as:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega), \boldsymbol{\psi}|_{\Gamma} = \boldsymbol{0}, \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{n}} \cdot \boldsymbol{n}|_{\Gamma} = 0 \right\}. \tag{5.4}$$

Note that if $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',\ell}(\Omega)$, then $\operatorname{div} \boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}_{\ell}(\Omega)$ and the range space of the normal derivative $\gamma_1 : \boldsymbol{Y}_{p',\ell}(\Omega) \longrightarrow \boldsymbol{W}^{1/p,p'}(\Gamma)$ is

$$\boldsymbol{Z}_{p'}(\Gamma) = \left\{ \boldsymbol{z} \in \boldsymbol{W}^{1/p,p'}(\Gamma);\ \ \boldsymbol{z} \cdot \boldsymbol{n} = 0 \right\}.$$

Secondly, we shall use the space:

$$\boldsymbol{T}^{\ell}_{r,p}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{-\ell}(\Omega);\ \Delta \boldsymbol{v} \in [\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]' \right\},$$

equipped with the norm:

$$\| \boldsymbol{v} \|_{\boldsymbol{T}^{\ell}_{r,p}(\Omega)} = \| \boldsymbol{v} \|_{\boldsymbol{W}^{0,p}_{-\ell}(\Omega)} + \| \Delta \boldsymbol{v} \|_{[\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]'} .$$

We also introduce the following space:

$$\boldsymbol{H}^{r}_{p,\ell}(\operatorname{div}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega); \operatorname{div} \boldsymbol{v} \in W^{0,r}_{\ell-1}(\Omega) \right\}.$$

This space is equipped with the graph norm. Moreover, we have the following result (see [6] for the proof):

Lemma 5.2. *(Amrouche and Meslameni [6]). Let* $\frac{3}{2} < p < \infty$ *and* $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$. *Then the mapping* $\gamma_\tau : \boldsymbol{v} \longrightarrow \boldsymbol{v}_\tau|_\Gamma$ *on the space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping, still denoted by* γ_τ, *from* $\boldsymbol{T}^\ell_{r,p}(\Omega)$ *into* $\boldsymbol{W}^{-1/p,p}(\Gamma)$ *for* $\ell = 0$ *and if* $p \neq 3$ *for* $\ell = 1$ *and we have the Green formula: for any* $\boldsymbol{v} \in \boldsymbol{T}^\ell_{r,p}(\Omega)$ *and* $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',\ell}(\Omega)$,

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\boldsymbol{X}^\ell_{r',p'}(\Omega)]' \times \boldsymbol{X}^\ell_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma, \tag{5.5}$$

where the duality on Γ *is defined by:* $\langle ., . \rangle_\Gamma = \langle ., . \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}$.

Finally, we have

Lemma 5.3. *(Amrouche and Meslameni [6]). Let* Ω *be a Lipschitz open set in* $\mathbb{R}^3$. *Suppose that* $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$ *and* $\ell = 0$ *or* $\ell = 1$. *Then*
i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *is dense in* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}, \Omega)$.

ii) The mapping $\gamma_n : \boldsymbol{v} \longrightarrow \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma$ *on the space* $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping, still denoted by* γ_n, *from* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,, \Omega)$ *into* $\boldsymbol{W}^{-1/p,p}(\Gamma)$. *If in addition* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *and* $\frac{3}{2} < p < \infty$, *we have the following Green formula: for any* $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,, \Omega)$ *and* $\varphi \in W^{1,p'}_{1-\ell}(\Omega)$,

$$\int_\Omega \boldsymbol{v} \cdot \nabla \varphi \, d\boldsymbol{x} + \int_\Omega \varphi \, \mathrm{div}\, \boldsymbol{v} \, d\boldsymbol{x} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, \varphi \rangle_\Gamma \,. \tag{5.6}$$

5.2 Very weak solutions in $\boldsymbol{L}^p(\Omega)$

To begin with we introduce the definition of very weak solution.

Let

$$\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]', h \in L^r(\Omega), \text{ and } \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{5.7}$$

with

$$\frac{3}{2} < p < \infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{3} = \frac{1}{r}, \tag{A$_1$}$$

yielding $1 < r < 3$.

Definition 5.4. *(**Very weak solution for the Oseen problem**) We suppose that* r *and* p *satisfy* (A$_1$) *and let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfy* (5.7) *and let* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$. *We say that* $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}_0(\Omega)$ *is a very weak solution of problem* (1.1) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$ *and* $\theta \in W^{1,p'}_0(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{\varphi})) \, d\boldsymbol{x} - \langle \pi, \nabla \cdot \boldsymbol{\varphi} \rangle_{W^{-1,p}_0(\Omega) \times \mathring{W}^{1,p'}_0(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma, \tag{5.8}$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \theta \, d\boldsymbol{x} = -\int_\Omega h \, \theta \, d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma \,, \tag{5.9}$$

where the duality on Ω *is defined by:*

$$\langle ., . \rangle_\Omega = \langle ., . \rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]' \times \boldsymbol{X}^0_{r',p'}(\Omega)} \,.$$

Note that if (A$_1$) is satisfied, we have:

$$W^{1,p'}_0(\Omega) \hookrightarrow L^{r'}(\Omega) \quad \text{and} \quad \boldsymbol{Y}_{p',0}(\Omega) \hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega),$$

and $\int_\Omega \boldsymbol{u} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}) \, d\mathbf{x}$ is well defined , which means that all the brackets and integrals have a sense.

Proposition 5.5. *Let* p *and* r *satisfy* (A$_1$) *and let* $\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]'$, $h \in L^r(\Omega)$, $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. *Then the following two statements are equivalent:*

i) $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}_0(\Omega)$ *is a very weak solution of* (1.1)

ii) $(\boldsymbol{u}, \pi)$ *satisfies* (1.1) *in the sense of distributions.*

Proof. *i)* $\Rightarrow$ *ii)* Let $(\boldsymbol{u},\pi)\in \boldsymbol{L}^p(\Omega)\times W_0^{-1,p}(\Omega)$ be a very weak solution of (1.1), then if we take $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega)$ and $\theta\in\mathcal{D}(\Omega)$ we can deduce by (5.8) and (5.9) that

$$-\Delta\boldsymbol{u}+\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f}\ \text{ in }\ \Omega\quad\text{and}\quad\nabla\cdot\boldsymbol{u}=h\ \text{ in }\ \Omega,$$

Since $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\Omega)$ and $\frac{1}{p}+\frac{1}{3}=\frac{1}{r}$, we can deduce by Hölder that $\boldsymbol{v}\otimes\boldsymbol{u}\in\boldsymbol{L}^r(\Omega)$. Moreover, we have $-\Delta\boldsymbol{u}=-\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u})-\nabla\,\pi+\boldsymbol{f}\in[\boldsymbol{X}^0_{r',p'}(\Omega)]'$ and $\boldsymbol{u}\in\boldsymbol{T}^0_{p,r}(\Omega)$. Now, let $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',0}(\Omega)\subset\boldsymbol{X}^0_{r',p'}(\Omega)$, it follows

$$\langle-\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega=\langle-\nabla\,\pi-\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u})+\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega\,.$$

Lemma 5.2 implies that

$$\langle-\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega=\int_\Omega\boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\mathbf{x}-\left\langle\boldsymbol{u}_\tau,\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma$$

and from (5.3) that

$$\langle\nabla\,\pi,\boldsymbol{\varphi}\rangle_\Omega=-\langle\pi,\nabla\cdot\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\Omega)\times\mathring{W}_0^{1,p'}(\Omega)}\,.$$

On the other hand, we have $\nabla\,\boldsymbol{\varphi}\in\boldsymbol{L}^{r'}(\Omega)$ and $\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi})=\boldsymbol{v}\cdot\nabla\,\boldsymbol{\varphi}\in\boldsymbol{L}^{p'}(\Omega)$. Then we obtain

$$\begin{aligned}\langle\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}),\boldsymbol{\varphi}\rangle_\Omega&=\langle\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}),\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,r}(\Omega)\times\mathring{\boldsymbol{W}}_0^{1,r'}(\Omega)}\\&=-\langle\boldsymbol{v}\otimes\boldsymbol{u},\nabla\,\boldsymbol{\varphi}\rangle_{\boldsymbol{L}^r(\Omega)\times\boldsymbol{L}^{r'}(\Omega)}\\&=-\int_\Omega\boldsymbol{u}\cdot\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi})\,d\mathbf{x}.\end{aligned}$$

Thus we have

$$\int_\Omega\boldsymbol{u}\Delta\boldsymbol{\varphi}\,d\mathbf{x}-\left\langle\boldsymbol{u}_\tau,\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma=\langle\pi,\nabla\cdot\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\Omega)\times\mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)}+\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega+\int_\Omega\boldsymbol{u}\cdot\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi})\,d\mathbf{x}.,$$

and we can deduce that for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',0}(\Omega)$

$$\left\langle\boldsymbol{u}_\tau,\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma=\left\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma.$$

Now let $\boldsymbol{\mu}\in\boldsymbol{W}^{1/p,p'}(\Gamma)$, then we have $\langle\boldsymbol{u}_\tau-\boldsymbol{g}_\tau,\boldsymbol{\mu}\rangle_\Gamma=\langle\boldsymbol{u}_\tau-\boldsymbol{g}_\tau,\boldsymbol{\mu}_\tau\rangle_\Gamma$.It is clear that $\boldsymbol{\mu}_\tau\in\boldsymbol{Z}_{p'}(\Omega)$ and it implies that there exists $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',0}(\Omega)$ such that $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}=\boldsymbol{\mu}_\tau$ on Γ. We can deduce that $\boldsymbol{u}_\tau=\boldsymbol{g}_\tau$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla\cdot\boldsymbol{u}=h$, we deduce that $\boldsymbol{u}\in\boldsymbol{H}^r_{p,1}(\mathrm{div},\Omega)$, then it follows from (5.6), that for any $\theta\in W_0^{1,p'}(\Omega)$,

$$\langle\boldsymbol{u}\cdot\boldsymbol{n},\theta\rangle_\Gamma=\langle\boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_\Gamma\,.$$

Consequently $\boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{g}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u}=\boldsymbol{g}$ on Γ.
ii) $\Rightarrow$ *i)* The converse is a simple consequence of (5.6), (5.3) and Lemma 5.2. □

Theorem 5.6. *Let Ω be an exterior domain with $C^{1,1}$ boundary and let p and r satisfy* (A_1) *and let $\boldsymbol{f}$, h, and $\boldsymbol{g}$ satisfy* (5.7), $\boldsymbol{v}\in\boldsymbol{H}_3(\Omega)$. *Then the Oseen problem* (1.1) *has a unique solution $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and $\pi\in W_0^{-1,p}(\Omega)$ if and only if for any $(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^{(p')*}(\Omega)$:*

$$\langle\boldsymbol{f},\boldsymbol{\lambda}\rangle-\langle h,\mu\rangle+\langle\boldsymbol{g},(\mu I-\nabla\,\boldsymbol{\lambda})\cdot\boldsymbol{n}\rangle_\Gamma=0.$$

Moreover, there exists a constant $C>0$ depending only on p, r and Ω such that:

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{L}^p(\Omega)}+\|\,\pi\,\|_{W_0^{-1,p}(\Omega)}\le C\big(1+\|\,\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}\big)^4\Big(\|\,\boldsymbol{f}\,\|_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'}+\|\,h\,\|_{L^r(\Omega)}+\|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big).\quad(5.10)$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u},\pi)\in\boldsymbol{L}^p(\Omega)\times W_0^{-1,p}(\Omega)$ such that for any $\boldsymbol{w}\in\boldsymbol{Y}_{p',0}(\Omega)$ and $\theta\in W_0^{1,p'}(\Omega)$ it holds:

$$\int_\Omega\boldsymbol{u}\cdot(-\Delta\boldsymbol{w}+\boldsymbol{v}\cdot\nabla\,\boldsymbol{w}+\nabla\,\theta)d\mathbf{x}-\langle\pi,\mathrm{div}\,\boldsymbol{w}\rangle_{W_0^{-1,p}(\Omega)\times\mathring{W}_0^{1,p'}(\Omega)}=$$

$$\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\left\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\right\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_\Gamma-\int_\Omega h\,\theta\,d\mathbf{x}.$$

Let T be a linear form defined by:

$$T: \quad \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega) \longrightarrow \mathbb{R}$$
$$(\mathbf{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x},$$

with $(\boldsymbol{w}, \theta) \in \boldsymbol{W}_0^{2,p'}(\Omega) \times W_0^{1,p'}(\Omega)$ is a solution of the following Oseen problem:

$$-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \theta = \mathbf{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see Theorem 4.1)

$$\inf_{(\boldsymbol{\xi},\eta) \in \mathcal{N}_0^{(p')*}(\Omega)} \|\boldsymbol{w} + \boldsymbol{\xi}\|_{\boldsymbol{W}_0^{2,p'}(\Omega)} + \|\theta + \eta\|_{W_0^{1,p'}(\Omega)} \leq$$

$$C\big(1 + \|\, \boldsymbol{v}\, \|_{\boldsymbol{L}^3(\Omega)}\big)^4 \left(\|\mathbf{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\varphi\|_{W_0^{1,p'}(\Omega)} \right). \tag{5.11}$$

Then we have for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)$ and for any $(\boldsymbol{\xi}, \eta) \in \mathcal{N}_0^{(p')*}(\Omega)$

$$\begin{aligned} |T(\mathbf{F}, \varphi)| = &\ | \langle \boldsymbol{f}, \boldsymbol{w} + \boldsymbol{\xi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial(\boldsymbol{w} + \boldsymbol{\xi})}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta + \eta \rangle_\Gamma - \int_\Omega h\, (\theta + \eta) d\mathbf{x}\, | \\ \leq &\ C \left(\|\, \boldsymbol{f}\, \|_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + \|\, \boldsymbol{g}\, \|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|\, h\, \|_{L^r(\Omega)} \right) \\ &\times \left(\|\, \boldsymbol{w} + \boldsymbol{\xi}\, \|_{\mathbf{W}_0^{2,p'}(\Omega)} + \|\, \theta + \eta\, \|_{W_0^{1,p'}(\Omega)} \right). \end{aligned}$$

Using (5.11), we prove that

$$\begin{aligned} |\, T(\mathbf{F}, \varphi)\, | \leq &\ C\big(1 + \|\, \boldsymbol{v}\, \|_{\boldsymbol{L}^3(\Omega)}\big)^4 \left(||\mathbf{F}||_{\mathbf{L}^{p'}(\Omega)} + ||\varphi||_{W_0^{1,p'}(\Omega)} \right) \\ &\times \left(||\boldsymbol{f}||_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + ||\boldsymbol{g}||_{\mathbf{W}^{-1/p,p}(\Omega)} + ||h||_{L^r(\Omega)} \right). \end{aligned}$$

It implies that the linear form T is continuous on $\mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)$ and moreover there exists a unique solution$(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ solution of the Oseen problem (1.1) satisfying estimate (5.10). □

5.3 Very weak solutions in $\boldsymbol{W}^{0,p}_{-1}(\Omega)$

Here, we are interested in the case of the following assumptions:

$$\boldsymbol{f} \in [\boldsymbol{X}^1_{r',p'}(\Omega)]', \ h \in W^{0,r}_{-1}(\Omega) \ \text{and} \ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{5.12}$$

with

$$\frac{3}{2} < p < \infty, \quad p \neq 3 \quad \text{and} \quad \frac{1}{p} + \frac{1}{3} = \frac{1}{r}, \tag{A$_2$}$$

yielding $1 < r < 3$.

Definition 5.7. ***(Very weak solution for the Oseen problem)*** *Suppose that* (A_2) *is satisfied and let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfying* (5.12) *and let* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$. *We say that* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ *is a very weak solution of* (1.1) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p',1}(\Omega)$ *and* $\theta \in W_1^{1,p'}(\Omega)$,

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{\varphi} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\varphi}))\, d\boldsymbol{x} - \langle \pi, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}_1^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \tag{5.13}$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \theta\, d\boldsymbol{x} = -\int_\Omega h\, \theta d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma \tag{5.14}$$

where the duality on Ω *is defined by:*

$$\langle ., . \rangle_\Omega = \langle ., . \rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]' \times \boldsymbol{X}^1_{r',p'}(\Omega)}.$$

Note that if $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$, we have:

$$W_1^{1,p'}(\Omega) \hookrightarrow W_1^{0,r'}(\Omega), \quad \text{and} \quad \boldsymbol{Y}_{p',1}(\Omega) \hookrightarrow \boldsymbol{X}^1_{r',p'}(\Omega),$$

and $\displaystyle\int_\Omega \boldsymbol{u} \cdot \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\varphi})\, d\mathbf{x}$ is well defined which means that all the brackets and integrals have a sense. As previously we prove under the assumption (A_2), that if $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (5.12) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$, then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ is a very weak solution of (1.1) if and only if $(\boldsymbol{u}, \pi)$ satisfy (1.1) in the sense of distributions.

Theorem 5.8. *Let Ω be an exterior domain with $C^{1,1}$ boundary. Suppose that (A_2) is satisfied and let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (5.12) and let $\boldsymbol{v} \in \boldsymbol{H}_3(\Omega)$. Then the Oseen problem (1.1) has a solution $\boldsymbol{u} \in \boldsymbol{W}^{0,p}_{-1}(\Omega)$ and $\pi \in W^{-1,p}_{-1}(\Omega)$ if and only if for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$:*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle - \langle h, \mu \rangle + \langle \boldsymbol{g}, (\eta I - \nabla \boldsymbol{\lambda}) \cdot \boldsymbol{n} \rangle_\Gamma = 0.$$

In $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$, each solution is unique up to an element of $\mathcal{N}_0^p(\Omega)$ and there exists a constant $C > 0$ depending only on p, r and Ω such that:

$$\inf_{(\boldsymbol{\xi},\eta) \in \mathcal{N}_0^p(\Omega)} (||\boldsymbol{u} + \boldsymbol{\xi}||_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + ||\pi + \eta||_{W^{-1,p}_{-1}(\Omega)})$$

$$\leq C\big(1 + ||\, \boldsymbol{v}\, ||_{\boldsymbol{L}^3(\Omega)}\big)^7 (||\boldsymbol{f}||_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'} + ||h||_{W^{0,r}_{-1}(\Omega)} + ||\boldsymbol{g}||_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{5.15}$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\theta \in W_1^{1,p'}(\Omega)$ the following equality holds:

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{w} + \nabla \theta) d\mathbf{x} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x}.$$

Let T be a linear form defined from $\Big(\mathbf{W}^{0,p'}_1(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)) \perp \mathcal{N}_0^p(\Omega)\Big)$ onto $\mathbb{R}$ by:

$$T(\mathbf{F}, \varphi) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x},$$

with $(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{2,p'}_1(\Omega) \times W^{1,p'}_1(\Omega)$ is a solution of the following Oseen problem:

$$-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{w} + \nabla\, \theta = \mathbf{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see Theorem 4.2)

$$\inf_{(\boldsymbol{\xi},\eta) \in \mathcal{N}_0^{p'}(\Omega)} (||\boldsymbol{w} + \boldsymbol{\xi}||_{\boldsymbol{W}^{2,p'}_1(\Omega)} + ||\theta + \eta||_{W^{1,p'}_1(\Omega)}) \leqslant C$$

$$\big(1 + ||\, \boldsymbol{v}\, ||_{\boldsymbol{L}^3(\Omega)}\big)^6 \left(||\mathbf{F}||_{\boldsymbol{W}^{0,p'}_1(\Omega)} + \big(1 + ||\, \boldsymbol{v}\, ||_{\boldsymbol{L}^3(\Omega)}\big)||\varphi||_{W^{1,p'}_1(\Omega)}\right). \tag{5.16}$$

Then for any pair $(\mathbf{F}, \varphi) \in (\mathbf{W}^{0,p'}_1(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)) \perp \mathcal{N}^{1,p}_0(\Omega)$ and for any $(\boldsymbol{\xi}, \eta) \in \mathcal{N}_0^{p'}(\Omega)$

$$\begin{aligned} |T(\mathbf{F}, \varphi)| = & \ | \langle \boldsymbol{f}, \boldsymbol{w} + \boldsymbol{\xi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial (\boldsymbol{w} + \boldsymbol{\xi})}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta + \eta \rangle_\Gamma - \int_\Omega h\, (\theta + \eta) d\mathbf{x}\, | \\ & \leq C \Big(\|\, \boldsymbol{f}\, \|_{[\mathbf{X}^1_{r',p'}(\Omega]'} + \|\, \boldsymbol{g}\, \|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|\, h\, \|_{W^{0,r}_{-1}(\Omega)} \Big) \\ & \times \Big(\|\, \boldsymbol{w} + \boldsymbol{\xi}\, \|_{\mathbf{W}^{2,p'}_1(\Omega)} + \|\, \theta + \eta\, \|_{W^{1,p'}_1(\Omega)} \Big). \end{aligned}$$

Using (5.16), we prove that

$$|T(\mathbf{F},\varphi)| \leq C\big(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)^6\left(||\mathbf{F}||_{\boldsymbol{W}_1^{0,p'}(\Omega)}+\big(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\big)||\varphi||_{W_1^{1,p'}(\Omega)}\right)$$
$$\times\left(\|\,\boldsymbol{f}\,\|_{[\mathbf{X}_{p'}(\Omega)]'}+\|\,\boldsymbol{g}\,\|_{\mathbf{W}^{-1/p,p}(\Omega)}+\|\,h\,\|_{W_{-1}^{0,r}(\Omega)}\right).$$

From this we can deduce that the linear form T is continuous on the following space $\mathbf{W}_1^{0,p'}(\Omega)\times\mathring{W}_1^{1,p'}(\Omega)\perp\mathcal{N}_0^p(\Omega)$ and we deduce that there exists $(\boldsymbol{u},\pi)\in(\mathbf{W}_{-1}^{0,p}(\Omega)\times W_{-1}^{-1,p}(\Omega))$ solution of the Oseen problem (1.1), which is unique up to an element of $\mathcal{N}_0^p(\Omega)$, satisfying the estimate (5.15). □

Remark 5.9. Observe that each solution $(\boldsymbol{\xi},\eta)\in\boldsymbol{W}_{-1}^{0,p}(\Omega)\times W_{-1}^{-1,p}(\Omega)$ to the Oseen problem (1.1) with null data obviously belongs to $\mathcal{N}_0^p(\Omega)$, in fact the proof is very similar to that of Lemma 3.11 of [6]. Moreover, if $p\neq 3$, we have $\mathcal{N}_0^p(\Omega)\subset\boldsymbol{W}_{-1}^{0,p}(\Omega)\times W_{-1}^{-1,p}(\Omega)$.

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Chérif Amrouche and Mohamed Meslameni
Laboratoire de Mathématiques et de leurs Applications, CNRS UMR 5142
Université de Pau et des Pays de l'Adour
64013 Pau, France,
cherif.amrouche@univ-pau.fr and mohamed.meslameni@univ-pau.fr

Šárka Nečasová
Mathematical Institute of the Academy of Sciences
Žitná 25, 115 67 Praha 1
Czech Republic,
matus@math.cas.cz