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TAYLOR FUNCTIONAL CALCULUS

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ABSTRACT. The notion of spectrum of an operator is one of the central concepts of operator theory. It is closely connected with the existence of a functional calculus which provides important information about the structure of Banach space operators.

The situation for commuting n-tuples of Banach space operators is much more complicated. There are many possible definitions of joint spectra. However, the joint spectrum introduced by J. L. Taylor has a distinguished property — there exists a functional calculus for functions analytic on a neighbourhood of this spectrum.

The present paper gives a survey of basic properties of the Taylor spectrum and Taylor functional calculus.

1. INTRODUCTION.

The functional calculus of single operators (or more generally, single elements of a Banach algebra) is a standard and very useful tool in operator theory. It is defined by the Cauchy formula: if $\mathcal{A}$ is a unital Banach algebra, $a \in \mathcal{A}$ and f a function analytic on a neighbourhood of the spectrum $\sigma(a)$, then $f(a) \in \mathcal{A}$ is defined by

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-a)^{-1} \mathrm{d}z,$$

where Γ is a suitable contour surrounding $\sigma(a)$. The mapping $f \mapsto f(a)$ is an algebra homomorphism, it satisfies the spectral mapping property $\sigma(f(a)) = f(\sigma(a))$ for all f, and it is continuous in the sense that if $f_n \to f$ uniformly on a neighbourhood of $\sigma(a)$ then $f_n(a) \to f(a)$.

The functional calculus for n-tuples of commuting elements is much more complicated. In commutative Banach algebras it was constructed by Shilov, Arens, Calderon and Waelbroeck. The main result is: if $\mathcal{A}$ is a commutative unital Banach algebra, $a_1, \dots, a_n \in \mathcal{A}$ and f a function analytic on a neighbourhood of the joint spectrum $\sigma(a_1, \dots, a_n)$ then it is possible to define $f(a_1, \dots, a_n) \in \mathcal{A}$ such that the functional calculus $f \mapsto f(a_1, \dots, a_n)$ satisfies the same properties as the functional calculus of single elements — it is additive, multiplicative, continuous and satisfies the spectral mapping property.

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If $a_1, \dots, a_n$ are mutually commuting elements in a non-commutative unital Banach algebra $\mathcal{A}$, then it is possible to choose a unital commutative subalgebra $\mathcal{A}_0$ containing the elements $a_1, \dots, a_n$ and consider the functional calculus in $\mathcal{A}_0$. However, the joint spectrum $\sigma^{\mathcal{A}_0}(a_1, \dots, a_n)$ and the functional calculus depend on the choice of $\mathcal{A}_0$ and there is no optimal candidate for the choice of $\mathcal{A}_0$.

The best functional calculus known at present was constructed by (Taylor, 1979a, 1970b). He defined a joint spectrum σ_T (called now the Taylor spectrum) for commuting Banach space operators $A_1, \dots, A_n$ and the functional calculus $f \mapsto f(A_1, \dots, A_n)$ for functions analytic on a neighbourhood of $\sigma_T(A_1, \dots, A_n)$ which satisfies the expected properties as the functional calculi above.

Although the Taylor functional calculus is defined only for n-tuples of commuting Banach space operators and not for commuting n-tuples of elements in a Banach algebra, the Taylor functional calculus implies easily the existence of the above mentioned functional calculus in commutative Banach algebras.

2. Taylor spectrum

Let $s = (s_1, \dots, s_n)$ be a system of indeterminates. Denote by $\Lambda[s]$ the exterior algebra generated by $s = (s_1, \dots, s_n)$, i.e., $\Lambda[s]$ is the free complex algebra generated by $s = (s_1, \dots, s_n)$, where the multiplication operation $\wedge$ in $\Lambda[s]$ satisfies the anticommutative relations $s_i \wedge s_j = -s_j \wedge s_i \quad (i, j = 1, \dots, n)$. In particular, $s_i \wedge s_i = 0$ for all i.

For $F \subset \{1, \dots, n\}$, $F = \{i_1, \dots, i_p\}$ with $1 \le i_1 < i_2 < \cdots < i_p \le n$ write $s_F = s_{i_1} \wedge \cdots \wedge s_{i_p}$. Every element of $\Lambda[s]$ can be written uniquely in the form

$$\sum_{F \subset \{1, \dots, n\}} \alpha_F s_F$$

with complex coefficients α_F. Clearly, $s_\emptyset$ is the unit in $\Lambda[s]$.

For $p = 0, 1, \dots, n$ let $\Lambda^p[s]$ be the set of all elements of $\Lambda[s]$ of degree p, i.e., $\Lambda^p[s, X]$ is the subspace generated by the elements s_F with $\operatorname{card} F = p$. Thus $\Lambda[s] = \bigoplus_{p=0}^n \Lambda^p[s]$, $\dim \Lambda^p[s] = \binom{n}{p}$ and $\dim \Lambda[s] = 2^n$.

Let X be a vector space. Write $\Lambda[s, X] = X \otimes \Lambda[s]$. So

$$\Lambda[s, X] = \left\{ \sum_{F \subset \{1, \dots, n\}} x_F s_F : x_F \in X \right\};$$

to simplify the notation, the symbol " $\otimes$ " is omitted. Similarly, for $p = 0, \dots, n$ write $\Lambda^p[s, X] = X \otimes \Lambda^p$; so

$$\Lambda^p[s, X] = \left\{ \sum_{\substack{F \subset \{1, \dots, n\} \\ \operatorname{card} F = p}} x_F s_F : x_F \in X \right\}.$$

Thus $\Lambda^p[s,X]$ is a direct sum of $\binom{n}{p}$ copies of X and $\Lambda[s,X]$ is a direct sum of 2^n copies of X.

In the following X will be a fixed complex Banach space. Then $\Lambda[s,X]$ can be considered to be also a Banach space. For the following considerations it is not essential which norm one takes on $\Lambda[s,X]$; one can assume it to be $\|\sum x_F s_F\| = \Bigl(\sum \|x_F\|^2\Bigr)^{1/2}$. This norm has an advantage that if X is a Hilbert space then so is $\Lambda[s,X]$.

For $j=1,\dots,n$ let $S_j : \Lambda[s,X] \to \Lambda[s,X]$ be the operators of left multiplication by s_j,

$$S_j\Bigl(\sum_F x_F s_F\Bigr) = \sum_F x_F s_j \wedge s_F = \sum_{\substack{F\subset\{1,\dots,n\}\\ j\notin F}} (-1)^{\operatorname{card}\{i\in F: i<j\}} x_F s_{F\cup\{j\}}.$$

Clearly, $S_jS_i = -S_iS_j \quad (i,j=1,\dots,n)$. In particular, $S_i^2 = 0$ for all i.

For an operator $T \in B(X)$ denote by the same symbol the operator $T : \Lambda[s,X] \to \Lambda[s,X]$ defined by

$$T\Bigl(\sum_F x_F s_F\Bigr) = \sum_F (Tx_F)s_F.$$

Obviously, $TS_j = S_jT$ for all j.

Let $A = (A_1,\dots,A_n)$ be an n-tuple of mutually commuting operators on X. Denote by δ_A the operator $\delta_A : \Lambda[s,X] \to \Lambda[s,X]$ defined by

$$\delta_A = \sum_{i=1}^n A_i S_i.$$

Clearly,

$$\bigl(\delta_A\bigr)^2 = \sum_{i=1}^n\sum_{j=1}^n A_iS_iA_jS_j = \sum_{1\le i<j\le n} A_iA_j(S_iS_j + S_jS_i) = 0,$$

and so $\operatorname{Im}\delta_A \subset \ker\delta_A$ (note that the commutativity of the operators A_i has been used).

Definition 2.1. An n-tuple $A = (A_1,\dots,A_n)$ of mutually commuting operators on a Banach space X is called *Taylor regular* if $\ker\delta_A = \operatorname{Im}\delta_A$.

The *Taylor spectrum* $\sigma_T(A)$ is the set of all $\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$ such that the n-tuple $A-\lambda = (A_1-\lambda_1,\dots,A_n-\lambda_n)$ is not Taylor regular.

Since $\delta_A\Lambda^p[s,X] \subset \Lambda^{p+1}[s,X] \quad (p=0,1,\dots,n-1)$, it is possible to define operators $\delta_A^p : \Lambda^p[s,X] \to \Lambda^{p+1}[s,X]$ as the restrictions of δ_A to $\Lambda^p[s,X]$. Thus δ_A defines the following sequence of operators

$$0 \to \Lambda^0[s,X] \xrightarrow{\delta_A^0} \Lambda^1[s,X] \xrightarrow{\delta_A^1} \cdots \xrightarrow{\delta_A^{n-1}} \Lambda^n[s,X] \to 0, \tag{1}$$

where $\delta^{p+1}\delta^p = 0$ for each p.

The sequence (1) is called the *Koszul complex* of A. It is easy to see that A is Taylor regular if and only if the Koszul complex is exact, i.e., if

$\operatorname{Im} \delta_A^p = \ker \delta_A^{p+1}$ for all p, where one sets formally δ_A^p to be the zero operators for $p < 0$ or $p \geq n$.

Remark 2.2. (i) Let $n = 1$. It is possible to identify $\Lambda^0[s, X]$ and $\Lambda^1[s, X]$ with X, and so the Koszul complex of a single operator $A_1 \in B(X)$ becomes

$$0 \to X \xrightarrow{A_1} X \to 0.$$

This complex is exact if and only if A_1 is invertible. Thus for single operators the Taylor spectrum coincides with the ordinary spectrum.

(ii) Let $n = 2$ and let $A = (A_1, A_2)$ be a commuting pair of operators on X. Then the Koszul complex of A becomes

$$0 \to X \xrightarrow{\delta_A^0} X \oplus X \xrightarrow{\delta_A^1} X \to 0,$$

where δ_A^0 and δ_A^1 are defined by $\delta_A^0 x = A_1 x \oplus A_2 x \quad (x \in X)$ and $\delta_A^1(x \oplus y) = -A_2 x + A_1 y \quad (x, y \in X)$.

(iii) The most important parts of the Koszul complex of an n-tuple $A = (A_1, \dots, A_n)$ are its ends. The first mapping δ_A^0 can be interpreted as $\delta_A^0 : X \to X^n$ defined by $\delta_A^0 x = \bigoplus_{i=1}^n A_i x \quad (x \in X)$. Thus the Koszul complex of A is exact at $\Lambda^0[s, X]$ if and only if $\bigcap_{i=1}^n \ker A_i = \{0\}$. Similarly, $\delta_A^{n-1} : X^n \to X$ is defined by $\delta_A^{n-1}(x_1 \oplus \cdots \oplus x_n) = \sum_{i=1}^n (-1)^{i-1} A_i x_i$, and so the exactness at $\Lambda^n[s, X]$ means that $\operatorname{Im} A_1 + \cdots + \operatorname{Im} A_n = X$.

Proposition 2.3. *Let $A_1, \dots, A_n, B_1, \dots, B_n$ be mutually commuting operators on a Banach space X satisfying $\sum_{i=1}^n A_i B_i = I$. Then the n-tuple $A = (A_1, \dots, A_n)$ is Taylor regular.*

Proof. For $j = 1, \dots, n$ let $H_j : \Lambda[s, X] \to \Lambda[s, X]$ be the operators defined by

$$H_j\Big(\sum_{F \subset \{1,\dots,n\}} x_F s_F\Big) = \sum_{\substack{F \subset \{1,\dots,n\} \\ j \in F}} (-1)^{\operatorname{card}\{i \in F : i < j\}} x_F s_{F \setminus \{j\}}. \tag{2}$$

It is easy to verify that $H_j S_j + S_j H_j = I \quad (j = 1, \dots, n)$ and $H_i S_j + S_j H_i = 0$ for $i \neq j$.

Suppose that $\sum_{i=1}^n A_i B_i = I$. As above, denote by the same symbols B_i the operators acting on $\Lambda[s, X]$. Let $\varepsilon_B : \Lambda[s, X] \to \Lambda[s, X]$ be the operator defined by $\varepsilon_B = \sum_{j=1}^n H_j B_j$. Then

$$\begin{aligned} \varepsilon_B \delta_A + \delta_A \varepsilon_B &= \sum_{i=1}^n \sum_{j=1}^n (H_j B_j S_i A_i + S_i A_i H_j B_j) \\ &= \sum_{i=1}^n (H_i S_i B_i A_i + S_i H_i A_i B_i) + \sum_{i \neq j} (H_j S_i + S_i H_j) B_j A_i \\ &= \sum_{i=1}^n B_i A_i = I_{\Lambda[s,X]}. \end{aligned}$$

Let $\psi \in \ker \delta_A$. Then $\psi = (\varepsilon_B \delta_A + \delta_A \varepsilon_B)\psi = \delta_A \varepsilon_B \psi$, and so $\psi \in \operatorname{Im} \delta_A$. Hence $\ker \delta_A = \operatorname{Im} \delta_A$ and the n-tuple A is Taylor regular. □

If $\mathcal{A}$ is a unital commutative Banach algebra and $a_1, \dots, a_n \in \mathcal{A}$ then the joint spectrum is defined by

$$\sigma^{\mathcal{A}}(a_1, \dots, a_n) = \{(f(a_1), \dots, f(a_n)) : f \in \mathcal{M}(\mathcal{A})\},$$

where $\mathcal{M}(\mathcal{A})$ is the set of all multiplicative functionals $f : \mathcal{A} \to \mathbb{C}$ (i.e., the maximal ideal space of $\mathcal{A}$).

Let $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ be a commuting n-tuple of elements. Denote by $\langle a \rangle$ the smallest closed unital algebra containing $a_1, \dots, a_n$. Clearly $\langle a \rangle$ is a unital commutative Banach algebra.

Proposition 2.3 implies that

$$\sigma_T(A) \subset \sigma^{\mathcal{A}}(A) \tag{3}$$

for any unital commutative Banach algebra $\mathcal{A} \subset B(X)$ containing the operators $A_1, \dots, A_n$. In particular, $\sigma_T(A) \subset \sigma^{\langle A \rangle}(A)$ for all commuting n-tuples $A \in B(X)^n$.

For the study of basic properties of the Taylor spectrum one needs the following lemma. Its proof is elementary and various formulations of the lemma can be found in (Taylor, 1970a, Lemma 2.1; Vasilescu, 1979a, Lemma 2.1) or (Müller, 2007, Lemma 11.3).

Lemma 2.4. *Let X, Y, Z be Banach spaces, let $T : X \to Y$ and $S : Y \to Z$ be operators satisfying $\operatorname{Im} T = \ker S$ and let $\operatorname{Im} S$ be closed. Then there exists $\varepsilon > 0$ such $\operatorname{Im} T' = \ker S'$ and $\operatorname{Im} S'$ is closed for all pairs of operators $T' : X \to Y$ and $S' : Y \to Z$ satisfying $\|T' - T\| < \varepsilon$, $\|S' - S\| < \varepsilon$ and $S'T' = 0$.*

Corollary 2.5. *The set of all commuting Taylor regular n-tuples is relatively open in the set of all commuting n-tuples. Consequently, $\sigma_T(A)$ is a compact subset of $\mathbb{C}^n$.*

Moreover, for each $n \in \mathbb{N}$ the mapping $A \mapsto \sigma_T(A)$ defined on commuting n-tuples $A = (A_1, \dots, A_n) \in B(X)^n$ is upper semi-continuous.

Proof. Consider the sequence

$$\Lambda[s, X] \xrightarrow{\delta_A} \Lambda[s, X] \xrightarrow{\delta_A} \Lambda[s, X].$$

Note that $\|\delta_A - \delta_B\| \le \sum_{i=1}^n \|A_i - B_i\|$ for all commuting n-tuples $A, B \in B(X)^n$. By the previous lemma, $\sigma_T(A)$ is closed. By (3), it is compact. Clearly the mapping $A \mapsto \sigma_T(A)$ is upper semicontinuous. □

A very important property of the Taylor spectrum is the projection property — that $\sigma_T(A_{i_1}, \dots, A_{i_k}) = P\sigma_T(A_1, \dots, A_n)$ for all $k \le n$, $1 \le i_1 < \dots < i_k \le n$, where $P : \mathbb{C}^n \to \mathbb{C}^k$ is the natural projection defined by $P(z_1, \dots, z_n) = (z_{i_1}, \dots, z_{i_k})$.

It is well known that the analogous projection property is satisfied both for the *surjective spectrum*

$$\sigma_{sur}(A_1,\dots,A_n)=\big\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n:(A_1-\lambda_1)X+\cdots+(A_n-\lambda_n)X\neq X\big\}$$

and the *approximate point spectrum*

$$\sigma_{ap}(A_1,\dots,A_n)=\Big\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n:\inf_{x\in X,\|x\|=1}\sum_{i=1}^n\|(A_i-\lambda_i)x\|=0\Big\},$$

see (Słodkowski and Żelazko, 1974). The proof of projection property for the Taylor spectrum follows (Słodkowski, 1977).

Lemma 2.6. *Let Z_1,Z_2 be Banach spaces, let $B:Z_1\to Z_1$, $D:Z_1\to Z_2$ and $C:Z_2\to Z_2$ be operators satisfying $DB=CD$. Suppose that $DZ_1\neq Z_2$. Then there exists a complex number λ such that $DZ_1+(C-\lambda)Z_2\neq Z_2$.*

Proof. It is possible to reduce the statement of Lemma 2.6 to the projection property of the surjective spectrum. Consider the Banach space $Z=Z_2\oplus Z_1\oplus Z_1\oplus\cdots$ (for example with the ℓ^1 norm) and operators $U,V\in B(Z)$ given in the matrix form by

$$U=\begin{pmatrix}0&D&0&0&\cdots\\0&0&I&0&\\0&0&0&I&\\\vdots&&&&\ddots\end{pmatrix}\qquad\text{and}\qquad V=\begin{pmatrix}C&0&0&\cdots\\0&B&0&\\0&0&B&\\\vdots&&&\ddots\end{pmatrix}.$$

It is easy to check that $UV=VU$. Furthermore, $UZ\neq Z$ since $DZ_1\neq Z_2$. By the projection property for the surjective spectrum, there exists $\lambda\in\mathbb{C}$ such that $UZ+(V-\lambda)Z\neq Z$. Since $UZ\supset 0\oplus Z_1\oplus Z_1\oplus\cdots$, this is equivalent to the condition $DZ_1+(C-\lambda)Z_2\neq Z_2$. ☐

To prove the projection property for the Taylor spectrum it is necessary to investigate the exactness of the Koszul complex in more details.

For $k=0,\dots,n$ denote by $\Gamma_k^{(n)}$ the set of all commuting n-tuples of operators $A=(A_1,\dots,A_n)\in B(X)^n$ such that the Koszul complex of A is exact at $\Lambda^k(X)$, i.e., $\operatorname{Im}\delta_A^{k-1}=\ker\delta_A^k$. In agreement with the previous convention set formally $\Gamma_{-1}^{(n)}$ to be the set of all commuting n-tuples of operators.

Proposition 2.7. *Let $A_1,\dots,A_n,A_{n+1}$ be commuting operators on a Banach space X, let $0\le k\le n$ and $(A_1,\dots,A_n)\notin\Gamma_k^{(n)}$. Then there exists $\lambda\in\mathbb{C}$ such that $(A_1,\dots,A_n,A_{n+1}-\lambda)\notin\Gamma_{k+1}^{(n+1)}$.*

Proof. Write for short $A=(A_1,\dots,A_n)$, $A_\lambda=(A_1,\dots,A_n,A_{n+1}-\lambda)$ $(\lambda\in\mathbb{C})$ and $s=(s_1,\dots,s_n)$. Suppose that $A\notin\Gamma_k^{(n)}$. Clearly, $A_{n+1}\ker\delta_A^k\subset\ker\delta_A^k$. Consider the following diagram

$$\begin{array}{ccc}\Lambda^{k-1}[s,X]&\xrightarrow{\delta_A^{k-1}}&\ker\delta_A^k\\ \downarrow{\scriptstyle A_{n+1}}&&\downarrow{\scriptstyle A_{n+1}}\\ \Lambda^{k-1}[s,X]&\xrightarrow{\delta_A^{k-1}}&\ker\delta_A^k.\end{array}$$

By Lemma 2.6, there is a $\lambda \in \mathbb{C}$ such that $\operatorname{Im} \delta_A^{k-1} + (A_{n+1} - \lambda) \ker \delta_A^k \neq \ker \delta_A^k$. It is sufficient to prove that $\operatorname{Im} \delta_{A_\lambda}^k \neq \ker \delta_{A_\lambda}^{k+1}$.

Let $\psi \in \ker \delta_A^k \setminus \big(\operatorname{Im} \delta_A^{k-1} + (A_{n+1} - \lambda) \ker \delta_A^k\big)$. Then

$$\begin{aligned} \delta_{A_\lambda}^{k+1} S_{n+1}\psi &= \Big((A_{n+1} - \lambda)S_{n+1} + \sum_{i=1}^{n} A_i S_i\Big) S_{n+1}\psi \\ &= \sum_{i=1}^{n} A_i S_i S_{n+1}\psi = -S_{n+1}\delta_A^k \psi = 0. \end{aligned}$$

So $S_{n+1}\psi \in \ker \delta_{A_\lambda}^{k+1}$. It is sufficient to show that $S_{n+1}\psi \notin \operatorname{Im} \delta_{A_\lambda}^k$.

Suppose on the contrary that there is a $\varphi \in \Lambda^k[s, s_{n+1}, X]$ with $\delta_{A_\lambda}^k \varphi = S_{n+1}\psi$. Write $\varphi = S_{n+1}\varphi_{k-1} + \varphi_k$ for $\varphi_{k-1} \in \Lambda^{k-1}[s, X]$, $\varphi_k \in \Lambda^k[s, X]$. Then $S_{n+1}\psi = \delta_{A_\lambda}^k \varphi = S_{n+1}\big(-\delta_A^{k-1}\varphi_{k-1} + (A_{n+1} - \lambda)\varphi_k\big) + \delta_A^k \varphi_k$. Thus $\varphi_k \in \ker \delta_A^k$ and $\psi = -\delta_A^{k-1}\varphi_{k-1} + (A_{n+1} - \lambda)\varphi_k \in \operatorname{Im} \delta_A^{k-1} + (A_{n+1} - \lambda) \ker \delta_A^k$, which is a contradiction. □

Proposition 2.8. *Let $A_1, \dots, A_n, A_{n+1}$ be commuting operators on a Banach space X, let $0 \le k \le n$. Suppose that $(A_1, \dots, A_n) \in \Gamma_{k-1}^{(n)} \cap \Gamma_k^{(n)}$. Then $(A_1, \dots, A_n, A_{n+1}) \in \Gamma_k^{(n+1)}$.*

Proof. Let $A = (A_1, \dots, A_n)$, $s = (s_1, \dots, s_n)$ and $A' = (A_1, \dots, A_n, A_{n+1})$. Suppose that $\ker \delta_A^{k-1} = \operatorname{Im} \delta_A^{k-2}$ and $\ker \delta_A^k = \operatorname{Im} \delta_A^{k-1}$. It is sufficient to prove that $\ker \delta_{A'}^k \subset \operatorname{Im} \delta_{A'}^{k-1}$ (the opposite inclusion is always true).

Let $\psi \in \ker \delta_{A'}^k$. Express $\psi = \eta_k + S_{n+1}\eta_{k-1}$ for some $\eta_k \in \Lambda^k[s, X]$ and $\eta_{k-1} \in \Lambda^{k-1}[s, X]$. Then

$$0 = \delta_{A'}^k \psi = \sum_{i=1}^{n+1} A_i S_i \eta_k + \sum_{i=1}^{n} A_i S_i S_{n+1}\eta_{k-1} = \delta_A^k \eta_k + S_{n+1}(A_{n+1}\eta_k - \delta_A^{k-1}\eta_{k-1}).$$

Thus $\delta_A^k \eta_k = 0$, and so $\eta_k = \delta_A^{k-1}\xi_{k-1}$ for some $\xi_{k-1} \in \Lambda^{k-1}[s, X]$. Further,

$$0 = A_{n+1}\eta_k - \delta_A^{k-1}\eta_{k-1} = \delta_A^{k-1}(A_{n+1}\xi_{k-1} - \eta_{k-1}),$$

and so $A_{n+1}\xi_{k-1} - \eta_{k-1} = \delta_A^{k-2}\xi_{k-2}$ for some $\xi_{k-2} \in \Lambda^{k-2}[s, X]$. Hence

$$\begin{aligned} \psi &= \eta_k + S_{n+1}\eta_{k-1} = \delta_A^{k-1}\xi_{k-1} + S_{n+1}A_{n+1}\xi_{k-1} - S_{n+1}\delta_A^{k-2}\xi_{k-2} \\ &= \delta_{A'}^{k-1}\xi_{k-1} + \delta_{A'}^{k-1}S_{n+1}\xi_{k-2} \in \operatorname{Im} \delta_{A'}^{k-1}. \end{aligned}$$

□

Corollary 2.9. *Let $A_1, \dots, A_{n+1} \in B(X)$ be commuting operators.*

(i) *If $(A_1, \dots, A_n)$ is Taylor regular then $(A_1, \dots, A_n, A_{n+1})$ is also Taylor regular.*

(ii) *If $(A_1, \dots, A_n)$ is Taylor singular then there exists $\lambda \in \mathbb{C}$ such that $(A_1, \dots, A_n, A_{n+1} - \lambda)$ is also Taylor singular.*

(iii) *Consequently, $\sigma_T(A_1, \dots, A_n) = P\sigma_T(A_1, \dots, A_n, A_{n+1})$, where $P : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is the natural projection onto the first n coordinates.*

In particular, since the Taylor spectrum of a single operator $A_1 \in B(X)$ is equal to the ordinary spectrum, which is non-empty, this corollary implies that $\sigma_T(A_1, \dots, A_n)$ is always non-empty for every commuting n-tuple $(A_1, \dots, A_n) \in B(X)^n$.

Since obviously $\sigma_T(A_{\pi(1)}, \dots, A_{\pi(n)}) = \big\{(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}) : (\lambda_1, \dots, \lambda_n) \in \sigma_T(A_1, \dots, A_n)\big\}$ for any permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$, one has the following theorem:

Theorem 2.10. *(projection property of the Taylor spectrum) Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple of operators, let $1 \le k \le n$ and $1 \le i_1 < i_2 < \dots < i_k \le n$. Then*

$$\sigma_T(A_{i_1}, \dots, A_{i_k}) = P_{i_1,\dots,i_k}\sigma_T(A_1, \dots, A_n),$$

where $P_{i_1,\dots,i_k} : \mathbb{C}^n \to \mathbb{C}^k$ is the projection defined by $P_{i_1,\dots,i_k}(\lambda_1, \dots, \lambda_n) = (\lambda_{i_1}, \dots, \lambda_{i_k})$.

A consequence of the projection property is the spectral mapping property for polynomial mappings.

Theorem 2.11. *(spectral mapping property) Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple of operators, let $k \in \mathbb{N}$ and let $p = (p_1, \dots, p_k)$ be a k-tuple of polynomials in n variables. Let $p(A) = (p_1(A), \dots, p_k(A))$. Then*

$$\sigma_T(p(A)) = p(\sigma_T(A)).$$

Proof. Consider the $(n+k)$-tuple $B = (A_1, \dots, A_n, p_1(A), \dots, p_k(A))$. Then

$$\begin{aligned} &\sigma_T(B) \subset \sigma^{\langle A\rangle}(B) \\ = \;& \{(f(A_1), \dots, f(A_k), f(p_1(A)), \dots, f(p_k(A))) : f \in \mathcal{M}(\langle A\rangle)\} \\ = \;& \{(z, p_1(z), \dots, p_k(z)) : z \in \sigma^{\langle A\rangle}(B) \subset \{(z, p(z)) : z \in \mathbb{C}^n\}. \end{aligned}$$

If $z \in \sigma_T(A)$ then there exists $z' \in \mathbb{C}^k$ such that $(z, z') \in \sigma_T(B)$. So $z' = p(z)$ and $z' \in \sigma_T(p(A))$ by the projection property.

Conversely, if $z' \in \sigma_T(p(A))$ then there exists $z \in \mathbb{C}^n$ such that $(z, z') \in \sigma_T(B) \subset \{(w, p(w)) : w \in \mathbb{C}^n\}$. So $z' = p(z)$ and $z \in \sigma_T(A)$ by the projection property. $\square$

Remark 2.12. Let $\mathcal{A}$ be a unital Banach algebra. By a *spectral system* we mean a mapping $\tilde\sigma$ that assigns to each commuting tuple $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ a nonempty compact subset $\tilde\sigma(a) \subset \mathbb{C}^n$ such that $\tilde\sigma(a) \subset \sigma^{\langle a\rangle}(a)$ and $\tilde\sigma$ satisfies the projection property, $\tilde\sigma(a_{i_1}, \dots, a_{i_k}) = P_{i_1,\dots,i_k}\tilde\sigma(a)$ for all a and $i_1, \dots, i_k$.

A spectral system $\tilde\sigma$ is *upper semicontinuous* if the mapping $(a_1, \dots, a_n) \mapsto \tilde\sigma(a_1, \dots, a_n)$ is upper semicontinuous for each n.

Theorem 2.10 and Proposition 2.3 imply that the Taylor spectrum is an upper semicontinuous spectral system. Further examples of spectral systems are the surjective spectrum and approximate point spectrum. Other examples of spectral systems will be discussed in the next section.

As in the previous theorem, one can prove that any spectral system satisfies also the spectral mapping property $\tilde{\sigma}(p(A)) = p(\tilde{\sigma}(A))$ for all k-tuples $p = (p_1, \ldots, p_k)$ of polynomials in n variables.

The Taylor spectrum has also a nice duality property.

Theorem 2.13. *Let $A = (A_1, \ldots, A_n) \in B(X)^n$ be a commuting n-tuple of operators. Then A is Taylor regular if and only if $A^* = (A_1^*, \ldots, A_n^*) \in B(X^*)^n$ is Taylor regular.*
Consequently, $\sigma_T(A) = \sigma_T(A^*)$.

The proof of the previous theorem is based on the following elementary lemma. For details see (Słodkowski, 1977).

Lemma 2.14. *Let X, Y, Z be Banach spaces, let $T : X \to Y$ and $S : Y \to Z$ be operators satisfying $ST = 0$. The following statements are equivalent:*

(i) $\operatorname{Im} T = \ker S$ *and* $\operatorname{Im} S$ *is closed;*
(ii) $\operatorname{Im} S^* = \ker T^*$ *and* $\operatorname{Im} T^*$ *is closed.*

Remark 2.15. If H is a Hilbert space then it is usual to identify its dual H^* with H. With this convention one has rather

$$\sigma_T(A_1^*, \ldots, A_n^*) = \{(\bar{z}_1, \ldots, \bar{z}_n) : (z_1, \ldots, z_n) \in \sigma_T(A_1, \ldots, A_n)\}$$

for all commuting n-tuples $(A_1, \ldots, A_n) \in B(H)^n$.

Remark 2.16. The precise name of complex (1) is the *cochain Koszul complex* of A. It is possible to assign to a commuting n-tuple $A = (A_1, \ldots, A_n) \in B(X)^n$ also another "dual" complex (called the *chain Koszul complex* of A). As in the proof of Proposition 2.3, for $j = 1, \ldots, n$ define operators $H_j : \Lambda[s, X] \to \Lambda[s, X]$ by (2) and set $\varepsilon_A = \sum_{j=1}^n A_j H_j : \Lambda[s, X] \to \Lambda[s, X]$. Equivalently, for $1 \le i_1 < i_2 < \cdots < i_p \le n$ one has

$$\varepsilon_A x s_{i_1} \wedge \cdots \wedge s_{i_p} = \sum_{k=1}^{p} (-1)^{k-1} A_{i_k} x s_{i_1} \wedge \cdots \wedge \widehat{s_{i_k}} \wedge \cdots \wedge s_{i_p},$$

where the hat denotes the omitted term.

It is easy to verify that $H_j B = B H_j$ for all $B \in B(X)$ and $H_i H_j = -H_j H_i \quad (1 \le i, j \le n)$. Thus $(\varepsilon_A)^2 = 0$. Clearly, $\varepsilon_A \Lambda^p[s, X] \subset \Lambda^{p-1}[s, X]$ for all p, and so ε_A defines a complex

$$0 \leftarrow \Lambda^0[s, X] \xleftarrow{\varepsilon_A^0} \Lambda^1[s, X] \xleftarrow{\varepsilon_A^1} \cdots \xleftarrow{\varepsilon_A^{n-1}} \Lambda^n[s, X] \leftarrow 0, \tag{4}$$

where ε_A^p is the restriction of ε_A to $\Lambda^{p+1}[s, X] \quad (p = 0, \ldots, n)$. Complex (4) is called the chain Koszul complex of A.

The chain complex can be also used for the definition of the Taylor spectrum of A (in fact this was the original definition of Taylor). Fortunately, these two definitions coincide since the chain Koszul complex of A is exact if and only if the cochain Koszul complex is exact, see (Eschmeier and Putinar, 1996, p. 32).

Note also that for $0 \le p \le n$ the exactness of one of the Koszul complexes at $\Lambda^p[s,X]$ is equivalent to the exactness of the other Koszul complex at $\Lambda^{n-p}[s,X]$.

3. Variants of the Taylor spectrum

Słodkowski's spectra

Recall that the sets $\Gamma_k^{(n)}$ were defined as the sets of all commuting n-tuples $(A_1,\dots,A_n) \in B(X)^n$ such that the Koszul complex (1) is exact at the k-th position, i.e., $\ker \delta_A^{k+1} = \operatorname{Im} \delta_A^k$.

For a commuting n-tuple $A = (A_1,\dots,A_n)$ of operators on X and $k = 0,1,\dots,n$ define

$$\sigma_{\delta,k}(A) = \left\{ \lambda \in \mathbb{C}^n : A - \lambda \notin \bigcap_{j=0}^{k} \Gamma_{n-j}^{(n)} \right\}.$$

The condition in the definition of $\sigma_{\delta,k}$ means that the Koszul complex of $A - \lambda$ is not exact at some of the last k positions. Clearly

$$\sigma_{sur}(A) = \sigma_{\delta,0}(A) \subset \sigma_{\delta,1}(A) \subset \cdots \subset \sigma_{\delta,n}(A) = \sigma_T(A),$$

where σ_{sur} is the surjective spectrum defined in the previous section.

Dually one can define

$$\sigma_{\pi,k}(A) = \left\{ \lambda \in \mathbb{C}^n : A - \lambda \notin \bigcap_{j=0}^{k} \Gamma_j^{(n)} \ \text{ or } \ \operatorname{Im} \delta_A^k \text{ is not closed} \right\}.$$

Evidently,

$$\sigma_{ap}(A) = \sigma_{\pi,0}(A) \subset \sigma_{\pi,1}(A) \subset \cdots \subset \sigma_{\pi,n}(A) = \sigma_T(A),$$

where σ_{ap} is the approximate point spectrum.

The Słodkowski spectra $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$ satisfy the following duality property, see (Słodkowski, 1977).

Theorem 3.1. *Let $A = (A_1,\dots,A_n)$ be a commuting n-tuple of operators on a Banach space X and let $0 \le k \le n$. Then:*

(i) $\sigma_{\delta,k}(A_1^*,\dots,A_n^*) = \sigma_{\pi,k}(A_1,\dots,A_n)$;
(ii) $\sigma_{\pi,k}(A_1^*,\dots,A_n^*) = \sigma_{\delta,k}(A_1,\dots,A_n)$.

Theorem 3.2. *$\sigma_{\delta,k}$ and $\sigma_{\pi,k}$ are upper semicontinuous spectral systems for each $k \ge 0$.*

Proof. By Propositions 2.7 and 2.8, $\sigma_{\delta,k}$ is a spectral system. The statements for $\sigma_{\pi,k}$ follow from the duality. □

The approximate point spectrum and the surjective spectrum give an important information about the n-tuple $(A_1,\dots,A_n)$; the meaning of the middle terms in the Koszul complex is not so clear. Therefore it is useful to know that great parts of the Taylor spectrum are in fact included in the surjective and approximate point spectra.

Theorem 3.3. *Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple of operators. Then $\sigma_{ap}(A) \cap \sigma_{sur}(A)$ contains the distinguished (Shilov) boundary of $\sigma_T(A)$, i.e., if $\lambda \in \sigma_T(A)$ satisfies that for each open neighbourhood U of λ there exists a polynomial p in n variables such that $\sup\{|p(z)| : z \in U \cap \sigma_T(A)\} > \sup\{|p(z)| : z \in \sigma_T(A) \setminus U\}$, then $\lambda \in \sigma_{ap}(A) \cap \sigma_{sur}(A)$.*

Consequently, the polynomially convex hulls of $\sigma_{sur}(A), \sigma_{ap}(A)$ and $\sigma_T(A)$ coincide.

Theorem 3.3 is a consequence of the spectral mapping property. Another interesting result is true for $n = 2$, see (Curto, 1986; Wrobel, 1986; Chō and Takagushi, 1981).

Theorem 3.4. *Let $(A_1, A_2) \in B(X)^2$ be a commuting pair of operators. Then $\partial\sigma_T(A_1, A_2) \subset \sigma_{ap}(A) \cup \sigma_{sur}(A)$, where ∂ denotes the topological boundary.*

Split spectrum

Definition 3.5. Let $A = (A_1, \dots, A_n)$ be an n-tuple of commuting operators on a Banach space X. We say that A is *split regular* if it is Taylor regular and the mapping $\delta_A : \Lambda[s, X] \to \Lambda[s, X]$ has a generalized inverse, i.e., there exists an operator $W : \Lambda[s, X] \to \Lambda[s, X]$ satisfying $\delta_A W \delta_A = \delta_A$.

The *split spectrum* $\sigma_S(A)$ is the set of all $\lambda \in \mathbb{C}^n$ such that the n-tuple $A - \lambda$ is not split regular.

The following result characterizes the split regular n-tuples of operators. The proof is simple and is omitted.

Proposition 3.6. *Let $A = (A_1, \dots, A_n)$ be an n-tuple of mutually commuting operators on a Banach space X. The following conditions are equivalent:*

(i) *A is split regular;*
(ii) *A is Taylor regular and $\ker \delta_A^p$ is a complemented subspace of $\Lambda^p[s, X]$ for each $p = 0, \dots, n-1$;*
(iii) *there exist operators $W_1, W_2 : \Lambda[s, X] \to \Lambda[s, X]$ such that $W_1\delta_A + \delta_A W_2 = I_{\Lambda[s,X]}$;*
(iv) *there exists an operator $V : \Lambda[s, X] \to \Lambda[s, X]$ such that $V\delta_A + \delta_A V = I$, $V^2 = 0$ and $V\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X] \quad (p = 0, \dots, n)$. Equivalently, there are operators $V_p : \Lambda^{p+1}[s, X] \to \Lambda^p[s, X]$ (see the diagram below) such that $V_{p-1}V_p = 0$ and $V_p\delta_A^p + \delta_A^{p-1}V_{p-1} = I_{\Lambda^p[s,X]}$ for every p (for $p = 0$ and $p = n$ this reduces to $V_0\delta_A^0 = I_{\Lambda^0[s,X]}$ and $\delta_A^{n-1}V_{n-1} = I_{\Lambda^n[s,X]}$, respectively).*

$$0 \to \Lambda^0[s, X] \underset{V_0}{\overset{\delta_A^0}{\rightleftarrows}} \Lambda^1[s, X] \underset{V_1}{\overset{\delta_A^1}{\rightleftarrows}} \cdots \underset{V_{n-1}}{\overset{\delta_A^{n-1}}{\rightleftarrows}} \Lambda^n[s, X] \to 0$$

Remark 3.7. For single operators on a Banach space the split spectrum coincides with the Taylor spectrum (and with the ordinary spectrum).

By Proposition 3.6 (ii), the split spectrum coincides with the Taylor spectrum also for n-tuples of commuting operators on a Hilbert space. For general Banach spaces the split spectrum may be bigger than the Taylor spectrum, see (Müller, 1997).

For $T \in B(X)$ define the operators $L_T, R_T : B(X) \to B(X)$ by $L_T A = TA$ and $R_T A = AT \quad (A \in B(X))$. For an n-tuple $A = (A_1, \dots, A_n) \in B(X)^n$ write $L_A = (L_{A_1}, \dots, L_{A_n})$ and $R_A = (R_{A_1}, \dots, R_{A_n})$.

It is possible to show that the Taylor spectra of A, L_A and R_A are related in the following way, see (Curto, 1991; Eschmeier and Putinar, 1996, Corollary 2.6.11) or (Müller, 2007, Theorem 26.7).

Theorem 3.8. *Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple of operators. Then*

$$\sigma_S(A) = \sigma_T(L_A) = \sigma_S(L_A) = \sigma_T(R_A) = \sigma_S(R_A).$$

Corollary 3.9. *The split spectrum σ_S is an upper semicontinuous spectral system.*

Essential Taylor spectrum

Definition 3.10. Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of operators on an infinite-dimensional Banach space X. We say that A is *essentially Taylor regular* if $\dim \ker \delta_A / \operatorname{Im} \delta_A < \infty$. The *essential Taylor spectrum* $\sigma_{Te}(A)$ is the set of all $\lambda \in \mathbb{C}^n$ such that $A - \lambda$ is not essentially Taylor regular.

The essentially Taylor regular n-tuples are an analogy of the Fredholm operators. The following result is easy to see.

Proposition 3.11. *Let $A = (A_1, \dots, A_n)$ be an essentially Taylor regular n-tuple of operators. Then $\operatorname{Im} \delta_A$ is closed.*

If $n = 1$ then (A_1) is essentially Taylor regular if and only if A_1 is Fredholm.

It is possible to show that the essential Taylor spectrum is also an upper semicontinuous spectral system.

For essentially Taylor regular n-tuples one can define the *index* which has similar perturbation properties as the index of Fredholm operators.

Definition 3.12. Let $A = (A_1, \dots, A_n) \in B(X)^n$ be an essentially Taylor regular n-tuple of operators. For $0 \le k \le n$ let $\alpha_k(A) = \dim \ker \delta_A^k / \operatorname{Im} \delta_A^{k-1}$. Let

$$\operatorname{ind} A = \sum_{i=0}^{n} (-1)^i \alpha_i(A).$$

Theorem 3.13. *Let $A = (A_1, \dots, A_n) \in B(X)^n$ be an essentially Taylor regular n-tuple of operators. Then:*

(i) *there exists* $\varepsilon > 0$ *such that* $\alpha_k(B) \le \alpha_k(A)$ *for all* $k, 0 \le k \le n$ *and every commuting* n*-tuple* $B = (B_1, \dots, B_n) \in B(X)^n$ *satisfying* $\max\{\|A_i - B_i\| : 0 \le i \le n\} < \varepsilon$;
(ii) *there exists* $\varepsilon > 0$ *such that* B *is essentially Taylor regular and* $\operatorname{ind} B = \operatorname{ind} A$ *for every commuting* n*-tuple* $B = (B_1, \dots, B_n) \in B(X)^n$ *satisfying* $\max\{\|A_i - B_i\| : 0 \le i \le n\} < \varepsilon$;
(iii) *let* $B = (B_1, \dots, B_n) \in B(X)^n$ *be a commuting* n*-tuple such that* $B_i - A_i$ *is compact for all* i. *Then* B *is essentially Taylor regular and* $\operatorname{ind} B = \operatorname{ind} A$.

For the proof of (i) and (ii) see (Vasilescu, 1979a). For (iii) see (Ambrozie, 1996).

4. Taylor functional calculus for the split spectrum

The most important property of the Taylor spectrum is the existence of the functional calculus for functions analytic on a neighbourhood of the Taylor spectrum.

As the construction of the Taylor functional calculus is rather technical, in this section a simpler version for functions analytic on a neighbourhood of the split spectrum is presented. Since the split spectrum contains the Taylor spectrum, this split Taylor functional calculus is less rich. However, the construction of the calculus is much simpler.

Note that for Hilbert space operators the split spectrum coincides with the Taylor spectrum and so the corresponding functional calculi also coincide. The split functional calculus is also sufficient for the construction of the functional calculus in commutative Banach algebras.

Theorem 4.1. *Let* $A = (A_1, \dots, A_n)$ *be an* n*-tuple of mutually commuting operators on a Banach space* X. *Suppose that* A *is split regular, i.e.,* $\ker \delta_A = \operatorname{Im} \delta_A$ *and* δ_A *has a generalized inverse. Then there exists a neighbourhood* U *of* 0 *in* $\mathbb{C}^n$ *and an analytic function* $V : U \to B(\Lambda[s, X])$ *such that* $V(z)\delta_{A-z} + \delta_{A-z}V(\lambda) = I_{\Lambda[s,X]}$ *for every* $z \in U$.

Moreover, one can assume that $V(z)^2 = 0 \quad (z \in U)$ *and*

$$V(z)\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X] \qquad (z \in U, p = 0, \dots, n).$$

Proof. By Proposition 3.6, there exists an operator $V : \Lambda[s, X] \to \Lambda[s, X]$ such that $V^2 = 0$, $\delta_A V + V\delta_A = I_{\Lambda[s,X]}$, and $V\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X]$ for every p.

For $z \in \mathbb{C}^n$ write $H_z = \delta_{A-z} - \delta_A$. Let U be the set of all $z \in \mathbb{C}^n$ such that $\|H_z\| < \|V\|^{-1}$. Clearly, U is a neighbourhood of 0 in $\mathbb{C}^n$ and, for $z \in U$, the operators $I + H_z V$ and $I + VH_z$ are invertible. One has $V(I + H_z V) = (I + VH_z)V$, and so $(I + VH_z)^{-1}V = V(I + H_z V)^{-1}$. For $z \in U$ set $V(z) = (I + VH_z)^{-1}V$. Then

$$\begin{aligned} & \delta_{A-z}V(z) + V(z)\delta_{A-z} \\ = \quad & (\delta_A + H_z)V(I + H_z V)^{-1} + (I + VH_z)^{-1}V(\delta_A + H_z) \end{aligned}$$

$$\begin{aligned} = \quad & (I+VH_z)^{-1}\big((I+VH_z)(\delta_A+H_z)V \\ & +V(\delta_A+H_z)(I+H_zV)\big)(I+H_zV)^{-1}. \end{aligned}$$

The expression in the middle is equal to

$$\begin{aligned} & \delta_A V + H_z V + VH_z\delta_A V + VH_z^2 V + V\delta_A + VH_z + V\delta_A H_z V + VH_z^2 V \\ = {} & (I+VH_z)(I+H_zV) + V(H_z\delta_A + \delta_A H_z + H_z^2)V \\ = {} & (I+VH_z)(I+H_zV) + V\big((\delta_A+H_z)^2 - (\delta_A)^2\big)V = (I+VH_z)(I+H_zV) \end{aligned}$$

since $(\delta_A)^2 = 0$ and $(\delta_A + H_z)^2 = (\delta_{A-z})^2 = 0$. Thus

$$\delta_{A-z}V(z) + V(z)\delta_{A-z} = I_{\Lambda[s,X]} \qquad (z \in U).$$

Further,

$$V(z)^2 = (I+VH_z)^{-1}V \cdot V(I+H_zV)^{-1} = 0.$$

Finally, $V(z) = \sum_{i=0}^{\infty}(-1)^i(VH_z)^iV$ where

$$(VH_z)\Lambda^p[s,X] \subset \Lambda^p[s,X] \qquad (p = 0,\dots,n),$$

and so

$$V(z)\Lambda^p[s,X] \subset \Lambda^{p-1}[s,X] \qquad (z \in U, p = 0,\dots,n).$$

□

Corollary 4.2. *Let $A = (A_1,\dots,A_n)$ be an n-tuple of mutually commuting operators on a Banach space X. Let $G = \mathbb{C}^n \setminus \sigma_S(A)$. Then there exists an operator-valued C^∞-function $V : G \to B(\Lambda[s,X])$ such that $\delta_{A-z}V(z) + V(z)\delta_{A-z} = I_{\Lambda[s,X]}$ and*

$$V(z)\Lambda^p[s,X] \subset \Lambda^{p-1}[s,X] \qquad (z \in G, p = 0,\dots,n).$$

Proof. For every $w \in G$ there exists a neighbourhood U_w of w and an analytic operator-valued function $V_w : U_w \to B(\Lambda[s,X])$ such that $V_w(z)\delta_{A-z} + \delta_{A-z}V_w(z) = I_{\Lambda[s,X]}$ and

$$V_w(z)\Lambda^p[s,X] \subset \Lambda^{p-1}[s,X] \qquad (z \in U_w, p = 0,\dots,n).$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be a C^∞-partition of unity subordinated to the cover $\{U_w, w \in G\}$ of G, i.e., ψ_i are C^∞-functions, $0 \le \psi_i \le 1$, $\operatorname{supp}\psi_i \subset U_{w_i}$ for some $w_i \in G$, for each $z \in G$ there exists a neighbourhood U of z such that all but finitely many of the functions ψ_i are 0 on U and $\sum_{i=1}^{\infty}\psi_i(z) = 1$ for each $z \in G$.

For $z \in G$ set $V(z) = \sum_{i=1}^{\infty}\psi_i(z)V_{w_i}(z)$. Then

$$\delta_{A-z}V(z) + V(z)\delta_{A-z} = \sum_{i=1}^{\infty}\big(\delta_{A-z}V_{w_i}(z) + V_{w_i}(z)\delta_{A-z}\big)\psi_i(z) = I_{\Lambda[s,X]}$$

and

$$V(z)\Lambda^p[s,X] \subset \Lambda^{p-1}[s,X]$$

for all $z \in G$ and $p = 0,1,\dots,n$. □

Remark 4.3. It is possible to require also that $V(z)\delta_{A-z}V(z) = V(z)$ and $V(z)^2 = 0$ for all $z \in G$. These additional properties of the generalized inverse V, however, are not essential and are not used in the sequel.

In the following fix a commuting n-tuple $A = (A_1, \dots, A_n)$ of bounded linear operators on a Banach space X, the set $G = \mathbb{C}^n \setminus \sigma_S(A)$ and a C^∞-function $V : G \to B(\Lambda[s, X])$ with the properties of Corollary 4.2.

Consider the space $C^\infty(G, \Lambda[s, X])$. Clearly, this space can be identified with the set $\Lambda[s, C^\infty(G, X)]$.

The function $V : G \to B(\Lambda[s, X])$ induces naturally the operator (denoted by the same symbol) $V : C^\infty(G, \Lambda[s, X]) \to C^\infty(G, \Lambda[s, X])$ by

$$(Vy)(z) = V(z)y(z) \qquad \big(z \in G, y \in C^\infty(G, \Lambda[s, X])\big).$$

Similarly, define the operator δ_{A-z} (or δ for short if no ambiguity can arise) acting in $C^\infty(G, \Lambda[s, X])$ by

$$(\delta y)(z) = \delta_{A-z}\, y(z) \qquad \big(z \in G, y \in C^\infty(G, \Lambda[s, X])\big).$$

Clearly, $\delta^2 = 0$, $V\delta + \delta V = I_{\Lambda[s, C^\infty(G,X)]}$ and both V and δ are "graded", i.e.,

$$\begin{array}{ll} V\Lambda^p[s, C^\infty(G, X)] \subset \Lambda^{p-1}[s, C^\infty(G, X)] \qquad & \text{and} \\ \delta\Lambda^p[s, C^\infty(G, X)] \subset \Lambda^{p+1}[s, C^\infty(G, X)]. & \end{array}$$

Consider now another set of indeterminates $\mathrm{d}\bar{z} = (\mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_n)$ and the space $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$. Let $\bar{\partial} : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the linear mapping defined by

$$\bar{\partial} f s_{i_1} \wedge \cdots \wedge s_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q} = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} \mathrm{d}\bar{z}_k \wedge s_{i_1} \wedge \cdots \wedge s_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}.$$

Obviously, $\bar{\partial}^2 = 0$.

The operators V and δ can be lifted to $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ in the natural way. Clearly, the properties of V and δ are preserved: $\delta^2 = 0$, $V\delta + \delta V = I$ and both V and δ are graded. Note also that $\delta\bar{\partial} = -\bar{\partial}\delta$ and $(\bar{\partial} + \delta)^2 = 0$.

Let $W : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the mapping defined in the following way: if $\psi \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$, $\psi = \psi_0 + \dots + \psi_n$, where ψ_j is the part of ψ of degree j in $\mathrm{d}\bar{z}$, then set $W\psi = \eta_0 + \dots + \eta_n$, where

$$\begin{aligned} \eta_0 &= V\psi_0, \\ \eta_1 &= V(\psi_1 - \bar{\partial}\eta_0), \\ &\vdots \\ \eta_n &= V(\psi_n - \bar{\partial}\eta_{n-1}). \end{aligned} \tag{5}$$

Note that η_j is the part of $W\psi$ of degree j in $\mathrm{d}\bar{z}$.

Lemma 4.4. *Let $W : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the mapping defined by (5). Then:*

(i) $\operatorname{supp} W\psi \subset \operatorname{supp}\psi$ *for all* ψ*;*

(ii) *if* G' *is an open subset of* G *and* $\psi \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ *satisfies* $(\bar{\partial} + \delta)\psi = 0$ *on* G', *then* $(\bar{\partial} + \delta)W\psi = \psi$ *on* G';

(iii) $(\bar{\partial} + \delta)W(\bar{\partial} + \delta) = \bar{\partial} + \delta$.

Proof. (i) Clear.

(ii) Let $\psi = \psi_0 + \cdots + \psi_n$, where ψ_j is the part of ψ of degree j in $\mathrm{d}\bar{z}$. The condition $(\bar{\partial} + \delta)\psi = 0$ on G' can be rewritten as

$$\begin{aligned} \delta\psi_0 &= 0, \\ \bar{\partial}\psi_0 + \delta\psi_1 &= 0, \\ &\vdots \\ \bar{\partial}\psi_{n-1} + \delta\psi_n &= 0 \end{aligned} \tag{6}$$

(the condition $\bar{\partial}\psi_n = 0$ is satisfied automatically).

Let $W\psi = \eta_0 + \cdots + \eta_n$, where η_j are defined by (5). The required condition $(\bar{\partial} + \delta)W\psi = \psi$ becomes

$$\begin{aligned} \delta\eta_0 &= \psi_0, \\ \bar{\partial}\eta_0 + \delta\eta_1 &= \psi_1, \\ &\vdots \\ \bar{\partial}\eta_{n-1} + \delta\eta_n &= \psi_n \end{aligned} \tag{7}$$

on G' (again, $\bar{\partial}\eta_n = 0$ automatically).

By (5) and (6), one has $\delta\eta_0 = \delta V\psi_0 = (\delta V + V\delta)\psi_0 = \psi_0$ and $\bar{\partial}\eta_0 + \delta\eta_1 = \bar{\partial}\eta_0 + \delta V(\psi_1 - \bar{\partial}\eta_0) = \bar{\partial}\eta_0 + (I - V\delta)(\psi_1 - \bar{\partial}\eta_0) = \psi_1 - V\delta(\psi_1 - \bar{\partial}\eta_0) = \psi_1$, since $\delta(\psi_1 - \bar{\partial}\eta_0) = \delta\psi_1 + \bar{\partial}\delta\eta_0 = \delta\psi_1 + \bar{\partial}\psi_0 = 0$.

It is possible to prove (7) by induction. Suppose that $\bar{\partial}\eta_{j-1} + \delta\eta_j = \psi_j$ for some $j \geq 1$. Then $\delta(\psi_{j+1} - \bar{\partial}\eta_j) = \delta\psi_{j+1} + \bar{\partial}\delta\eta_j = \delta\psi_{j+1} + \bar{\partial}\psi_j = 0$ and, by the induction assumption, $\bar{\partial}\eta_j + \delta\eta_{j+1} = \bar{\partial}\eta_j + \delta V(\psi_{j+1} - \bar{\partial}\eta_j) = \bar{\partial}\eta_j + (I - V\delta)(\psi_{j+1} - \bar{\partial}\eta_j) = \psi_{j+1}$.

(iii) Since $(\bar{\partial} + \delta)^2 = 0$, the statement follows from (ii). □

The differential form

$$(2i)^{-n}\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{8}$$

will be interpreted as the Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$.

Let P be the natural projection $P : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[\mathrm{d}\bar{z}, C^\infty(G, X)]$ that annihilates all terms containing at least one of the indeterminates $s_1, \ldots, s_n$ and leaves invariant all the remaining terms.

Let U be a neighbourhood of $\sigma_S(A)$. Let f be a function analytic in U. It is possible to find a compact neigbourhood Δ of $\sigma_S(A)$ such that $\Delta \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Define $f(A) : X \to X$ by

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_{\partial\Delta} Pf(z)Wxs \wedge \mathrm{d}z \qquad (x \in X), \tag{9}$$

where $\mathrm{d}z$ stands for $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and $s = s_1 \wedge \cdots \wedge s_n$. By the Stokes formula,

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_\Delta \bar{\partial}\varphi P f(z) W x s \wedge \mathrm{d}z,$$

where φ is a C^∞-function equal to 0 on a neighbourhood of $\sigma_S(A)$ and to 1 on a neighbourhood of $\mathbb{C}^n \setminus \Delta$.

On $\mathbb{C}^n \setminus \Delta$ one has

$$\bar{\partial}\varphi P f W x s = P f(\bar{\partial} + \delta) W x s = P f x s = 0.$$

Thus it is possible to write

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi P f(z) W x s \wedge \mathrm{d}z. \tag{10}$$

It is clear from the Stokes theorem that the definition of $f(A)x$ does not depend on the choice of the function φ and, by (10), it is independent of Δ.

Moreover, $f(A)$ does not depend on the choice of the mapping W.

Suppose that W_1, W_2 are two operators satisfying

$$(\bar{\partial} + \delta) W_i x s = x s \qquad (i = 1, 2).$$

For those z where $\varphi \equiv 1$ one has

$$(\bar{\partial} + \delta)\varphi f(z)(W_1 - W_2) x s = 0,$$

and so the form $\eta = (\bar{\partial} + \delta)\varphi f(z)(W_1 - W_2) x s$ has a compact support. One has

$$\begin{aligned} &\int_{\mathbb{C}^n} \bar{\partial}\varphi P f(z) W_1 x s \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar{\partial}\varphi P f(z) W_2 x s \wedge \mathrm{d}z \\ &= \int_{\mathbb{C}^n} P \bar{\partial}\varphi f(z)(W_1 - W_2) x s \wedge \mathrm{d}z = \int_{\mathbb{C}^n} P(\bar{\partial} + \delta)\varphi f(z)(W_1 - W_2) x s \wedge \mathrm{d}z \\ &= \int_{\mathbb{C}^n} P\eta \wedge \mathrm{d}z = \int_{\mathbb{C}^n} P(\bar{\partial} + \delta) W_1 \eta \wedge \mathrm{d}z = \int_{\mathbb{C}^n} \bar{\partial} P W_1 \eta \wedge \mathrm{d}z = 0 \end{aligned}$$

by the Stokes theorem.

In fact, in the same way it is possible to show that

$$f(A)x = \frac{-1}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi f P \psi \wedge \mathrm{d}z \tag{11}$$

for any form ψ satisfying $(\bar{\partial} + \delta)\psi = x s$ on $\mathbb{C}^n \setminus \sigma_S(A)$.

It is possible to express the mapping PW that appears in the definition of the functional calculus more explicitly. By the definition of W, one has

$$P W x s = (-1)^{n-1} V(\bar{\partial} V)^{n-1} x s = (-1)^{n-1} V_0 \bar{\partial} V_1 \bar{\partial} \cdots \bar{\partial} V_{n-1} x s.$$

Note that it is possible to write formulas (9) and (10) also globally:

$$\begin{aligned} f(A) &= \tfrac{-1}{(2\pi i)^n} \int\limits_{\partial\Delta} P f(z) W I s \wedge \mathrm{d}z = \tfrac{-1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \bar{\partial}\varphi P f(z) W I s \wedge \mathrm{d}z \\ &= \tfrac{(-1)^n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \bar{\partial}\varphi f V(\bar{\partial} V)^{n-1} I s \wedge \mathrm{d}z, \end{aligned} \tag{12}$$

where $I = I_X$ is the identity operator on X. The coefficients of forms in (12) are $B(X)$-valued C^∞-functions. Therefore $f(A) \in B(X)$.

Proposition 4.5. *For $n = 1$, the functional calculus defined by (12) coincides with the classical functional calculus given by the Cauchy formula.*

Proof. Let $A \in \mathcal{B}(X)$ and let f be a function analytic on a neighbourhood of $\sigma(A)$. Then $Wxs = Vxs = (A - z)^{-1}x$. Thus, for a suitable contour Σ surrounding $\sigma(A)$, one has

$$f(A) = \frac{-1}{2\pi i}\int_\Sigma PfWIs \wedge \mathrm{d}z = \frac{1}{2\pi i}\int_\Sigma f(z)(z - A)^{-1}\mathrm{d}z,$$

which is the Cauchy formula. □

The proof of basic properties of this functional calculus is postponed to the next section where they will be proved more generally, for functions analytic on a neighbourhood of the Taylor spectrum.

It is worth to note that this simpler split functional calculus is sufficient for introducing the functional calculus for n-tuples of elements in commutative Banach algebras. Indeed, let $\mathcal{A}$ be a commutative Banach algebra and $a = (a_1, \dots, a_n) \in \mathcal{A}^n$. Consider the n-tuple $L_a = (L_{a_1}, \dots, L_{a_n}) \in B(\mathcal{A})^n$. Then $\sigma_S(L_a) = \sigma^{\mathcal{A}}(a)$ and for any function f analytic on a neighbourhood of $\sigma^{\mathcal{A}}(a)$ one can define $f(L_a) \in B(\mathcal{A})$. Then the functional calculus for a may be defined by $f(a) = f(L_a)(1_{\mathcal{A}})$. The details are postponed to the next section.

Remark 4.6. If $A = (A_1, \dots, A_n)$ is a commuting tuple of Hilbert space operators then it is possible to choose $V(z) = (\delta_{A-z} + \delta^*_{A-z})^{-1}$ (this mapping does not satisfy that $V(z)\Lambda^p[s, X] \subset \Lambda^{p-1}[s, X]$ but this property is not essential for the construction). Formula (12) is then quite explicit.

The split functional calculus for Hilbert space operators was constructed by (Vasilescu, 1979b). For Banach space operators this was generalized in (Kordula and Müller, 1995).

5. Taylor functional calculus

The most important property of the Taylor spectrum is the existence of the analytic functional calculus. The calculus was constructed in (Taylor, 1970b). For simplified versions of the calculus see (Levi, 1982; Helemskii, 1981; 1989; Albrecht, 1993; Eschmeier and Putinar, 1996) and (Andersson, 1997). The construction below follows (Müller, 2007) which is based on (Vasilescu, 1979b).

Let $A = (A_1, \dots, A_n)$ be an n-tuple of commuting operators on a Banach space X. Let $G = \mathbb{C}^n \setminus \sigma_T(A)$.

The key fact is the following theorem.

Theorem 5.1. *Let $G' \subset G$ be an open subset. Let $\eta \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G', X)]$ satisfy $(\bar\partial + \delta)\eta = 0$. Then there exists $\psi \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G', X)]$ such that $(\bar\partial + \delta)\psi = \eta$.*

Moreover, it is possible to find ψ such that its support is contained in any given neighbourhood of $\operatorname{supp}\eta$.

For each $\lambda \in \mathbb{C}^n \setminus \sigma_T(A)$ one can find first a neighbourhood U of λ and a local solution $\psi_U \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(U, X)]$ satisfying $(\bar{\partial} + \delta)\psi_U = \eta$ on U. The problem is to glue the local solutions and find a global solution on the whole set G'. The details are omitted here; for a proof see (Vasilescu, 1979b, Theorem 8.1) or (Müller, 2007, Theorem 29.9), see also (Frunza, 1975).

Corollary 5.2. *Let $x \in X$. Then there exists $\psi_x \in \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$ such that $(\bar{\partial} + \delta)\psi_x = xs$.*

Let f be a function analytic on a neighbourhood of $\sigma_T(A)$. As in (11), the form ψ_x can be used to define the vector $f(A)x \in X$. However, this definition of $f(A)$ is local, defined for each $x \in X$ separately, and it is not clear at the first glance that $f(A)$ defined in this way is continuous and linear.

For functions f analytic on a neighbourhood of $\sigma_S(A)$ it was possible to find a mapping W acting on $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(\mathbb{C}^n \setminus \sigma_S(A), B(X)]$ such that Wxs served as ψ_x. Thus all the considerations were done in the Banach space $B(X)$.

For functions f analytic on a neighbourhood of $\sigma_T(A)$ this is no longer possible. To simplify the situation, it is possible to consider the Banach space $\mathcal{H}(X)$ of all bounded homogeneous mappings $\varphi : X \to X$, i.e., the mappings satisfying $\varphi(\lambda x) = \lambda x \quad (\lambda \in \mathbb{C}, x \in X)$ and $\|\varphi\| := \sup\{\|\varphi(x)\| : x \in X, \|x\| \le 1\} < \infty$ (no additivity is assumed).

For $i = 1, \dots, n$ let $L'_{A_i} : \mathcal{H}(X) \to \mathcal{H}(X)$ be defined by $L'_{A_i}\varphi = A_i\varphi \quad (\varphi \in \mathcal{H}(X))$. Let $L'_A = (L'_{A_1}, \dots, L'_{A_n})$. Clearly L'_A is a commuting n-tuple of bounded linear operators acting on the Banach space $\mathcal{H}(X)$.

Moreover, it is possible to show that $\sigma_T(L'_A) = \sigma_T(A)$. Thus one has

Corollary 5.3. *There is a form $W_A \in \Lambda^{n-1}[s, \mathrm{d}\bar{z}, C^\infty(G, \mathcal{H}(X))]$ such that $(\bar{\partial} + \delta_{L'_{A-\lambda}})W_A(\lambda) = Is$, where I is the identity operator on X.*

The form W_A can be also considered to be a mapping $W_A : X \to \Lambda^{n-1}[s, \mathrm{d}\bar{z}, C^\infty(G, X)]$. Then $(\bar{\partial} + \delta_{A-\lambda})W_A(\lambda)x = xs$ for all $x \in X$.

The definition of the Taylor functional calculus is analogous to the definition of the split functional calculus.

Recall that P is the projection $P : \Lambda[s, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda[\mathrm{d}\bar{z}, C^\infty(G, X)]$ that annihilates all terms containing at least one of the indeterminates $s_1, \dots, s_n$ and leaves invariant all the remaining terms.

Let U be a neighbourhood of $\sigma_T(A)$ and let f be a function analytic on U. It is possible to find a compact neighbourhood Δ of $\sigma_T(A)$ such that $\Delta \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Define $f(A) : X \to X$ by

$$f(A) = \frac{-1}{(2\pi i)^n} \int_{\partial\Delta} P f W_A \wedge \mathrm{d}z. \tag{13}$$

By the Stokes formula,

$$f(A)=\frac{-1}{(2\pi i)^n}\int_\Delta \bar\partial\varphi P f W_A\wedge \mathrm{d}z,$$

where φ is a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ and to 1 on a neighbourhood of $\mathbb{C}^n\setminus\Delta$.

On $\mathbb{C}^n\setminus\Delta$ one has $\bar\partial\varphi PfW_A=Pf(\bar\partial+\delta)W_A=PfIs=0$. Thus it is possible to write

$$f(A)=\frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n} \bar\partial\varphi P f W_A\wedge \mathrm{d}z. \tag{14}$$

It is clear from the Stokes theorem that the definition of $f(A)$ does not depend on the choice of the function φ and, by (14), it is independent of Δ.

It is possible to show that $f(A)$ does not depend on the choice of the form W_A.

The following simple lemma will be used frequently.

Proposition 5.4. *Let $\eta\in\Lambda[s,d\bar z,C^\infty(G,X)]$ be a differential form with compact support disjoint with $\sigma_T(A)$ such that $(\bar\partial+\delta)\eta=0$. Then*

$$\int_{\mathbb{C}^n}P\eta\wedge\mathrm{d}z=0.$$

Proof. By Theorem 5.1, there exists $\psi\in\Lambda[s,\mathrm{d}\bar z,C^\infty(G,X)]$ with a compact support disjoint with $\sigma_T(A)$ such that $(\delta+\bar\partial)\psi=\eta$. Then

$$P\eta=P(\bar\partial+\delta)\psi=P\bar\partial\psi.$$

By the Stokes theorem,

$$\int_{\mathbb{C}^n}P\eta\wedge\mathrm{d}z=\int_{\mathbb{C}^n}\bar\partial P\psi\wedge\mathrm{d}z=0.$$

□

Let $x\in X$ and let $\psi_1,\psi_2\in\Lambda[s,\mathrm{d}\bar z,C^\infty(G,X)]$ satisfy $(\delta+\bar\partial)\psi_1=xs=(\delta+\bar\partial)\psi_2$. Let φ be a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ and to 1 on a neighbourhood of $\mathbb{C}^n\setminus U$. Then

$$\int\bar\partial\varphi Pf\psi_1\wedge\mathrm{d}z-\int\bar\partial\varphi Pf\psi_2\wedge\mathrm{d}z=\int P(\delta+\bar\partial)\varphi f(\psi_1-\psi_2)\wedge\mathrm{d}z.$$

On $\mathbb{C}^n\setminus\Delta$ one has $\varphi\equiv1$, and so $(\delta+\bar\partial)\varphi f(\psi_1-\psi_2)=f(\delta+\bar\partial)(\psi_1-\psi_2)=0$. Thus the form $(\delta+\bar\partial)\varphi f(\psi_1-\psi_2)$ has a compact support disjoint with $\sigma_T(A)$. By Proposition 5.4, $\int P(\delta+\bar\partial)\varphi f(\psi_1-\psi_2)\wedge\mathrm{d}z=0$.

In particular, the definition of $f(A)$ does not depend on the choice of W_A.

Note that for the definition of $f(A)x$ one can use any form ψ satisfying $(\bar\partial+\delta_{A-z})\psi=xs$ on a neighbourhood of $\operatorname{supp}\varphi$. This implies that for functions analytic on a neighbourhood of $\sigma_S(A)$ the Taylor functional calculus coincides with the split functional calculus introduced in the previous section. By Proposition 4.5, for $n=1$ the Taylor functional calculus coincides with the usual functional calculus for single operators.

Lemma 5.5. $f(A) \in B(X)$.

Proof. Clearly $f(A) \in \mathcal{H}(X)$, so it is sufficient to show the additivity of $f(A)$.

Let $x, y \in X$. Then $(\delta + \bar{\partial})(W_A x + W_A y) = (x+y)s$, and so $f(A)(x+y) = \int_{\mathbb{C}^n} \bar{\partial}\varphi P f(W_A x + W_A y) \wedge \mathrm{d}z = f(A)x + f(A)y$. □

The following result is the first step to show the multiplicativity of the Taylor functional calculus.

Proposition 5.6. *Let f be a function analytic on a neighbourhood of $\sigma_T(A)$, $1 \le j \le n$ and $g(z) = z_j f(z)$. Then $g(A) = A_j f(A)$.*

Proof. The statement is well-known for $n = 1$. Suppose that $n \ge 2$. Then

$$\begin{aligned} -(2\pi i)^n \big(A_j f(A) - g(A)\big) &= A_j \int_{\mathbb{C}^n} \bar{\partial}\varphi P f W_A \wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar{\partial}\varphi P g z_j W_A \wedge \mathrm{d}z \\ &= \int_{\mathbb{C}^n} \bar{\partial}\varphi f \cdot (A_j - z_j) P W_A \wedge \mathrm{d}z. \end{aligned}$$

Express $W_A \in \Lambda^{n-1}\big[s, \mathrm{d}\bar{z}, C^\infty(G, \mathcal{H}(X))\big]$ as

$$W_A = \sum_{F \subset \{1,\dots,n\}} s_F \wedge \xi_F,$$

where ξ_F contains no variable from $s_1, \dots, s_n$. Since $(\bar{\partial} + \delta_{A-z})W_A = Is$, for each $F \ne \{1, \dots, n\}$ one has

$$\bar{\partial}\xi_F + \sum_{k \in F} (-1)^{\operatorname{card}\{k' \in F : k' < k\}} (A_k - z_k)\xi_{F \setminus \{k\}} = 0.$$

In particular, for $F = \{j\}$ one has

$$\bar{\partial}\xi_{\{j\}} = -(A_j - z_j)\xi_\emptyset = -(A_j - z_j) P W_A.$$

Thus

$$\begin{aligned} \int_{\mathbb{C}^n} \bar{\partial}\varphi f \cdot (A_j - z_j) P W_A \wedge \mathrm{d}z &= - \int_{\mathbb{C}^n} \bar{\partial}\varphi f \bar{\partial}\xi_{\{j\}} \wedge \mathrm{d}z \\ &= -\int_{\mathbb{C}^n} \bar{\partial}\big(\varphi \bar{\partial} f \xi_{\{j\}} - \bar{\partial}\varphi f \xi_{\{j\}}\big) \wedge \mathrm{d}z = 0 \end{aligned}$$

by the Stokes theorem. Hence $g(A) = A_j f(A)$. □

Proposition 5.6 implies that the definition of the Taylor functional calculus for polynomials coincides with the usual definition.

Proposition 5.7. *Let $A = (A_1, \dots, A_n) \in \mathcal{B}(X)^n$, $B = (B_1, \dots, B_m) \in \mathcal{B}(X)^m$. Suppose that $(A, B) = (A_1, \dots, A_n, B_1, \dots, B_m)$ is a commuting $(n+m)$-tuple and let f and g be functions analytic on a neighbourhood of $\sigma_T(A)$ and $\sigma_T(B)$, respectively. Let h be defined by $h(z, w) = f(z) \cdot g(w)$. Then $h(A, B) = g(B) f(A)$.*

Proof. Write $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_m)$. Denote by $\bar\partial_z$, $\bar\partial_w$ and $\bar\partial_{z,w}$ the $\bar\partial$ mapping corresponding to z,w and (z,w), respectively. Associate with B another system $t=(t_1,\dots,t_m)$ of exterior indeterminates when defining the operator δ_{B-w}.

Choose forms W_A, W_B and $W_{A,B}$ corresponding to the tuples A,B and (A,B). Let Δ' and Δ'' be compact neighbourhoods of $\sigma_T(A)$ and $\sigma_T(B)$ contained in the domains of definition of f and g, respectively. Let φ,ψ and χ be C^∞-functions equal to 0 on a neighbourhood of $\sigma_T(A)$ $\big(\sigma_T(B),\ \sigma_T(A,B)\big)$ and to 1 on a neighbourhood of $\mathbb{C}^n\setminus\Delta'$ $\big(\mathbb{C}^m\setminus\Delta''$ and $\mathbb{C}^{n+m}\setminus\Delta'\times\Delta''$, respectively$\big)$.

Denote by P_s and P_t the projections which annihilate all terms containing at least one of the variables $s_1,\dots,s_n$ $(t_1,\dots,t_m$, respectively) and leave invariant the remaining terms. Set $P=P_sP_t$.

Let $x\in X$. Then

$$f(A)x=\frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n}\bar\partial_z\varphi P_s fW_Ax\wedge \mathrm{d}z=\frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n}P_s\xi\wedge\mathrm{d}z,$$

where $\xi=(\bar\partial_z+\delta_{A-z})\varphi fW_Ax-fxs$. If $\varphi\equiv 1$, then $\xi\equiv 0$. Thus $\operatorname{supp}\xi$ is compact, $\operatorname{supp}\xi\subset\operatorname{Int}\Delta'$. Further,

$$\text{(15)}\quad g(B)f(A)x=\frac{1}{(2\pi i)^{n+m}}\int_{\mathbb{C}^m}P_t(\bar\partial_w+\delta_{B-w})\psi gW_B\Big(\int_{\mathbb{C}^n}P_s\xi\wedge\mathrm{d}z\Big)\wedge\mathrm{d}w.$$

On the other hand, $-(2\pi i)^{m+n}h(A,B)x=\int P\eta_1\wedge\mathrm{d}z\wedge\mathrm{d}w$, where

$$\eta_1=(\bar\partial_{z,w}+\delta_{A-z,B-w})\chi hW_{A,B}x-hxs\wedge t.$$

Clearly, $\operatorname{supp}\eta_1\subset\Delta'\times\Delta''$.

One has $(\bar\partial_{z,w}+\delta_{A-z,B-w})\xi\wedge t=(\bar\partial_z+\delta_{A-z})\xi\wedge t=0$. By Theorem 5.1, there exists $\alpha\in\Lambda[s,t,\mathrm{d}\bar z,\mathrm{d}\bar w,C^\infty(\mathbb{C}^{n+m}\setminus\sigma_T(A,B),X)]$ such that $(\bar\partial_{z,w}+\delta_{A-z,B-w})\alpha=\xi\wedge t$. Moreover, one can assume that $\operatorname{supp}\alpha\subset\Delta'\times\mathbb{C}^m$. Let

$$\eta_2=(\bar\partial_{z,w}+\delta_{A-z,B-w})\psi g\alpha-g\xi\wedge t.$$

Then $(\bar\partial_{z,w}+\delta_{A-z,B-w})(\eta_1-\eta_2)=0$. Clearly, $\operatorname{supp}\eta_2\subset\Delta'\times\mathbb{C}^m$. Moreover, if $\psi\equiv 1$, then $\eta_2\equiv 0$, and so $\operatorname{supp}\eta_2$ is compact. On a neighbourhood of $\sigma_T(A,B)$ one has $\eta_2=-g\xi\wedge t=fgxs\wedge t=-\eta_1$. By Proposition 5.4, $\int P(\eta_1+\eta_2)\wedge\mathrm{d}z\wedge\mathrm{d}w=0$, and so

$$\begin{aligned}(2\pi i)^{m+n}h(A,B)x&=\int_{\mathbb{C}^{n+m}}P\eta_2\wedge\mathrm{d}z\wedge\mathrm{d}w\\ &=(-1)^{mn}\int_{\mathbb{C}^m}\Big(\int_{\mathbb{C}^n}P_t(\bar\partial_{z,w}+\delta_{B-w})\psi gP_s\alpha\wedge\mathrm{d}z\Big)\wedge\mathrm{d}w\end{aligned}$$

by the Fubini theorem (the factor $(-1)^{mn}$ is caused by convention (8) defining the Lebesgue measures in $\mathbb{C}^n$, $\mathbb{C}^m$ and $\mathbb{C}^{m+n}$, respectively). By the Stokes theorem, one has

$$(2\pi i)^{m+n}h(A,B)x=(-1)^{mn}\int_{\mathbb{C}^m}P_t(\bar\partial_w+\delta_{B-w})g\Big(\int_{\mathbb{C}^n}\psi P_s\alpha\wedge\mathrm{d}z\Big)\wedge\mathrm{d}w.$$

Consider the form

$$\eta_3 = (-1)^{mn}(\bar\partial_w + \delta_{B-w})g\int_{\mathbb{C}^n}\psi P_s\alpha\wedge \mathrm{d}z - (\bar\partial_w + \delta_{B-w})\psi g W_B\int_{\mathbb{C}^n} P_s\xi\wedge \mathrm{d}z.$$

Clearly, $(\bar\partial_w + \delta_{B-w})\eta_3 = 0$. If $\psi \equiv 1$, then, by the Stokes theorem,

$$\begin{aligned}\eta_3 &= (-1)^{mn}g\int_{\mathbb{C}^n} P_s(\bar\partial_{z,w} + \delta_{A-z,B-w})\alpha\wedge \mathrm{d}z - (-1)^{mn}g\int_{\mathbb{C}^n}\bar\partial_z P_s\alpha\wedge\mathrm{d}z \\ &-g\int_{\mathbb{C}^n} P_s\xi\wedge\mathrm{d}z\wedge t = (-1)^{mn}g\int_{\mathbb{C}^n} P_s\xi\wedge t\wedge\mathrm{d}z - g\int_{\mathbb{C}^n} P_s\xi\wedge\mathrm{d}z\wedge t = 0.\end{aligned}$$

Thus $\operatorname{supp}\eta_3$ is compact and disjoint with $\sigma_T(B)$. Hence $\int P_t\eta_3\wedge\mathrm{d}w = 0$ and

$$\begin{aligned}(2\pi i)^{n+m}h(A,B)x &= \int_{\mathbb{C}^m} P_t(\bar\partial_w + \delta_{B-w})\psi g W_B\int_{\mathbb{C}^n} P_s\xi\wedge\mathrm{d}z\wedge\mathrm{d}w \\ &= (2\pi i)^{m+n}g(B)f(A)x\end{aligned}$$

by (15). Hence $h(A,B) = g(B)f(A)$. □

The following simple lemma will be used later:

Lemma 5.8. *Let K be a compact subset of $\mathbb{C}^n$ and let f be a function analytic on an open neighbourhood of K. Then there are functions h_j $(j = 1,\dots,n)$ analytic on a neighbourhood of the set $D = \{(z,z) : z\in K\}$ such that*

$$f(z) - f(w) = \sum_{j=1}^n (z_j - w_j)\cdot h_j(z,w).$$

Proof. For $j = 1,\dots,n$ define g_j by

$$\begin{aligned}&g_j(z_1,\dots,z_n,w_1,\dots,w_n)\\ = \;&f(z_1,\dots,z_j,w_{j+1},\dots,w_n) - f(z_1,\dots,z_{j-1},w_j,\dots,w_n).\end{aligned}$$

It is easy to see that g_j is analytic on a neighbourhood of D.

Let $h_j(z,w) = \frac{g_j(z,w)}{z_j - w_j}$. Clearly, h_j is analytic at each point (z,w) with $z_j \neq w_j$. By the Weierstrass division theorem, see (Gunning and Rossi, 1965, p. 70), h_j can be defined and is analytic also on a neighbourhood of each point (z,w) with $z_j = w_j$. Thus h_j is analytic on a neighbourhood of D. Hence

$$\sum_{j=1}^n (z_j - w_j)\cdot h_j(z,w) = \sum_{j=1}^n g_j(z,w) = f(z) - f(w).$$

□

Denote by H_K the algebra of all functions analytic on a neighbourhood of a compact set $K\subset\mathbb{C}^n$ (more precisely, the algebra of all germs of functions analytic on a neighbourhood of K).

Theorem 5.9. *Let $A = (A_1,\dots,A_n)$ be an n-tuple of mutually commuting operators on X. Then:*

(i) *the mapping* $f \mapsto f(A)$ *is linear and multiplicative, i.e., the Taylor functional calculus is a homomorphism from* $H_{\sigma_T(A)}$ *to* $\mathcal{B}(X)$*;*

(ii) *if* p *is a polynomial,* $p(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$*, then* $p(A) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha A^\alpha$*;*

(iii) *if* $f_n \to f$ *uniformly on a compact neighbourhood of* $\sigma_T(A)$*, then* $f_n(A) \to f(A)$ *in the norm topology;*

(iv) $f(A) \in (A)''$ *for each* $f \in H_{\sigma_T(A)}$*, where* $(A)''$ *denotes the bicommutant of the set* $\{A_1, \dots, A_n\}$*.*

Proof. (i) The linearity of the mapping $f \mapsto f(A)$ is clear. Let f and g be functions analytic on a neighbourhood of $\sigma_T(A)$. Consider the $(2n)$-tuple (A, A). It is easy to see that $\sigma_T(A, A) = \{(z, z) : z \in \sigma_T(A)\}$. Define functions $h_1(z, w) = f(z)g(w)$ and $h_2(z, w) = f(z)g(z)$. By Lemma 5.8, it is possible to write $g(z) - g(w) = \sum_{i=1}^n (z_i - w_i) q_i(z, w)$ for some functions $q_1, \dots, q_n$ analytic on a neighbourhood of $\sigma_T(A, A)$. By Proposition 5.7, one has $h_1(A, A) = f(A)g(A)$ and $h_2(A, A) = (fg)(A)$. Thus, by Proposition 5.6,

$$(fg)(A) - f(A)g(A) = h_2(A, A) - h_1(A, A) = \sum_{i=1}^n (A_i - A_i)(fq_i)(A, A) = 0.$$

Hence $(fg)(A) = f(A)g(A)$.

(ii) The statement follows from Proposition 5.6.

(iii) Follows from the definition.

(iv) Let $S \in B(X)$ be an operator commuting with $A_1, \dots, A_n$. By Proposition 5.7, it is possible to consider $f(A)$ to be a function of the $(n+1)$-tuple $(A_1, \dots, A_n, S)$. Therefore $f(A)$ commutes with its argument S. Hence $f(A) \in (A)''$. $\square$

It follows from the general theory that the Taylor spectrum satisfies the spectral mapping property for all polynomials (and consequently, for all functions that can be approximated by polynomials uniformly on a neighbourhood of the Taylor spectrum). In fact, the spectral mapping property is true for all analytic functions.

The next lemma shows that each operator A_j behaves as the zero on the quotient $\ker \delta_A / \operatorname{Im} \delta_a$.

Lemma 5.10. *Let* $A = (A_1, \dots, A_n)$ *be a commuting* n*-tuple of operators acting on a Banach space* X*. Let* $j \in \{1, \dots, n\}$*. Then* $A_j \ker \delta_A \subset \operatorname{Im} \delta_A$*.*

Proof. Let $\psi \in \ker \delta_A$. Write $\psi = s_j \wedge \psi_1 + \psi_2$, where ψ_2 does not contain s_j. Then

$$0 = \delta_A \psi = s_j \wedge A_j \psi_2 + \sum_{i \neq j} s_i \wedge s_j \wedge A_i \psi_1 + \sum_{i \neq j} s_i \wedge A_i \psi_2.$$

In particular, $A_j \psi_2 - \sum_{i \neq j} s_i \wedge A_i \psi_1 = 0$. Thus

$$\delta_A \psi_1 = s_j A_j \psi_1 + \sum_{i \neq j} s_i \wedge A_i \psi_1 = s_j \wedge A_j \psi_1 + A_j \psi_2 = A_j \psi.$$

□

It is natural to expect that $f(A)$ behaves as $f(0)$ on the quotient space $\ker \delta_A / \operatorname{Im} \delta_A$. However, there is a technical difficulty because in general $\operatorname{Im} \delta_A$ is not closed, and so the quotient $\ker \delta_A / \operatorname{Im} \delta_A$ is not a Banach space. Therefore the proof is a little bit more complicated.

Lemma 5.11. *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of operators on X, let $c = (c_1, \dots, c_n) \in \sigma_T(A)$ and let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Consider exterior indeterminates $t = (t_1, \dots, t_n)$ and the operator $\delta_{A-c,t} : \Lambda[t, X] \to \Lambda[t, X]$ defined by $\delta_{A-c,t}\psi = \sum_{j=1}^n (A_j - c_j) t_j \wedge \psi$ for all $\psi \in \Lambda[t, X]$. Let $\eta \in \ker \delta_{A-c,t}$. Then $(f(A) - f(c))\eta \in \delta_{A-c,t}\Lambda[t, X]$.*

Proof. To define $f(A)$, consider exterior indeterminates $s = (s_1, \dots, s_n)$, the mapping δ_{A-z} acting on $\Lambda[s, \mathrm{d}\bar{z}, C^\infty(\mathbb{C}^n \setminus \sigma_T(A), X)]$ defined by the formula $\delta_{A-z}\psi = \sum_{j=1}^n (A_j - z_j) s_j \wedge \psi$, and the mapping W_A corresponding to A. Note that δ_{A-z} and W_A are connected with variables s; the mapping $\delta_{A-c,t}$ is related to variables t.

Without loss of generality one can assume that η is homogeneous of degree p, $0 \le p \le n$.

Since $\eta \in \Lambda^p[t, X]$ and $\Lambda[t, X]$ is a direct sum of $\binom{n}{p}$ copies of X, it is possible to define the form $\xi_0 := W_A \eta \in \Lambda[s, t, \mathrm{d}\bar{z}, C^\infty(G, X)]$ coordinatewise. Then $(\bar\partial + \delta_{A-z})\xi_0 = s \wedge \eta$ and $(\bar\partial + \delta_{A-z})\delta_{A-c,t}\xi_0 = -\delta_{A-c,t}(\bar\partial + \delta_{A-z})\xi_0 = 0$. Thus there exists $\xi_1 \in \Lambda[s, t, \mathrm{d}\bar{z}, C^\infty(G, X)]$ such that $(\bar\partial + \delta_{A-z})\xi_1 = \delta_{A-c,t}\xi_0$.

Similarly one can construct forms $\xi_1, \dots, \xi_{n-p} \in \Lambda[s, t, \mathrm{d}\bar{z}, C^\infty(G, X)]$ such that $(\bar\partial + \delta_{A-z})\xi_{k+1} = \delta_{A-c,t}\xi_k$. Clearly the degree of ξ_k in t is $p + k$.

Set $\xi = \sum_{k=0}^{n-p} (-1)^k \xi_k$ Then

$$(\bar\partial + \delta_{A-z} + \delta_{A-c,t})\xi = \sum_{k=0}^{n-p} (-1)^k (\bar\partial + \delta_{A-z})\xi_k + \sum_{k=0}^{n-p} (-1)^k \delta_{A-c,t}\xi_k = s \wedge \eta,$$

since $\delta_{A-c,t}\xi_{n-p} = 0$.

Let Δ be a compact neighbourhood of $\sigma_T(A)$ contained in the domain of definition of f. Let φ be a C^∞-function equal to 0 on a neighbourhood of $\sigma_T(A)$ and to 1 on a neighbourhood of $\mathbb{C}^n \setminus \Delta$. Let P_s be the projection annihilating all terms that contain at least one of the variables $s_1, \dots, s_n$ and leaving invariant all other terms.

Consider the integral

$$\int (\bar\partial + \delta_{A-c,t}) P_s \varphi \xi \wedge \mathrm{d}z = \int (\bar\partial + \delta_{A-c,t}) P_s \varphi \sum_{k=0}^{n-p} (-1)^k \xi_k \wedge \mathrm{d}z.$$

Since ξ_k has degree $p+k$ in t and $n-k-1$ in $(s, \mathrm{d}\bar z)$, the only relevant term in the integral above is ξ_0. Thus

$$\begin{aligned} & \int(\bar\partial+\delta_{A-c,t})P_s\varphi\xi\wedge\mathrm{d}z = \int(\bar\partial+\delta_{A-c,t})P_s\varphi\xi_0\wedge\mathrm{d}z \\ = & \int\bar\partial P_s\varphi W_A\eta\wedge\mathrm{d}z = -(2\pi i)^n f(A)\eta. \end{aligned}$$

Consider now the n-tuple $B=(c_1I,\dots,c_nI)\in B(X)^n$. Since f can be approximated by polynomials uniformly on a neighbourhood of c, one has $f(B)=f(c)\cdot I$.

As above, consider the mappings δ_{B-z} and W_B connected with variables s. Let $\xi'_0 = W_B\eta$ and inductively define $\xi'_k\in\Lambda[s,t,\mathrm{d}\bar z, C^\infty(G,X)]$ satisfying $(\bar\partial+\delta_{B-z})\xi'_{k+1}=\delta_{A-c,t}\xi'_k$.

Let $\xi'=\sum_{k=0}^{n-p}(-1)^k\xi'_k$. As above, one has $(\bar\partial+\delta_{B-z}+\delta_{A-c,t})\xi' = s\wedge\eta$ and

$$\begin{aligned} & \int(\bar\partial+\delta_{A-c,t})P_s\varphi\xi'\wedge\mathrm{d}z = \int(\bar\partial+\delta_{A-c,t})P_s\varphi W_B\eta\wedge\mathrm{d}z \\ = & \int\bar\partial P_s\varphi W_B\eta\wedge\mathrm{d}z = -(2\pi i)^n f(B)\eta = -(2\pi i)^n f(c)\eta. \end{aligned}$$

To show that $(f(A)-f(c))\eta\in\delta_{A-c,t}\Lambda[t,X]$, consider the linear mapping U acting on $\Lambda[s,t,\mathrm{d}\bar z, C^\infty(\mathbb{C}^n\setminus\sigma_T(A),X)]$ defined by

$$U\Big(t_{i_1}\wedge\dots\wedge t_{i_m}\wedge\psi\Big) = (t_{i_1}-s_{i_1})\wedge\dots\wedge(t_{i_m}-s_{i_m})\wedge\psi$$

for all $i_1,\dots,i_m$ and $\psi\in\Lambda[s,\mathrm{d}\bar z, C^\infty(\mathbb{C}^n\setminus\sigma_T(A),X)]$. Then $P_sU=P_s$ and, for each $\psi\in\Lambda[s,t,\mathrm{d}\bar z, C^\infty(\mathbb{C}^n\setminus\sigma_T(A),X)]$,

$$\begin{aligned} & U(\bar\partial+\delta_{A-z}+\delta_{A-c,t})\psi \\ = & \bar\partial U\psi + \sum(A_j-z_j)s_j\wedge U\psi + \sum(A_j-c_j)(t_j-s_j)\wedge U\psi \\ = & (\bar\partial+\delta_{B-z}+\delta_{A-c,t})U\psi. \end{aligned}$$

One has

$$\begin{aligned} & -(2\pi i)^n f(A)\eta = \int(\bar\partial+\delta_{A-c,t})P_s\varphi\xi\wedge\mathrm{d}z = \int P_s(\bar\partial+\delta_{A-z}+\delta_{A-c,t})\varphi\xi\wedge\mathrm{d}z \\ & = \int P_sU(\bar\partial+\delta_{A-z}+\delta_{A-c,t})\varphi\xi\wedge\mathrm{d}z = \int P_s(\bar\partial+\delta_{B-z}+\delta_{A-c,t})\varphi U\xi\wedge\mathrm{d}z. \end{aligned}$$

Thus

$$-(2\pi i)^n\big(f(A)-f(c)\big)\eta = \int P_s(\bar\partial+\delta_{B-z}+\delta_{A-c,t})\varphi(U\xi-\xi')\wedge\mathrm{d}z = \int P_s\theta\wedge\mathrm{d}z,$$

where $\theta=(\bar\partial+\delta_{B-z}+\delta_{A-c,t})\varphi(U\xi-\xi')$. If $\varphi\equiv 1$, then θ is equal to

$$(\bar\partial+\delta_{B-z}+\delta_{A-c,t})U\xi - s\wedge\eta = U(\bar\partial+\delta_{A-z}+\delta_{A-c,t})\xi - s\wedge\eta = U(s\wedge\eta)-s\wedge\eta = 0;$$

so $\operatorname{supp}\theta\subset\operatorname{Int}\Delta$. Furthermore, θ can be written as $\theta=(\bar\partial+\delta_{B-z}+\delta_{A-c,t})\psi$ for some form $\psi\in\Lambda[s,t,\mathrm{d}\bar z, C^\infty(\mathbb{C}^n,X)]$ with compact support. Indeed, by Theorem 5.1, there exists a form $\vartheta\in\Lambda[s,t,\mathrm{d}\bar z,\mathrm{d}\bar w, C^\infty(\mathbb{C}^{2n},X)]$ with $\operatorname{supp}\vartheta\subset\Delta\times\mathbb{C}^n$ such that $(\bar\partial_{z,w}+\delta_{B-z}+\delta_{A-c,t})\vartheta=\theta$.

Set $\psi(z) = \vartheta_0(z,c)$, where ϑ_0 is the part of ϑ containing none of the variables $\mathrm{d}\bar{w}_j$. Then $\operatorname{supp}\psi \subset \Delta$ and $(\bar{\partial}_z + \delta_{B-z} + \delta_{A-c,t})\psi = \theta$. By the Stokes theorem,

$$\int P_s\theta \wedge \mathrm{d}z = \int P_s(\bar{\partial}_z + \delta_{B-z} + \delta_{A-c,t})\psi \wedge \mathrm{d}z$$

$$= \int \bar{\partial}_z P_s\psi \wedge \mathrm{d}z + \int P_s\delta_{A-c,t}\psi \wedge \mathrm{d}z = \delta_{A-c,t}\int P_s\psi \wedge \mathrm{d}z \in \delta_{A-c,t}\Lambda[t,X].$$

□

Proposition 5.12. *Let $A = (A_1,\dots,A_n)$ be a commuting n-tuple of operators on X, $c = (c_1,\dots,c_n) \in \sigma_T(A)$ and let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Then the $(n+1)$-tuple $\big(A_1 - c_1,\dots,A_n - c_n, f(A)\big)$ is Taylor regular if and only if $f(c) \neq 0$.*

Proof. Relate exterior variables $s_1,\dots,s_{n+1}$ to the $(n+1)$-tuple $(A-c,f(A))$. Write for short $s = (s_1,\dots,s_n)$. Let $\delta_{A-c} : \Lambda[s,X] \to \Lambda[s,X]$ be defined by $\delta_{A-c}\psi = \sum(A_j - c_j)s_j \wedge \psi \quad (\psi \in \Lambda[s,X])$. One has $\Lambda[s,s_{n+1},X] = \Lambda[s,X] \oplus s_{n+1} \wedge \Lambda[s,X]$. The operator $\delta_{A-c,f(A)}$ corresponding to the $(n+1)$-tuple $(A-c,f(A))$ can be written in this decomposition in the matrix form

$$\delta_{A-c,f(A)} = \begin{pmatrix} \delta_{A-c} & 0 \\ f(A) & -\delta_{A-c} \end{pmatrix}.$$

Consider the following two cases:

(a) $f(c) = 0$.

Since $c \in \sigma_T(A)$, there is a $\psi \in \Lambda[s,X]$ such that $\delta_{A-c}\psi = 0$ and $\psi \notin \delta_{A-c}\Lambda[s,X]$. By the preceding lemma, there is an $\eta \in \Lambda[s,X]$ such that $f(A)\psi = \delta_{A-c}\eta$. Then $\delta_{A-c,f(A)}(\psi + s_{n+1} \wedge \eta) = 0$ and $(\psi + s_{n+1} \wedge \eta) \notin \delta_{A-c,f(A)}\Lambda[s,s_{n+1},X]$ since $\psi \notin \delta_{A-c}\Lambda[s,X]$.

Thus the $(n+1)$-tuple $(A-c,f(A))$ is Taylor singular.

(b) $f(c) \neq 0$. Without loss of generality one can assume that $f(c) = 1$.

Let $\psi,\xi \in \Lambda[s,X]$, $\delta_{A-c,f(A)}(\psi + s_{n+1} \wedge \xi) = 0$. Then $\delta_{A-c}\psi = 0$ and $f(A)\psi - \delta_{A-c}\xi = 0$. By the preceding lemma, $f(A)\psi - \psi \in \delta_{A-c}\Lambda[s,X]$. Since $f(A)\psi \in \delta_{A-c}\Lambda[s,X]$, one has $\psi = \delta_{A-c}\eta$ for some $\eta \in \Lambda[s,X]$.

Further, $\delta_{A-c}(f(A)\eta - \xi) = f(A)\psi - \delta_{A-c}\xi = 0$. Thus there is a $\theta \in \Lambda[s,X]$ with $f(A)(f(A)\eta - \xi) - (f(A)\eta - \xi) = \delta_{A-c}\theta$. Set $\eta' = \eta - (f(A)\eta - \xi)$. Then $\delta_{A-c}\eta' = \delta_{A-c}\eta = \psi$ and $f(A)\eta' - \delta_{A-c}\theta = f(A)\eta - f(A)(f(A)\eta - \xi) + \delta_{A-c}\theta = f(A)\eta - (f(A)\eta - \xi) = \xi$. Hence $\delta_{A-c,f(A)}(\eta' - s_{n+1} \wedge \theta) = (\psi + s_{n+1} \wedge \xi)$ and the $(n+1)$-tuple $(A-c,f(A))$ is Taylor regular. □

Lemma 5.13. *Let $A = (A_1,\dots,A_n)$ be a commuting n-tuple of operators on X, let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Denote by $\mathcal{A}$ the commutative Banach algebra generated by $A_1,\dots,A_n$ and $f(A)$. Let φ be a multiplicative functional on $\mathcal{A}$ such that $\varphi(B) \in \sigma_T(B)$ for all tuples $B = (B_1,\dots,B_m)$ of operators in $\mathcal{A}$. Then $\varphi(f(A)) = f(\varphi(A))$.*

Proof. Consider the $(n+1)$-tuple $\big(A_1-\varphi(A_1),\dots,A_n-\varphi(A_n),f(A)-\varphi(f(A))\big)$. By assumption, this $(n+1)$-tuple is Taylor singular. By the previous proposition, one has $f(\varphi(A))-\varphi(f(A))=0$. $\square$

Corollary 5.14. *(spectral mapping property) Let $\tilde\sigma$ be a spectral system on $B(X)$ which is contained in the Taylor spectrum. Let $A=(A_1,\dots,A_n)$ be a commuting n-tuple of operators on X and let $f=(f_1,\dots,f_m)$ be an m-tuple of functions analytic on a neighbourhood of $\sigma_T(A)$. Then $\tilde\sigma(f(A))=f(\tilde\sigma(A))$.*

In particular, $\sigma_T(f(A))=f(\sigma_T(A))$. Similarly, $\sigma_{\pi k}(f(A))=f(\sigma_{\pi k}(A))$ and $\sigma_{\delta k}(f(A))=f(\sigma_{\delta k}(A))$ for all $k=0,\dots,n$.

Proof. Consider the commutative Banach algebra $\mathcal{A}$ generated by $A_1,\dots,A_n$ and $f_1(A),\dots,f_m(A)$. Since the restriction of $\tilde\sigma$ to $\mathcal{A}$ is again a spectral system, there is a compact subset $K\subset\mathcal{M}(\mathcal{A})$ such that $\tilde\sigma(B)=\{\varphi(B):\varphi\in K\}$ for each tuple $B=(B_1,\dots,B_k)\subset\mathcal{A}$, see (Żelazko, 1979) or (Müller, 2007, Theorem 7.12).

Then

$$
\begin{aligned}
&\tilde\sigma(f(A))=\{(\varphi(f_1(A),\dots\varphi(f_m(A))):\varphi\in K\}\\
=\;&\{(f_1(\varphi(A)),\dots,f_m(\varphi(A))):\varphi\in K\}=\{f(c):c\in\tilde\sigma(A)\}=f(\tilde\sigma(A)).
\end{aligned}
$$

$\square$

Corollary 5.15. *Let $A=(A_1,\dots,A_n)$ be a commuting n-tuple of operators on X. Suppose that $\sigma_T(A)\subset U_1\cup U_2$, where U_1,U_2 are open disjoint sets. Then there exists closed subspaces $X_1,X_2\subset X$ invariant for $A_1,\dots,A_n$ such that $X=X_1\oplus X_2$ and $\sigma_T(A_1|X_j,\dots,A_n|X_j)\subset U_j$ for $j=1,2$.*

Proof. Consider the function $f\equiv 1$ on U_1 and $f\equiv 0$ on U_2. It is easy to see that $f(A)$ is a projection, Set $X_1=f(A)X$ and $X_2=(1-f)(A)X$. $\square$

The following theorem was proved by (Putinar, 1982).

Theorem 5.16. *(superposition principle) Let $A=(A_1,\dots,A_n)$ be a commuting n-tuple of operators on X, let $f=(f_1,\dots,f_m)$ be an m-tuple of function analytic on a neighbourhood of $\sigma_T(A)$, let $B=f(A)$, let g be a function analytic on a neighbourhood of $\sigma_T(B)$ and let $h(z)=g(f_1(z),\dots,f_m(z))$. Then $h(A)=g(B)$.*

Proof. By Lemma 5.8, $g(v)-g(w)=\sum_{j=1}^m(v_j-w_j)r_j(v,w)$ for some functions $r_1,\dots,r_m$ analytic on a neighbourhood of the set $\big\{(v,v):v\in\sigma_T(B)\big\}$. So $g(f(z))-g(w)=\sum_{j=1}^m(f_j(z)-w_j)r_j'(z,w)$, where $r_j'(z,w)=r_j(f(z),w))$ are functions analytic on certain neighbourhood of the set $\sigma_T(A,f(A))=\big\{(z,f(z)):z\in\sigma_T(A)\big\}$. Thus $h(A)-g(B)=\sum_{j=1}^m(f_j(A)-B_j)r_j'(A,B)=0$. Hence $h(A)=g(B)$. $\square$

As a corollary of the Taylor functional calculus it is possible to obtain the properties of the functional calculus in commutative Banach algebras.

Theorem 5.17. *Let $\mathcal{A}$ be a commutative Banach algebra. To each finite family $a = (a_1, \dots, a_n)$ of elements of $\mathcal{A}$ and each function $f \in H_{\sigma(a)}$ it is possible to assign an element $f(a) \in \mathcal{A}$ such that the following conditions are satisfied:*

(i) *if $f(z_1, \dots, z_n) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ is a polynomial in n indeterminates, then $f(a_1, \dots, a_n) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n}$;*

(ii) *the mapping $f \mapsto f(a_1, \dots, a_n)$ is an algebra homomorphism from the algebra $H_{\sigma(a_1, \dots, a_n)}$ to $\mathcal{A}$;*

(iii) *if U is a neighbourhood of $\sigma(x_1, \dots, x_n)$, f, f_k $(k \in \mathbb{N})$ are analytic in U and f_k converge to f uniformly on U, then*

$$f_k(a_1, \dots, a_n) \to f(a_1, \dots, a_n);$$

(iv) *if $\varphi \in \mathcal{M}(\mathcal{A})$ and $f \in H_{\sigma(a_1, \dots, a_n)}$, then*

$$\varphi\big(f(a_1, \dots, a_n)\big) = f\big(\varphi(a_1), \dots, \varphi(a_n)\big);$$

(v) *$\tilde{\sigma}\big(f(a_1, \dots, a_n)\big) = f\big(\tilde{\sigma}(a_1, \dots, a_n)\big)$ for each compact-valued spectral system in $\mathcal{A}$;*

(vi) *if $a_1, \dots, a_m \in \mathcal{A}$, $n < m$, $f \in H_{\sigma(a_1, \dots, a_n)}$ and $\tilde{f} \in H_{\sigma(a_1, \dots, a_m)}$ satisfy $\tilde{f}(z_1, \dots, z_m) = f(z_1, \dots, z_n)$ for all $z_1, \dots, z_m$ in a neighbourhood of $\sigma(a_1, \dots, a_m)$, then*

$$\tilde{f}(a_1, \dots, a_m) = f(a_1, \dots, a_n);$$

(vii) *if $f_1, \dots, f_m \in H_{\sigma(a)}$, $b_i = f_i(a)$, $g \in H_{\sigma(b_1, \dots, b_m)}$ and $h \in H_{\sigma(a)}$ is defined by $h(z) = g(f_1(z), \dots, f_m(z)$, then $h(a) = g(b)$.*

Proof. For an n-tuple $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ consider the left multiplication operators $L_{a_i} \in B(\mathcal{A})$ defined by $L_{a_i} x = a_i x \quad (x \in \mathcal{A}, i = 1, \dots, n)$. Then $L_a = (L_{a_1}, \dots, L_{a_n})$ is a commuting n-tuple of operators. It is easy to show that $\sigma(a) = \sigma_T(L_a)$.

For a function f analytic on a neighbourhood of $\sigma(a)$ set $f(a) = f(L_a)1_{\mathcal{A}}$. Since $f(L_a) \in (L_a)''$, for each $b \in \mathcal{A}$ one has $f(L_a)(b) = f(L_a)L_b(1_{\mathcal{A}}) = L_b f(L_a)(1_{\mathcal{A}}) = b \cdot f(a) = L_{f(a)}(b)$. Thus $f(L_a) = L_{f(a)}$.

Properties (i), (ii), (iii), (vi) and (vii) follow from the corresponding properties of the Taylor functional calculus; the multiplicativity follows from the observation that

$$(fg)(a) = (fg)(L_a)(1_{\mathcal{A}}) = f(L_a)g(L_a)(1_{\mathcal{A}}) = L_{f(a)}g(a) = f(a)g(a).$$

Property (iv) follows from Lemma 5.13; this implies also (v). □

It is possible to show that properties (i), (ii), (iii) and (vi) determine the functional calculus uniquely, see (Zame, 1979). For the unicity of the Taylor functional calculus see (Putinar, 1983).

6. Concluding remarks

The Taylor spectrum and the corresponding functional calculus are defined for n-tuples of commuting Banach space operators. It is a natural question whether it is possible to define something similar for commuting n-tuples of elements of a Banach algebra.

Let $\mathcal{A}$ be a unital Banach algebra and $a_1, \dots, a_n \in \mathcal{A}$ mutually commuting elements. The first idea is of course to define the Taylor spectrum of $(a_1, \dots, a_n)$ as the Taylor spectrum of the n-tuple $(L_{a_1}, \dots, L_{a_n}) \in B(\mathcal{A})^n$. However, if $\mathcal{A} = B(X)$ for some Banach space X and $A_1, \dots, A_n \in B(X)$ commuting operators, then $\sigma_T(L_{A_1}, \dots, L_{A_n})$ is equal to the split spectrum $\sigma_S(A_1, \dots, A_n)$, and not to the Taylor spectrum $\sigma_T(A_1, \dots, A_n)$. So this natural definition of the Taylor spectrum in Banach algebras is not proper.

Problem 6.1. *Does there exist a reasonable definition of the Taylor spectrum and corresponding functional calculus for commuting n-tuples of Banach algebra elements?*

Further problems concern relations between the Taylor functional calculus and other types of spectra.

It was mentioned above that the split spectrum is in general bigger than the Taylor spectrum. However, it is not clear whether the Taylor functional calculus is really richer than the split functional calculus.

Problem 6.2. Let $A = (A_1, \dots, A_n) \in B(X)^n$ be a commuting n-tuple of operators. Let f be a function analytic on a neighbourhood of $\sigma_T(A)$. Is it possible to extend f analytically to a neighbourhood of $\sigma_S(A)$?

Note that for each polynomial p one has

$$p(\sigma_S(A)) = \sigma_s(p(A)) = \sigma(p(A)) = \sigma_T(p(A)) = p(\sigma_T(A)).$$

So $\sigma_T(A)$ can be smaller than $\sigma_S(A)$ but not "much smaller".

The following problem is similar. Note that the Taylor spectrum is not the smallest set for which there exists an analytic functional calculus. For example, for $n = 2$, one has $\sigma_{ap}(A) \cup \sigma_{sur}(A) \supset \partial\sigma_T(A)$. So any function analytic on a connected neighbourhood of $\sigma_{ap}(A) \cup \sigma_{sur}(A)$ can be extended analytically to a neighbourhood of $\sigma_T(A)$.

Problem 6.3. Is it possible to find a reasonable subset of the Taylor spectrum such that each function analytic on its neighbourhood can be extended analytically to a neighbourhood of the Taylor spectrum?

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