Energy dissipation and well-posedness for some problems in fluid dynamics

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Abstract conservation laws

System of equations (conservation laws)

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F}(\mathbf{U}) = 0,$$

$\mathbf{U}$ .... state variable

$\mathbb{F}$ .... flux

"Entropies"

$$\partial_t E_i(\mathbf{U}) + \mathrm{div}_x \mathbf{F}_{E_i}(\mathbf{U}) = \boxed{(\leq)}\, 0, \ \ i = 1, 2, \dots$$

E_i .... entropy

$\mathbf{F}_i$ .... entropy flux

A priori bounds

$$\int E_i(\mathbf{U}) \ \mathrm{d}x \text{ bounded in terms of the initial data}, \ \ i = 1, 2, \dots$$

Weak vs. strong solutions

Lack of regularity

- bounds available only in L^p (L^∞)
- presence of oscillations
- discontinuities (shocks) appearing in finite time even for initial states

Weak solutions

$$\int_\Omega \mathbf{U} \cdot \varphi(\tau_2, \cdot) - \mathbf{U}\varphi(\tau_1, \cdot) \ \mathrm{d}x$$

$$= \int_{\tau_1}^{\tau_2} \int_\Omega \left[\mathbf{U} \cdot \partial_t \varphi + \mathbb{F}(\mathbf{U}) : \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

Weak continuity

$$t \mapsto \mathbf{U}(t, \cdot) \text{ weakly continuous}$$

Compensated compactness - DiPerna, Tartar

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

$$\partial_t E_i + \mathrm{div}_x \mathbf{F}_i \leq 0, \ i = 1, 2, \dots$$

Nonlinear constitutive equations

$$\mathbb{F} = \mathbb{F}(\mathbf{U}), \ E_i = E_i(\mathbf{U}), \ \mathbf{F}_i = \mathbf{F}_i(\mathbf{U}), \ i = 1, 2, \dots$$

Compensated compactness

- linear field equations yield constraints on possible oscillations described by Young measure
- nonlinear constrained imposed by constitutive equations reduce the Young measures to Dirac masses (no oscillations)

Convex integration

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

Replacing constitutive equation

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \ \text{“implicit”}$$

$$\Lambda(\mathbf{U}, \mathbb{F}) \ \text{convex}, \ \Lambda(\mathbf{U}, \mathbb{F}) \geq E(\mathbf{U})$$

Relaxation of constitutive equation

$$E(\mathbf{U}) \leq \Lambda(\mathbf{U}, \mathbb{F}) \leq e, \ e \ \text{given “energy”}$$

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \Leftarrow E(\mathbf{U}) = e$$

Oscillatory lemma

Subsolutions

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0, \ \ E(\mathbf{U}) \leq \Lambda(\mathbf{U}, \mathbb{F}) \boxed{<} e$$

Oscillatory increments

$$\partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{G}_\varepsilon = 0, \ \ \Lambda(\mathbf{U} + \mathbf{w}_\varepsilon, \mathbb{F} + \mathbb{G}_\varepsilon) < e$$

$$\mathbf{w}_\varepsilon, \ \mathbb{G}_\varepsilon \text{ compactly supported in } Q, \ \mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(Q)$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{w}_\varepsilon|^2 \ \mathrm{d}x \geq C \int_B \Big(e - E(\mathbf{U}) \Big)^\alpha \ \mathrm{d}x$$

$$\Rightarrow$$

$$\liminf_{\varepsilon \to 0} \int_B E(\mathbf{U} + \mathbf{w}_\varepsilon) \ \mathrm{d}x \geq \int_B E(\mathbf{U}) \ \mathrm{d}x + C \int_B \Big(e - E(\mathbf{U}) \Big)^\alpha \ \mathrm{d}x$$

Admissibility criteria

Entropy admissibility criterion - Second law

$$\partial_t E(\mathbf{U}) + \mathrm{div}_x \mathbf{F}(\mathbf{U}) \leq 0$$

Entropy rate admissibility criterion - Dafermos

$\mathbf{U}$ maximal with respect to the relation $\succ$

$$\mathbf{U} \succ \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$\int_\Omega E(\mathbf{U}(t,\cdot)) \ \mathrm{d}x \leq \int_\Omega E(\mathbf{V})(t,\cdot) \ \mathrm{d}x \text{ for a.a. } t \in (\tau, \tau + \delta)$$

for some $\delta > 0$

Pointwise maximal entropy rate criterion

Maximal dissipation admissibility criterion I

$$\mathbf{U} \succ_{\max} \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$E(\mathbf{U}(t,\cdot)) \leq E(\mathbf{V})(t,\cdot) \text{ for a.a. } t \in (\tau, \tau + \delta)$$

$$\text{for some } \delta > 0$$

$$\mathbf{U} \succ_{\max-\mathrm{sharp}} \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$E(\mathbf{U}(t,\cdot)) < E(\mathbf{V})(t,\cdot) \text{ for a.a. } t \in (\tau, \tau + \delta)$$

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = \boxed{0} \text{ (inviscid)}$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \boxed{\Delta\vartheta} = -\varrho\vartheta\mathrm{div}_x\mathbf{u} \text{ (heat conductive)}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta \ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Application of the convex integration method, I

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

"Energy"

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

Application of the convex integration method, II

Construction of solutions

1. Fix ϱ and compute the acoustic potential Ψ

$$-\Delta \Psi = \partial_t \varrho$$

2. Compute $\vartheta = \vartheta[\mathbf{v}]$ for $\mathbf{v} \in L^\infty$

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x\Big(\vartheta(\mathbf{v} + \nabla_x \Psi)\Big)\right) - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x\left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho}\right)$$

3. Observe that $0 < \vartheta < \overline{\vartheta}$, $\overline{\vartheta}$ independent of $\mathbf{v}$
4. Take

$$e = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}]$$

and use a non-local variant of the results od DeLellis and Székelyhidi for the *incompressible* Euler system to find $\mathbf{v}$

Conservative solutions to the Euler-Fourier system

Total energy conservation

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{3}{2} \varrho \vartheta \right) (\tau, \cdot) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{3}{2} \varrho_0 \vartheta_0 \right) \ \mathrm{d}x$$

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many conservative weak solutions in $(0, T)$

Maximal dissipation criterion?

Entropy production rate

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$s(\varrho, \vartheta) = -\log \left(\frac{\varrho}{\vartheta^{3/2}} \right)$$

$$\sigma \boxed{\geq} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta^2}$$

Maximal dissipation

- Maximize the entropy production rate
- Maximize the total entropy $\int_\Omega \varrho s(\varrho, \vartheta) \ \mathrm{d}x$
- Maximize the entropy $\varrho s(\varrho, \vartheta)$