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Abstract

We examine a hydrodynamic model of the motion of ions in plasma in the regime of small Debye length, a small ratio of the ion/electron temperature, and high Reynolds number. We analyze the associated singular limit and identify the limit problem - the incompressible Euler system. The result leans on careful analysis of the oscillatory component of the solutions by means of Fourier analysis.

Key words: Compressible Navier-Stokes-Poisson system, singular limit, quasi-neutral limit

Contents

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1 Introduction

We study the singular limit in a hydrodynamic model of plasma in the regime of small Debye lenght, small ion/electron temperature ratio, and high Reynolds number.

1.1 Motivation

A simple hydrodynamic model of plasma consisting of electrons and a single family of positively charged ions reads as follows:

$$\partial_t \varrho_e + \mathrm{div}_x(\varrho_e \mathbf{u}_e) = 0,\ \partial_t \varrho_i + \mathrm{div}_x(\varrho_i \mathbf{u}_i) = 0,$$

$$m_e \left[\partial_t (n_e \mathbf{u}_e) + \mathrm{div}_x (n_e \mathbf{u}_e \otimes \mathbf{u}_e) \right] + k_B \nabla_x (T_e n_e) = \mathrm{div}_x \mathbb{S}_e + q n_e \nabla_x \Phi,$$

$$m_i \left[\partial_t (n_i \mathbf{u}_i) + \mathrm{div}_x (n_i \mathbf{u}_i \otimes \mathbf{u}_i) \right] + k_B \nabla_x (T_i n_i) = \mathrm{div}_x \mathbb{S}_i - q n_i \nabla_x \Phi,$$

$$-\Delta \Phi = q(n_i - n_e),$$

where we have denoted

- ion/electron density . . . . . n_i/n_e,
- ion/electron mass . . . . . m_i/m_e,

- ion/electron mass density .... $\varrho_i = m_i n_i / \varrho_e = m_e n_e$,
- ion/electron velocity .... $\mathbf{u}_i/\mathbf{u}_e$,
- ion/electron temperature .... T_i/T_e,
- ion/electron viscous stress tensor .... $\mathbb{S}_i/\mathbb{S}_e$,
- electric potential .... Φ,
- unite electric charge .... q,
- Boltzmann constant .... k_B,

see Juengel and Peng [8, Appendix A].

Neglecting the electron mass $m_e \approx 0$ as well as $\mathbb{S}_e \approx 0$, we arrive at a single particle system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho = \frac{1}{\mathrm{Re}} \mathrm{div}_x \mathbb{S}_i - T_i \varrho \nabla_x \Phi$$

$$-\lambda^2 \Delta \Phi + \exp\left(\frac{T_i}{T_e}\Phi\right) = \varrho,$$

where we have set $\varrho = \varrho_i$, $\mathbf{u} = \mathbf{u}_i$, $\Phi \approx q\Phi/T_i k_B$, and where we have introduced two characteristic numbers: the Reynolds number Re and the Debye length λ. Our goal is to study the behavior of the above system in the regime

$$\mathrm{Re} \to \infty, \ \lambda \to 0, \ \frac{T_i}{T_e} \to 0.$$

Small Debye length is characteristic for the quasi-neutral limit. The hypothesis $T_i/T_e \approx 0$ is motivated by the observed fact that the electrons in many types of plasma are close enough to thermal equilibrium while the massive ions have usually the much lower ambient temperature, see e.g. Goebel and Katz [6, Chapter 3]. Besides its physical relevance, a rigorous derivation of such a singular limit gives rise to interesting theoretical issues, among which the behavior of acoustic-electric waves.

Simplifying slightly the problem, we shall study the singular limit for $\varepsilon \to 0$ of the following Navier-Stokes-Poisson system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varepsilon^\alpha \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - \varrho \nabla_x \Phi, \tag{1.2}$$

$$-\varepsilon^2 \Delta \Phi + \varepsilon^\beta \Phi = \varrho - 1, \tag{1.3}$$

where the viscous stress is given by NEWTON'S RHEOLOGICAL LAW

$$\mathbb{S}(\nabla_x \mathbf{u}) = \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right). \tag{1.4}$$

This system describes the motion of ions in a discharged plasma in the turbulent regime of high Reynolds number $\varepsilon^{-\alpha}$, with the small Debye length ε and the ion/electron temperature ratio proportional to ε^{β}.

1.2 Target system

Formally, it is not difficult to perform the asymptotic limit $\varepsilon \to 0$ in the system (1.1 - 1.4). Seeing that $\varrho \to 1$ we conclude that $\mathbf{u} \to \mathbf{v}$, where the limit velocity field $\mathbf{v}$ satisfies the inviscid EULER SYSTEM:

$$\mathrm{div}_x \mathbf{v} = 0, \tag{1.5}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0. \tag{1.6}$$

In order to formulate rigorously such a result, we have to specify the physical domain Ω, the initial conditions for $t = 0$ as well as the boundary conditions if $\partial\Omega \neq \emptyset$. Given the physical background of the problem, we consider $\Omega \subset R^3$ the infinite slab

$$\Omega = R^2 \times [0, 1],$$

on the boundary of which the velocity $\mathbf{u}$ satisfies the slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\nabla_x \mathbf{u})\mathbf{n}]_{\tan}|_{\partial\Omega} = 0. \tag{1.7}$$

Similarly, we assume

$$\nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.8}$$

where $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega$.

The initial state of the system will be determined by the so-called ill-prepared initial data, specifically,

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon}, \ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.9}$$

where

$$\|\varrho_{0,\varepsilon}\|_{L^\infty \cap L^2(\Omega)} \leq c, \ \varrho_{0,\varepsilon} \to 1 \text{ in } L^2(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3). \tag{1.10}$$

In addition to (1.9 - 1.10), the initial distribution of the electric potential will be subjected to certain "compatibility" restrictions specified below.

1.3 Mathematics of the singular limit

Wang and Jiang [12] studied a similar problem (i.e. small Debye length-high Reynolds number limit) with the space-periodic boundary conditions and with (1.3) replaced by

$$-\varepsilon^2 \Delta \Phi = \varrho - 1. \tag{1.11}$$

In such a case, the acoustic component of the velocity together with the electric field undergo fast oscillations in the time variable. Accordingly, the authors of [12] establish *weak* convergence towards the limit Euler system, using the description of the acoustic-electric waves, similar to the description proposed by Masmoudi [10] in the context of the low Mach - high Reynolds number limit and the compressible Navier-Stokes system.

On the other hand, the same problem considered in an *unbounded* physical domain seems to offer the possibility to use the *dispersion effects* in order to show strong convergence even for the *ill-prepared initial data.* In [2] the authors investigated the quasineutral limit of the isentropic Navier-Stokes-Poisson system in the whole space and obtained the convergence of the weak solution of the Navier-Stokes-Poisson system to the weak solution of the incompressible Navier-Stokes equations by means of dispersive estimates of Strichartz's type under the assumption that the Mach number is related to the Debye length by a power law. In [1] the authors got the same type of result in the case of the Debye length of the same order of the Mach number. However, even in the presence of dispersion, *the low Debye length - high Reynolds* number limit problem remains open in the case of *ill- prepared initial data.* The present paper has for aim to fill this gap.

In contrast with the formally similar problems arising in the low Mach number limit, however, the associated acoustic-electric waves in the present setting are described by the standard Klein-Gordon equation without any obvious time scaling that would yield the desired dispersive estimates. *To the best of our knowledge, a rigorous derivation of the asymptotic limit for the system (1.1), (1.2), and (1.11) in a unbounded media is an open problem.*

Our analysis starts with the nowadays standard estimates based on the modulated energy inequality, see Wang and Jiang [12], Masmoudi [10], put in the general framework of *relative entropy inequality* introduced in [5]. As observed by Wang and Jiang [12] and mentioned above, the case of the ill-prepared data is more delicate than for the low Mach number limit since the density deviations from the equilibrium state $\varrho = 1$ are controlled in a weaker norm. In this paper, we solve this problem in the case of scaling mentioned in Section 1.1, namely $\lambda \approx \varepsilon$, $\mathrm{Re} \approx \varepsilon^{-\alpha}$, $T_i/T_e \approx \varepsilon^{\beta}$, $\alpha, \beta > 0$. The key ingredient of our approach are the dispersive decay estimates for acoustic-electric waves governed by a pseudodifferential wave-like equation. It is interesting to note that these estimates are effective only in the regime

$$0 < \beta < 2,$$

meaning when the temperature quotient T_i/T_e vanishes but still dominates the Debye length λ in the asymptotic limit.

The paper is organized as follows. In Section 2, we collect the necessary preliminary material and formulate our main result. In Section 3, we introduce the relative entropy inequality and establish uniform bounds on the family of solutions independent of the scaling parameter ε. The heart of the paper is Section 4, where we derive the acoustic-electric equation along with the associated dispersive estimates. At this stage, the simple geometry of the physical domain plays a crucial role making possible the use of the Fourier analysis of certain oscillatory integrals. The proof is concluded in Section 5 by another application of the relative entropy inequality.

2 Preliminaries, main result

We start with the basic hypothesis concerning the pressure $p = p(\varrho)$, specifically,

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \tag{2.1}$$

for a certain $\gamma > 3/2$.

2.1 Existence of solutions for the primitive and target systems

We say that $[\varrho, \mathbf{u}, \Phi]$ is a *weak solution* of the Navier-Stokes-Poisson system (1.1 - 1.4), (1.7 - 1.10) in $(0,T) \times \Omega$ if:

- $\varrho \geq 0$ a.a. in $(0,T) \times \Omega$,

$$(\varrho - 1) \in C_{\rm weak}(0,T; L^2 + L^\gamma(\Omega)),\ \varrho\mathbf{u} \in C_{\rm weak}(0,T; L^2 + L^{2\gamma/(\gamma+1)}(\Omega; R^3)),$$
$$\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3));$$

- $$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} \equiv u_3|_{\partial\Omega} = 0,$$

and
$$\int_0^\tau \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi]\ {\rm d}x\ {\rm d}t = \int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_{0,\varepsilon}\varphi(0,\cdot)\ {\rm d}x \tag{2.2}$$
for $0 < \tau < T$ and any test function $\varphi \in C^\infty_c([0,T] \times \overline{\Omega})$;

- $$\int_0^\tau \int_\Omega [\varrho\mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho){\rm div}_x \varphi - \varepsilon^\alpha \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi - \varrho \nabla_x \Phi \cdot \varphi]\ {\rm d}x\ {\rm d}t \tag{2.3}$$
$$= \int_\Omega \varrho\mathbf{u}(\tau,\cdot) \cdot \varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \cdot \varphi(0,\cdot)\ {\rm d}x$$
for any $\tau > 0$ and any test function $\varphi \in C^\infty_c([0,T] \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$;

- the electric potential Φ is the unique (strong) solution of the elliptic equation

$$-\varepsilon^2 \Delta \Phi + \varepsilon^\beta \Phi = \varrho - 1, \ \Phi \in C_{\rm weak}([0,T]; (W^{2,2} + W^{2,\gamma})(\Omega)), \ \nabla_x \Phi \cdot \mathbf{n} = 0; \tag{2.4}$$

the *energy inequality*

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + (H(\varrho) - H'(1)(\varrho - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi|^2 + \frac{\varepsilon^\beta}{2} \Phi^2 \right] (\tau, \cdot) \ {\rm d}x \tag{2.5}$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ {\rm d}x \ {\rm d}t$$

$$\leq \int_\Omega \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + (H(\varrho_{0,\varepsilon}) - H'(1)(\varrho_{0,\varepsilon} - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_{0,\varepsilon}|^2 + \frac{\varepsilon^\beta}{2} \Phi_{0,\varepsilon}^2 \right] {\rm d}x$$

holds for a.a. $\tau \in (0,T)$, where we have set

$$H(\varrho) = \varrho \int_0^\varrho \frac{p(z)}{z^2} \ {\rm d}z,$$

and

$$\Phi_{0,\varepsilon} = \left(-\varepsilon^2 \Delta_N + \varepsilon^\beta \right)^{-1} [\varrho_{0,\varepsilon} - 1], \ \Delta_N - \text{ the Neumann Laplacian in } \Omega.$$

The following result can be proved by the method of [3]:

Proposition 2.1 *Let the pressure p satisfy the hypothesis (2.1), with*

$$\gamma > \frac{3}{2}.$$

Let

$$(\varrho_{0,\varepsilon} - 1) \in L^2 \cap L^\infty(\Omega), \ \varrho_{0,\varepsilon} > 0, \ \mathbf{u}_{0,\varepsilon} \in L^2(\Omega; R^3).$$

Then for any $T > 0$ and $\varepsilon > 0$ the Navies-Stokes-Poisson system (1.1 - 1.4), (1.7 - 1.10) admits a weak solution $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Phi_\varepsilon]$ in $(0,T) \times \Omega$.

As for the target Euler system (1.5), (1.6), supplemented with the boundary condition

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.6}$$

and the initial condition

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ {\rm div}_x \mathbf{v}_0 = 0, \ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{2.7}$$

we may adapt the classical method of Kato and Lai [9] to obtain the following local existence result:

Proposition 2.2 *Let*

$$\mathbf{v}_0 \in W^{m,2}(\Omega; R^3),\ \mathrm{div}_x \mathbf{v}_0 = 0,\ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

be given, with

$$m > \frac{5}{2}.$$

Then there exists a time interval $(0, T_{\max})$, $T_{\max} > 0$ *such that the Euler system (1.5), (1.6), (2.6), (2.7) admits a unique solution* $\mathbf{v}$ *in* $[0, T_{\max}) \times \Omega$ *belonging to the class*

$$\mathbf{v} \in C([0, T_{\max}); W^{m,2}(\Omega; R^3)),\ \partial_t \mathbf{v},\ \nabla_x \Pi \in C([0, T_{\max}); W^{m-1,2}(\Omega; R^3)).$$

2.2 Main result

Let $\mathbf{H}$ denote the standard Helmholtz projection onto the space of solenoidal functions in $L^2(\Omega; R^3)$. Having collected all the necessary material, we are ready to formulate the main result of the present paper:

Theorem 2.1 *Let the pressure p satisfy (2.1). Let*

$$\alpha > 0,\ 0 < \beta < 2 \tag{2.8}$$

be given exponents. Suppose that the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \Phi_{0,\varepsilon}]$ *satisfy*

$$0 < \varrho_{0,\varepsilon} < c,\ (\varrho_{0,\varepsilon} - 1) \to 0,\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \textit{in}\ L^2(\Omega; R^3), \tag{2.9}$$

$$\varepsilon^{\beta/2}\Phi_{0,\varepsilon} \to s_0,\ \varepsilon\nabla_x\Phi_{0,\varepsilon} \to 0\ \textit{in}\ L^2(\Omega; R^3), \tag{2.10}$$

as $\varepsilon \to 0$*, where*

$$\mathbf{v}_0 \equiv \mathbf{H}[\mathbf{u}_0] \in W^{m,2}(\Omega; R^3)\ \textit{for a certain}\ m > \frac{5}{2}.$$

Let $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Phi_\varepsilon]$ *be a family of weak solutions of the Navier-Stokes-Fourier system (1.1 - 1.4), (1.7 - 1.10) , and let* $\mathbf{v}$ *be the solution of the inviscid Euler system (1.5), (1.6), (2.6), (2.7) in* $[0, T_{\max})$*, emanating from the initial data* $\mathbf{v}_0$*.*

Then

$$\varrho_\varepsilon \to 1\ \textit{in}\ L^\infty(0,T; L^2 + L^\gamma(\Omega)),$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v}\ \textit{in}\ L^2(0,T; L^2_{\rm loc}(\Omega; R^3))\ \textit{and weakly-(*) in}\ L^\infty(0,T; L^2_{\rm loc}(\Omega; R^3))$$

$$\varepsilon^{\beta/2}\Phi_\varepsilon \to 0\ \textit{in}\ L^2(0,T; L^2_{\rm loc}(\Omega))\ \textit{and weakly-(*) in}\ L^\infty(0,T; L^2_{\rm loc}(\Omega))$$

for any $0 < T < T_{\max}$*.*

Remark 2.1 *Since* $\beta < 2$*, the latter hypothesis*

$$\varepsilon\nabla_x\Phi_{0,\varepsilon} \to 0\ \textit{in}\ L^2(\Omega; R^3)$$

in (2.10) is enforced by the former as soon as we require $\varepsilon\nabla_x\Phi_{0,\varepsilon}$ *to be precompact in* L^2*.*

The rest of the paper is devoted to the proof of Theorem 2.1.

3 Uniform bounds

Motivated by [5], we introduce the relative entropy functional in its general form

$$\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + (H(\varrho) - H'(r)(\varrho - r) - H(r)) + \frac{\varepsilon^2}{2}|\nabla_x\Phi|^2 + \frac{\varepsilon^\beta}{2}\Phi^2\right] \, \mathrm{d}x,$$

where we have set

$$H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z, \;\; P(\varrho) = H'(\varrho).$$

As shown in [4, Theorem 1], any weak solution $[\varrho, \mathbf{u}, \Phi]$ of the Navier-Stokes-Poisson system (in the sense specified in Section 2) satisfies the *relative entropy inequality*:

$$\left[\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u}\Big| r, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \tag{3.1}$$

$$+\varepsilon^\alpha\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \leq \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) \, \mathrm{d}t,$$

where the remainder

$$\mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) = \int_\Omega \varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\Big) \, \mathrm{d}x + \int_\Omega \varepsilon^\alpha\mathbb{S}(\nabla_x\mathbf{U}) : (\nabla_x\mathbf{U} - \nabla_x\mathbf{u}) \, \mathrm{d}x \tag{3.2}$$

$$+\int_\Omega \Big((r-\varrho)\partial_t P(r) + \nabla_x P(r)\cdot(r\mathbf{U} - \varrho\mathbf{u})\Big) \, \mathrm{d}x - \int_\Omega \Big(p(\varrho) - p(r)\Big)\mathrm{div}_x\mathbf{U} \, \mathrm{d}x$$

$$+\int_0^\tau\int_\Omega \varrho\nabla_x\Phi\cdot\mathbf{U} \, \mathrm{d}x \, \mathrm{d}t$$

for any sufficiently smooth test functions r, $\mathbf{U}$,

$$r > 0, \;\; \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega} = 0, \;\; r \to 1, \;\; \mathbf{U} \to 0 \text{ as } |x| \to \infty.$$

3.1 Uniform bounds

Taking $r \equiv 1$, $\mathbf{U} \equiv 0$ in (3.1), we may use the hypotheses (2.9), (2.10) to deduce the following uniform bounds:

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega;R^3)} \leq c, \tag{3.3}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon - 1\|_{L^2+L^\gamma(\Omega)} \leq c, \tag{3.4}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varepsilon\nabla_x\Phi_\varepsilon\|_{L^2(\Omega;R^3)} \leq c,\ \operatorname{ess}\sup_{t\in(0,T)} \left\|\varepsilon^{\beta/2}\Phi_\varepsilon\right\|_{L^2(\Omega)} \leq c. \tag{3.5}$$

Seeing that Φ_ε, ϱ_ε are interrelated through (2.4), we get

$$\frac{\varrho-1}{\varepsilon^{\beta/2}} = G_{1,\varepsilon} + \varepsilon^{1-\frac{\beta}{2}} G_{2,\varepsilon} \tag{3.6}$$

where $G_{1,\varepsilon}$ are bounded in $L^\infty(0,T;L^2(\Omega))$ while $G_{2,\varepsilon}$ are bounded in $L^\infty(0,T;W^{-1,2}(\Omega))$.

4 Acoustic-electric waves

Decomposing

$$\varrho\mathbf{u} = \mathbf{H}[\varrho\mathbf{u}] + \nabla_x\Psi,$$

where Ψ is the acoustic potential, we may formally derive the acoustic-electric equation (*AE system*) writing the Navier-Stokes-Poisson system in the form:

$$\varepsilon^{\beta/2}\partial_t\left(\frac{\varrho-1}{\varepsilon^{\beta/2}}\right) + \operatorname{div}_x(\varrho\mathbf{u}) = 0,$$

$$\varepsilon^{\beta/2}\partial_t(\varrho\mathbf{u}) + \varepsilon^{\beta/2}\nabla_x\Phi = \varepsilon^{\beta/2}\left(\varepsilon^\alpha \operatorname{div}_x\mathbb{S} - \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \nabla_x p(\varrho) + (1-\varrho)\nabla_x\Phi\right).$$

Consequently, neglecting "small" terms on the right-hand side and revoking the relation between ϱ and Φ given by the Poisson equation, we arrive at the system:

AE SYSTEM

$$\varepsilon^{\beta/2}\partial_t s + \Delta\Psi = 0, \tag{4.1}$$

$$\varepsilon^{\beta/2}\partial_t\nabla_x\Psi + \nabla_x\left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s] = 0, \tag{4.2}$$

$$\nabla_x\Psi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{4.3}$$

with

$$s \approx \frac{\varrho-1}{\varepsilon^{\beta/2}},\ \nabla_x\Psi \approx \varrho\mathbf{u} - \mathbf{H}[\varrho\mathbf{u}].$$

4.1 Energy conservation

The functional

$$t \mapsto \frac{1}{2}\int_\Omega \left[s\left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s] + |\nabla_x \Psi|^2\right] \, \mathrm{d}x \tag{4.4}$$

represent energy for the AE system; it is straightforward to check that this quantity is conserved in time for any $\varepsilon > 0$.

4.2 Dispersive estimates

The simple geometry of the domain Ω combined with the homogeneous Neumann boundary condition (4.3) enables us to express solutions of (4.1 - 4.3) in terms of the continuous Fourier transform in the horizontal variable $x_h = (x_1, x_2)$ and the discrete Fourier transform in x_3. Indeed replacing Ω by

$$\hat{\Omega} = R^2 \times \mathcal{T}^1,$$

where

$$\mathcal{T}^1 = [-1, 1]_{\{-1,1\}},$$

we introduce the Fourier representation of a function $r = r(t, x_h, x_3)$,

$$\hat{r}(t,\xi,k) = \frac{1}{\sqrt{2}}\int_{-1}^{1}\int_{R^2} \exp\left(-\mathrm{i}\xi\cdot x_h\right) r(t,x_h,x_3)\exp\left(-\mathrm{i}kx_3\right) \, \mathrm{d}x_h \, \mathrm{d}x_3, \ \xi = (\xi_1,\xi_2)\in R^2, \ k \in Z, \ k \neq 0.$$

Accordingly, the AE equation can be reformulated as

$$\varepsilon^{\beta/2}\partial_t \hat{s}(t,\xi,k) - \left(|\xi|^2 + k^2\right)\hat{\Psi}(t,\xi,k) = 0, \tag{4.5}$$

$$\varepsilon^{\beta/2}\partial_t \hat{\Psi}(t,\xi,k) + \left(\varepsilon^{2-\beta}(|\xi|^2 + k^2) + 1\right)^{-1}\hat{s}(t,\xi,k) = 0. \tag{4.6}$$

The system (4.5), (4.6) can be explicitly integrated, namely

$$\hat{s}(t,\xi,k) = \hat{a}(\xi,k)\exp\left(\mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right) + \hat{b}(\xi,k)\exp\left(-\mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right), \tag{4.7}$$

$$\hat{\Psi}(t,\xi,k) = \hat{c}(\xi,k)\exp\left(\mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right) + \hat{d}(\xi,k)\exp\left(-\mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right), \tag{4.8}$$

with

$$\tau = \frac{t}{\varepsilon^{\beta/2}},$$

and the functions $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ uniquely determined in terms of the initial data $\hat{s}_0$, $\hat{\Psi}_0$, namely

$$\hat{a}(\xi,k) = \frac{\hat{s}_0(\xi,k) + i\Gamma(|\xi|,k)\hat{\Psi}_0}{2}, \quad \hat{b}(\xi,k) = \frac{\hat{s}_0(\xi,k) - i\Gamma(|\xi|,k)\hat{\Psi}_0}{2},$$

$$\hat{c}(\xi,k) = \frac{\hat{\Psi}_0(\xi,k) + i[\Gamma(|\xi|,k)]^{-1}\hat{s}_0}{2}, \quad \hat{d}(\xi,k) = \frac{\hat{\Psi}_0(\xi,k) - i[\Gamma(|\xi|,k)]^{-1}\hat{s}_0}{2},$$

where

$$\Gamma(|\xi|,k) = \sqrt{(|\xi|^2+k^2)(1+\varepsilon^{2-\beta}(|\xi|^2+k^2))}.$$

4.2.1 $L^1 - L^\infty$ estimates

Our goal is to establish the $L^1 - L^\infty$ estimates well known for the standard wave equation. To this end we consider a generic quantity,

$$\hat{Z}(\tau,\xi,k) = \mathcal{K}(|\xi|,k)\hat{A}(\xi,k)\exp\left(\pm \mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right)$$

and compute

$$Z(\tau,x_h,k) = \mathcal{F}^{-1}_{\xi\to x_h}\left[\mathcal{K}(|\xi|,k)\hat{A}(\xi,k)\exp\left(\pm \mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right)\right]$$

$$= \mathcal{F}^{-1}_{\xi\to x_h}\left[\mathcal{K}(|\xi|,k)\exp\left(\pm \mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right)\right] * A(x_h,k),$$

where $\mathcal{F}$ denotes the Fourier transform in the horizontal variable. Consequently,

$$\|Z(\tau,\cdot,k)\|_{L^\infty(R^2)} \le \left\|\mathcal{F}^{-1}_{\xi\to x_h}\left[\mathcal{K}(|\xi|,k)\exp\left(\pm \mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right)\right]\right\|_{L^\infty(\Omega)} \|A(\cdot,k)\|_{L^1(R^2)}.$$

Next, we use the idea of Guo, Peng, and Wang [7] computing explicitly

$$\mathcal{F}^{-1}_{\xi\to x_h}\left[\mathcal{K}(|\xi|,k)\exp\left(\pm \mathrm{i}\sqrt{\frac{|\xi|^2+k^2}{\varepsilon^{2-\beta}(|\xi|^2+k^2)+1}}\tau\right)\right](x_h,k) \tag{4.9}$$

$$= \frac{\pi\sqrt{2}}{2} \int_0^\infty \exp\left(\pm \mathrm{i} \sqrt{\frac{z + k^2}{\varepsilon^{2-\beta}(z+k^2)+1}} \tau \right) \mathcal{K}(\sqrt{z}, k) J_0\left(\sqrt{z}|x_h|\right) \ \mathrm{d}z,$$

where J_m, $m = 0, \dots$ denote the Bessel functions.

The value of the oscillatory integral (4.9) can be estimated by the standard *van der Corput lemma*, see Stein [11, Chapter 8.1.2, Proposition 2 and Corollary]:

Lemma 4.1 *Let $\Lambda = \Lambda(z)$ be a smooth function away from the origin,*

$$\partial_z \Lambda(z) \ \textit{monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a, b]$, $0 < a < b < \infty$. Let Φ be a smooth function on $[a, b]$.

Then

$$\left| \int_a^b \exp\left(\mathrm{i}\Lambda(z)t\right) \Phi(z) \ \mathrm{d}z \right| \leq c \frac{1}{t\Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ \mathrm{d}z \right],$$

where c is an absolute constant independent of Λ and Φ.

Now, suppose that the function $\mathcal{K}$ represent a frequency "cut-off" meaning $\mathcal{K} \in C_c((0, \infty) \times R)$. It can checked by a direct computation that the integrand in (4.9) satisfies the hypotheses of Lemma 4.1, with Λ_0 depending on F, k but *independent* of the scaling parameter ε. Thus summing up the previous considerations, we obtain the following result:

Lemma 4.2 *Suppose that s, Ψ is a solution of the AE equation (4.1 - 4.3) emanating from the initial data*

$$s(0, \cdot) = s_0, \ \Psi(0, \cdot) = \Psi_0,$$

where s_0, Ψ_0 are such that

$$\hat{s}_0(\xi, k) \equiv 0, \ \hat{\Psi}_0(\xi, k) \equiv 0 \ \textit{whenever} \ |\xi| > M, \ |\xi|^{-1} > M, \ |k| > M \tag{4.10}$$

for a certain $M > 0$.

Then there exists a constant $c = c(M)$ such that

$$\|s(t, \cdot)\|_{L^\infty(\Omega)} + \|\Psi(t, \cdot)\|_{L^\infty(\Omega)} \leq c(M) \frac{\varepsilon^{\beta/2}}{t} \left(\|s_0\|_{L^1(\Omega)} + \|\Psi_0\|_{L^1(\Omega)} \right) \ \textit{for all} \ t > 0. \tag{4.11}$$

4.2.2 Dispersive estimates for regularized data

In view of (4.7), (4.8), (4.10), the expressions

$$4\pi^2 \sum_{k\in Z} \int_{R^2} \hat{s}(t,\xi,k)(\varepsilon^{2-\beta}(|\xi|^2+k^2)+1)^{-1}\hat{s}(t,\xi,k)\mathrm{d}\xi = \int_\Omega s(t,\cdot)(-\varepsilon^{\beta-2}\Delta_N+1)^{-1}[s](t,\cdot)\ \mathrm{d}x$$

$$\text{and}\quad \|\hat{s}\|^2_{L^2(\Omega)} = \|s\|^2_{L^2(\Omega)}$$

as well as

$$4\pi^2 \sum_{k\in Z} \int_{R^2} (|\xi|^2+k^2)\Psi(t,\xi,k)\mathrm{d}\xi = \int_\Omega |\nabla_x \Psi|^2\ \mathrm{d}x \text{ and } \sum_{k\in Z}\int_{R^2} |\hat{\Psi}(t,\xi,k)|^2 \mathrm{d}\xi = \int_\Omega |\Psi(t,x)|^2\ \mathrm{d}x$$

are equivalent uniformly with respect to $t \in [0,T)$ and $\varepsilon \in (0,1)$.

Thus, the estimate (4.11) can be interpolated with the energy conservation (4.4), to obtain

$$\|s(t,\cdot)\|_{L^p(\Omega)} + \|\Psi(t,\cdot)\|_{L^p(\Omega)} \le c(M)\left[\frac{\varepsilon^{\beta/2}}{t}\right]^{1-\frac{2}{p}} \left(\|s_0\|_{L^{p'}(\Omega)} + \|\Psi_0\|_{L^{p'}(\Omega)}\right) \text{ for all } t>0,\ p\ge 2, \quad (4.12)$$

$$\frac{1}{p}+\frac{1}{p'}=1,$$

whenever s_0, Ψ_0 belong to the class (4.10).

Summarizing the above we derive uniform decay estimates for the AE equation with regularized initial data. Given a function g, we define its regularization $[g]_\delta$ as

$$[g]_\delta(x_h,x_3) = \frac{1}{\sqrt{2}} \sum_{|k|\le 1/\delta} \mathcal{F}^{-1}_{\xi\to x_h}\left[\hat{F}_\delta(|\xi|)\widehat{G_\delta g}(\xi,k)\right]\exp\left(-\mathrm{i}kx_3,\right), \quad (4.13)$$

where

$$G_\delta = \phi_\delta(x_h) \in C^\infty_c(R^2),\ 0 \le G_\delta \le 1,\ \phi_\delta \nearrow 1 \text{ as } \delta \to 0,$$
$$\hat{F}_\delta \in C^\infty_c(0,\infty), 0\le \hat{F}_\delta \le 1,\ \hat{F}_\delta \nearrow 1 \text{ as } \delta\to 0.$$

Lemma 4.3 *Let $s_{\varepsilon,\delta}$, $\Psi_{\varepsilon,\delta}$ be a solution of the AE equation (4.1 - 4.3), with the initial data*

$$s_{\varepsilon,\delta}(0,\cdot) = [s_0]_\delta,\ \Psi_{\varepsilon,\delta}(0,\cdot) = [\Psi_0]_\delta,\ \varepsilon,\delta>0.$$

Then

$$\|s_{\varepsilon,\delta}(\tau,\cdot)\|_{W^{m,p}(\Omega)} + \|\Psi_{\varepsilon,\delta}(\tau,\cdot)\|_{W^{m,p}(\Omega)} \le h_1(m,\delta,p)\left(\|s_0\|_{L^2(\Omega)} + \|\Psi_0\|_{L^2(\Omega)}\right) \quad (4.14)$$

for all $\tau \geq 0$, $m = 0, 1, \ldots$, $1 < p \leq \infty$,

$$\|s_{\varepsilon,\delta}(\tau,\cdot)\|_{W^{m,\infty}(\Omega)} + \|\Psi_{\varepsilon,\delta}(\tau,\cdot)\|_{W^{m,\infty}(\Omega)} \leq h_2(\varepsilon, m, \delta) \left(\|s_0\|_{L^2(\Omega)} + \|\Psi_0\|_{L^2(\Omega)} \right), \ m = 0, 1, \ldots \tag{4.15}$$

where

$$h_2(\varepsilon, m, \delta) \to 0 \ \textit{as} \ \varepsilon \to 0 \ \textit{for all} \ m, \delta > 0 \ \textit{fixed},$$

uniformly in τ *belonging compact subintervals of* $(0, \infty)$.

5 Convergence

In order to show convergence of $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Phi_\varepsilon]$ to the solution of the target system we apply once more the relative entropy inequality (3.1), this time for

$$r = 1, \ \mathbf{U}_{\varepsilon,\delta} = \mathbf{v} + \nabla_x \Psi_{\varepsilon,\delta}, \tag{5.1}$$

where $\mathbf{v}$ is the (smooth) solution of the target system (1.5), (1.6), with the initial data

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0],$$

and $\Psi_{\varepsilon,\delta}$, together with the associated $s_{\varepsilon,\delta}$, is the solution of the AE equation (4.1 - 4.3), with the initial data

$$\Psi_{\varepsilon,\delta}(0,\cdot) = [\Psi_0]_\delta, \ \nabla_x \Psi_0 = \mathbf{u}_0 - \mathbf{H}[\mathbf{u}_0], \ s_{\varepsilon,\delta}(0,\cdot) = [s_0]_\delta,$$

where s_0 is specified in (2.10).

5.1 Another application of the relative entropy inequality

Taking the ansatz (5.1) in the relative entropy inequality (3.1) we get

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + [H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)] + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau,\cdot) \ \mathrm{d}x \tag{5.2}$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon,\delta}) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon,\delta}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{v}_0 - \nabla_x \Psi_{0,\delta}|^2 \ \mathrm{d}x$$

$$+ \int_\Omega [H(\varrho_{0,\varepsilon}) - H'(1)(\varrho_{0,\varepsilon} - 1) - H(1)] \ \mathrm{d}x + \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla_x \Phi_{0,\varepsilon}|^2 + \frac{\varepsilon^\beta}{2} \Phi_{0,\varepsilon}^2 \right) \ \mathrm{d}x$$

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \Big(\partial_t \mathbf{U}_{\varepsilon,\delta} + \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{U}_{\varepsilon,\delta} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varepsilon^\alpha \mathbb{S}(\nabla_x \mathbf{U}_{\varepsilon,\delta}) : (\nabla_x \mathbf{U}_{\varepsilon,\delta} - \nabla_x \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho_\varepsilon) \mathrm{div}_x \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

Our ultimate goal is to perform the limit passage, first for $\varepsilon \to 0$ then for $\delta \to 0$ in (5.2) absorbing all terms on the right-hand side by means of a Gronwall type argument.

5.1.1 Initial data

In accordance with (2.9), (2.10), we have

$$\limsup_{\varepsilon \to 0} \int_\Omega \frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{v}_0 - \nabla_x [\Psi_0]_\delta|^2 \ \mathrm{d}x \leq c \|\nabla_x \Psi_0 - \nabla_x [\Psi_0]_\delta \|^2_{L^2(\Omega;R^3)} = \chi(\delta),$$

where the symbol $\chi(\delta)$ denotes a generic function, $\chi(\delta) \to 0$ for $\delta \to 0$.

Similarly,

$$\int_\Omega \left[H(\varrho_{0,\varepsilon}) - H'(1)(\varrho_{0,\varepsilon} - 1) - H(1) \right] \ \mathrm{d}x \leq c \|\varrho_{0,\varepsilon} - 1\|^2_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Finally, by virtue of the hypothesis (2.10), the relative entropy inequality (5.2) can be written as

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.3}$$

$$+ \varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon,\delta}) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}_{\varepsilon,\delta}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon \Big(\partial_t \mathbf{U}_{\varepsilon,\delta} + \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{U}_{\varepsilon,\delta} \Big) \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \Big) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varepsilon^\alpha \mathbb{S}(\nabla_x \mathbf{U}_{\varepsilon,\delta}) : (\nabla_x \mathbf{U}_{\varepsilon,\delta} - \nabla_x \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho_\varepsilon) \mathrm{div}_x \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

5.1.2 Viscous stresses

Next, we have

$$\int_0^\tau \int_\Omega \varepsilon^\alpha \mathbb{S}(\nabla_x \mathbf{U}_{\varepsilon,\delta}) : (\nabla_x \mathbf{U}_{\varepsilon,\delta} - \nabla_x \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \frac{1}{2} \int_0^\tau \int_\Omega \varepsilon^\alpha \mathbb{S}(\nabla_x \mathbf{U}_{\varepsilon,\delta}) : \left(\nabla_x (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) + \nabla_x^t (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) - \frac{2}{3} \mathrm{div}_x (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \mathbb{I} \right) \ \mathrm{d}x \ \mathrm{d}t;$$

whence (5.3) reduces to

$$\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2}|\nabla_x\Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2}\Phi_\varepsilon^2\right](\tau,\cdot)\ \mathrm{d}x \qquad (5.4)$$

$$\leq \chi(\delta) + \frac{1}{2}\int_\Omega |s_0|^2\ \mathrm{d}x$$

$$+\int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{U}_{\varepsilon,\delta}+\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\Big)\ \mathrm{d}x\ \mathrm{d}t-\int_0^\tau\int_\Omega p(\varrho_\varepsilon)\mathrm{div}_x\mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x+\int_0^\tau\int_\Omega \varrho_\varepsilon\nabla_x\Phi_\varepsilon\cdot\mathbf{U}_{\varepsilon,\delta}\ \mathrm{d}x\ \mathrm{d}t.$$

5.1.3 Convective term

We rewrite

$$\int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{U}_{\varepsilon,\delta}+\mathbf{u}_\varepsilon\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t = \int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{U}_{\varepsilon,\delta}+\mathbf{U}_{\varepsilon,\delta}\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{U}_{\varepsilon,\delta})\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t,$$

where, furthermore,

$$\int_0^\tau\int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{U}_{\varepsilon,\delta})\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t = \qquad (5.5)$$

$$\int_0^\tau\int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{U}_{\varepsilon,\delta})\cdot\nabla_x\mathbf{v}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho_\varepsilon(\mathbf{u}_\varepsilon-\mathbf{U}_{\varepsilon,\delta})\cdot\nabla_x^2\Psi_{\varepsilon,\delta}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t.$$

The first integral on the right-hand side of (5.5) can be absorbed by the left-hand side of (5.4) by a Gronwall type argument, while the second one vanishes for $\varepsilon \to 0$ in view of the dispersive estimates (4.15).

Next, we have

$$\int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{U}_{\varepsilon,\delta}+\mathbf{U}_{\varepsilon,\delta}\cdot\nabla_x\mathbf{U}_{\varepsilon,\delta}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_\Omega \varrho_\varepsilon\partial_t\nabla_x\Psi_{\varepsilon,\delta}\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho_\varepsilon\left[\nabla_x\Psi_{\varepsilon,\delta}\cdot\nabla_x(\mathbf{v}+\nabla_x\Psi_{\varepsilon,\delta})+\mathbf{v}\cdot\nabla_x^2\Psi_{\varepsilon,\delta}\right]\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t,$$

where, in accordance with (4.15),

$$\int_0^\tau\int_\Omega \varrho_\varepsilon\left[\nabla_x\Psi_{\varepsilon,\delta}\cdot\nabla_x(\mathbf{v}+\nabla_x\Psi_{\varepsilon,\delta})+\mathbf{v}\cdot\nabla_x^2\Psi_{\varepsilon,\delta}\right]\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t \to 0 \text{ as } \varepsilon\to 0.$$

Furthermore, we have

$$\int_0^\tau\int_\Omega \varrho_\varepsilon\Big(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\Big)\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t = -\int_0^\tau\int_\Omega \varrho_\varepsilon\nabla_x\Pi\cdot(\mathbf{U}_{\varepsilon,\delta}-\mathbf{u}_\varepsilon)\ \mathrm{d}x\ \mathrm{d}t,$$

where, in accordance with the uniform bounds established in (3.3), (3.6),

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t \to \int_0^\tau \int_\Omega \overline{\varrho \mathbf{u}} \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t = 0.$$

Similarly,

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{v} \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t \to \int_0^\tau \int_\Omega \mathbf{v} \cdot \nabla_x \Pi \ \mathrm{d}x \ \mathrm{d}t = 0,$$

and, finally,

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Pi \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \nabla_x \Pi \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \Pi \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t,$$

where both terms on the right-hand side can be controlled by the dispersive estimates established in Lemma 4.3.

In view of the above arguments, the relation (5.4) reads

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.6}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E}_\varepsilon \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega p(\varrho_\varepsilon) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

5.1.4 Pressure dependent terms

As the next step, we observe that

$$\int_0^\tau \int_\Omega p(\varrho_\varepsilon) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega (p(\varrho_\varepsilon) - p(1)) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega (p(\varrho_\varepsilon) - p'(1)(\varrho_\varepsilon - 1) - p(1)) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + p'(1) \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t,$$

where, by taking into account the dispersive estimates (4.15), the last integral can be handled as

$$\int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta \Phi_\varepsilon + \varepsilon^\beta \Phi_\varepsilon \right) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0.$$

Consequently, relation (5.6) simplifies to

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.7}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E}_\varepsilon \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

5.1.5 The terms depending on the electric potential

We write

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{U}_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t =$$

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t,$$

and,

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \nabla_x \Phi_\varepsilon \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta \Phi_\varepsilon + \varepsilon^\beta \Phi_\varepsilon \right) \nabla_x \Phi_\varepsilon \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t = -\varepsilon^2 \int_0^\tau \int_\Omega \Delta \Phi_\varepsilon \nabla_x \Phi_\varepsilon \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \frac{\varepsilon^2}{2} \int_0^\tau \int_\Omega \nabla_x |\nabla_x \Phi_\varepsilon|^2 \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^2 \int_0^\tau \int_\Omega \nabla_x \mathbf{v} \cdot \nabla_x \Phi_\varepsilon \cdot \nabla_x \Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t = \varepsilon^2 \int_0^\tau \int_\Omega \nabla_x \mathbf{v} \cdot \nabla_x \Phi_\varepsilon \cdot \nabla_x \Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t,$$

where the last integral can be absorbed by the left-hand side of (5.7) via a Gronwall type argument.

Similarly,

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

where

$$\int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta \Phi_\varepsilon + \varepsilon^\beta \Phi_\varepsilon \right) \nabla_x \Phi_\varepsilon \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \frac{\varepsilon^2}{2} \int_0^\tau \int_\Omega \nabla_x |\nabla_x \Phi_\varepsilon|^2 \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^2 \int_0^\tau \int_\Omega \nabla_x^2 \Psi_{\varepsilon,\delta} \cdot \nabla_x \Phi_\varepsilon \cdot \nabla_x \Phi_\varepsilon \ \mathrm{d}x \ \mathrm{d}t$$

$$- \frac{\varepsilon^\beta}{2} \int_0^\tau \int_\Omega \Phi_\varepsilon^2 \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0.$$

Thus we have arrived at

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.8}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E} \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot (\mathbf{U}_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

As the next step, we have

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t = \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t =$$

$$= \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta \Phi_\varepsilon + \varepsilon^\beta \Phi_\varepsilon \right) \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t$$

$$= - \int_0^\tau \int_\Omega \left(-\varepsilon^{2-\frac{\beta}{2}} \Delta \Phi_\varepsilon + \varepsilon^{\beta/2} \Phi_\varepsilon \right) \nabla_x \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \cdot \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0$$

for $\delta > 0$ fixed as a consequence of (3.6).

We have

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.9}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E} \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot (\nabla_x \Psi_{\varepsilon,\delta} - \mathbf{u}_\varepsilon) \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

Furthermore,

$$- \int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \Phi_\varepsilon \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \frac{1}{\varepsilon^{\beta/2}} \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \ \mathrm{d}t - \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta_N + \varepsilon^\beta \right)^{-1} (\varrho_\varepsilon - 1) \Delta \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \frac{1}{\varepsilon^{\beta/2}} \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^{\beta/2} \int_0^\tau \int_\Omega \left(-\varepsilon^2 \Delta_N + \varepsilon^\beta \right)^{-1} (\varrho_\varepsilon - 1) \partial_t s_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$

$$= \frac{1}{\varepsilon^{\beta/2}} \int_0^\tau \int_\Omega \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] + \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} (\varrho_\varepsilon - 1) \partial_t s_{\varepsilon,\delta} \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$= \left[\int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon^{\beta/2}} \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \right]_{t=0}^{t=\tau},$$

where we used the weak formulation (2.2). Thus, we may infer that

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.10}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E}_\varepsilon \ \mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t + \left[\int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon^{\beta/2}} \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \right]_{t=0}^{t=\tau},$$

where

$$\int_0^\tau \int_\Omega \varrho_\varepsilon \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t$$
$$= \left[\int_\Omega \frac{1}{2} |\nabla_x \Psi_{0,\delta}|^2 \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t.$$

Making use of the AE system (4.1 - 4.3) we get

$$\int_0^\tau \int_\Omega (\varrho_\varepsilon - 1) \partial_t \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t = -\int_0^\tau \int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon^{\beta/2}} \nabla_x \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \cdot \nabla_x \Psi_{\varepsilon,\delta} \ \mathrm{d}x \ \mathrm{d}t \to 0$$

as $\varepsilon \to 0$.

Thus we have obtained

$$\int_\Omega \left[\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2} |\nabla_x \Phi_\varepsilon|^2 + \frac{\varepsilon^\beta}{2} \Phi_\varepsilon^2 \right] (\tau, \cdot) \ \mathrm{d}x \tag{5.11}$$

$$\leq \chi(\delta) + \frac{1}{2} \int_\Omega |s_0|^2 \ \mathrm{d}x + c \int_0^\tau \mathcal{E}_\varepsilon \ \mathrm{d}t$$
$$+ \left[\int_\Omega \frac{1}{2} |\nabla_x \Psi_{0,\delta}|^2 \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \left[\int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon^{\beta/2}} \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \right]_{t=0}^{t=\tau},$$

where by the properties of the energy functional (4.4) it holds that

$$\left[\int_\Omega \frac{1}{2} |\nabla_x \Psi_{0,\delta}|^2 \ \mathrm{d}x \right]_{t=0}^{t=\tau} = -\left[\int_\Omega \frac{1}{2} s_{\varepsilon,\delta} \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] \ \mathrm{d}x \right]_{t=0}^{t=\tau}.$$

Recalling the energy identity (4.4) and seeing that the expressions

$$\|s_{\varepsilon,\delta}\|_{W^{m,2}(\Omega)} \text{ and } \|z_{\varepsilon,\delta}\|_{W^{m,2}(\Omega)}, \ m = 0, 1, \ldots, \quad z_{\varepsilon,\delta} = (-\varepsilon^{\beta-2} + 1)^{-1} [s_{\varepsilon,\delta}],$$

are equivalent uniformly with respect to $t \in [0, T)$ and $\varepsilon \in (0, 1)$, we deduce from the identities

$$\int_\Omega \varepsilon^{2-\beta} |\nabla_x z_{\varepsilon,\delta}|^2 \ \mathrm{d}x + \int_\Omega |z_{\varepsilon,\delta}|^2 \ \mathrm{d}x = \int_\Omega s_{\varepsilon,\delta} z_{\varepsilon,\delta} \ \mathrm{d}x,$$

$$\int_\Omega \varepsilon^{2-\beta} \nabla_x z_{\varepsilon,\delta} \cdot \nabla_x s_{\varepsilon,\delta} \ \mathrm{d}x + \int_\Omega z_{\varepsilon,\delta} s_{\varepsilon,\delta} \ \mathrm{d}x = \int_\Omega |s_{\varepsilon,\delta}|^2 \ \mathrm{d}x$$

that

$$\int_\Omega \frac{1}{2} s_{\varepsilon,\delta}^2 (\tau, \cdot) \ \mathrm{d}x - \int_\Omega \frac{1}{2} s_{\varepsilon,\delta} \left(-\varepsilon^{2-\beta} \Delta_N + 1 \right)^{-1} [s_{\varepsilon,\delta}] (\tau, \cdot) \ \mathrm{d}x \to 0$$

and

$$\int_\Omega \frac{1}{2} s^2_{\varepsilon,\delta}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \frac{1}{2}\Big| \left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s_{\varepsilon,\delta}]\Big|^2(\tau,\cdot)\ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0$$

provided $\delta > 0$ is kept fixed. Using in addition condition (2.4), (2.10) (3.5), (3.6) we deduce

$$\frac{1}{2}\int_\Omega |s_0|^2\ \mathrm{d}x - \left[\int_\Omega \frac{1}{2} s_{\varepsilon,\delta}\left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+\left[\int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon^{\beta/2}}\left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s_{\varepsilon,\delta}]\ \mathrm{d}x\right]_{t=0}^{t=\tau} - \int_\Omega \frac{\varepsilon^\beta}{2}\Phi^2_\varepsilon(\tau,\cdot)\ \mathrm{d}x =$$

$$-\int_\Omega \frac{1}{2}|s_{\varepsilon,\delta}|^2(\tau,\cdot)\ \mathrm{d}x + \int_\Omega \varepsilon^{\beta/2}\Phi_\varepsilon s_{\varepsilon,\delta}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \frac{\varepsilon^\beta}{2}\Phi^2_\varepsilon(\tau,\cdot)\ \mathrm{d}x + \eta(\varepsilon,\delta)$$

$$= -\frac{1}{2}\int_\Omega \left|\varepsilon^{\beta/2}\Phi_\varepsilon - s_{\varepsilon,\delta}\right|^2(\tau,\cdot)\ \mathrm{d}x$$

$$+\int_\Omega \frac{1}{2} s^2_{\varepsilon,\delta}(\tau,\cdot)\ \mathrm{d}x - \int_\Omega \frac{1}{2} s_{\varepsilon,\delta}\left(-\varepsilon^{2-\beta}\Delta_N + 1\right)^{-1}[s_{\varepsilon,\delta}](\tau,\cdot)\ \mathrm{d}x + \eta(\varepsilon,\delta),$$

where $\eta(\varepsilon,\delta) \to \chi(\delta)$ as $\varepsilon \to 0$.

Consequently, we conclude that

$$\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon - \mathbf{U}_{\varepsilon,\delta}|^2 + (H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1)) + \frac{\varepsilon^2}{2}|\nabla_x\Phi_\varepsilon|^2 + \left|\varepsilon^{\beta/2}\Phi_\varepsilon - s_{\varepsilon,\delta}\right|^2\right](\tau,\cdot)\ \mathrm{d}x$$
$$\le \chi(\delta) + c\int_0^\tau \mathcal{E}_\varepsilon\ \mathrm{d}t. \tag{5.12}$$

Using Gronwall's lemma, we may perform the limit for $\varepsilon \to 0$ and then for $\delta \to 0$ to obtain the conclusion of Theorem 2.1.

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