

Guaranteed and robust error bounds for singularly perturbed problems in arbitrary dimension

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$$\begin{aligned} -\Delta u + \kappa^2 u &= f && \text{in } \Omega \subset \mathbb{R}^d \\ u &= g_N && \text{on } \partial\Omega \end{aligned} \quad \kappa > 0$$

- ▶ Arbitrary dimension
- ▶ Neumann boundary conditions
- ▶ Guaranteed and robust upper bound on error
- ▶ Guaranteed bounds of trace constants

M. Ainsworth, T. Vejchodský: *Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems*, Numer. Math. 119 (2), 2011, 219–243.

Model problem

Classical formulation:

$$\begin{aligned}
 -\Delta u + \kappa^2 u &= f \quad \text{in } \Omega \subset \mathbb{R}^d \\
 u &= g_N \quad \text{on } \partial\Omega \qquad \qquad \qquad \kappa > 0
 \end{aligned}$$

Weak formulation:

$$u \in V : \quad \mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V$$

Linear FEM on d -dimensional simplices:

$$u_h \in V_h : \quad \mathcal{B}(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h$$

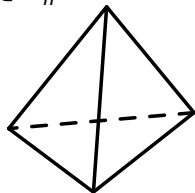
Notation:

$$V = H_0^1(\Omega)$$

$$\mathcal{B}(u, v) = (\nabla u, \nabla v)_\Omega + \kappa^2 (u, v)_\Omega$$

$$(f, v)_\Omega = \int_\Omega f v \, dx$$

$$V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$$



Main result

Upper bound:

$$\|u - u_h\|^2 \leq \sum_{K \in \mathcal{T}_h} [\eta_K(\mathbf{y}) + \text{osc}_K(f) + \text{osc}_{\partial\Omega \cap \partial K}(g_N)]^2$$

$$\forall \mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : \mathbf{y} \cdot \mathbf{n} = \Pi_\gamma^K g_N \text{ on all } \gamma \subset \partial\Omega \cap \partial K$$

Local efficiency:

$$\eta_K(\mathbf{y}) \leq C \left(\|u - u_h\|_{\tilde{K}} + \min\{h_K, \kappa^{-1}\} \|f - \Pi f\|_{\tilde{K}} + \min\{h_K, \kappa^{-1}\}^{1/2} \|g_N - \Pi_\gamma^K g_N\|_{\partial\Omega \cap \partial K} \right) \text{ for a special } \mathbf{y}$$

- ▶ $\eta_{\tilde{K}}^2(\mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_{0,K}^2 + \kappa^{-2} \|\Pi_K f - \kappa^2 u_h + \text{div } \mathbf{y}\|_{0,K}^2$
- ▶ $\text{osc}_K(f) = \min\{h_K/\pi, \kappa^{-1}\} \|f - \Pi_K f\|_{0,K}$
- ▶ $\text{osc}_{\partial\Omega \cap \partial K}(g_N) = \min\{C_T^K, C_T^{K,\kappa}\} \|g_N - \Pi_\gamma^K g_N\|_{\partial\Omega \cap \partial K}$
- ▶ $\Pi_K f \in P^1(K) : (f - \Pi_K f, \varphi)_K = 0 \quad \forall \varphi \in P^1(K)$
- ▶ $\Pi_\gamma^K g_N \in P^1(\gamma) : (f - \Pi_\gamma^K g_N, \varphi)_\gamma = 0 \quad \forall \varphi \in P^1(\gamma)$

- ▶ Compute robust inter-element fluxes g_K
 - ▶ $g_K \approx \nabla u \cdot \mathbf{n}_K$ on ∂K
[Ainsworth, Babuška, 1999], [Ainsworth, Vejchodský, 2011]
- ▶ For all elements K with $\kappa\rho_K \leq 1$ construct $\mathbf{y}_K^{(1)}$:
 - ▶ $\mathbf{y}_K^{(1)} \cdot \mathbf{n}_K = g_K$ on ∂K
 - ▶ $\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K^{(1)} = 0$
- ▶ For all elements K with $\kappa\rho_K > 1$ construct $\mathbf{y}_K^{(2)}$:
 - ▶ $\mathbf{y}_K^{(2)} \cdot \mathbf{n}_K = g_K$ on ∂K
 - ▶ Correct asymptotic behavior w.r.t. h and κ
- ▶ $\mathbf{y}|_K = \begin{cases} \mathbf{y}_K^{(1)} & \text{if } \kappa\rho_K \leq 1, \\ \mathbf{y}_K^{(2)} & \text{if } \kappa\rho_K > 1, \end{cases}$

Flux reconstruction #1

Definition:

$$\mathbf{y}_K^{(1)} = \nabla u_h + \mathbf{y}_K^L + \mathbf{y}_K^Q$$

$$\mathbf{y}_K^L = - \sum_{n=1}^{d+1} \lambda_n \sum_{\substack{m=1 \\ m \neq n}}^{d+1} R_{|\gamma_m}(\mathbf{x}_n) |\nabla \lambda_m| \mathbf{t}_{nm}$$

$$\mathbf{y}_K^Q = \frac{1}{d+1} \sum_{n=1}^{d+1} \sum_{\substack{m=2 \\ m > n}}^{d+1} \lambda_m \lambda_n \mathbf{t}_{mn} \mathbf{t}_{mn}^T \nabla r(\bar{\mathbf{x}}_K)$$

Notation:

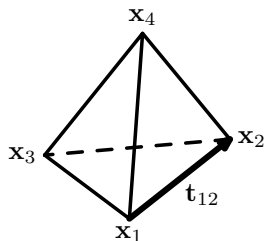
$$R = g_K - \nabla u_h \cdot \mathbf{n}_K$$

$$r = \Pi_K f - \kappa^2 u_h$$

$\lambda_n \dots$ barycentric coords in K

$\mathbf{t}_{mn} = \mathbf{x}_m - \mathbf{x}_n \dots$ edge vector

$\bar{\mathbf{x}}_K \dots$ barycentre of K



Definition:

$$\mathbf{y}_K^{(2)} = \nabla u_h + \mathbf{y}_K^O$$

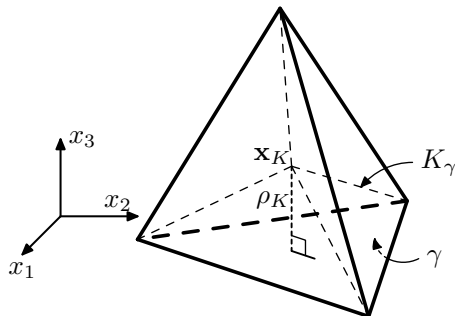
$$\mathbf{y}_K^O|_{K_\gamma} = \frac{1}{\rho_K} e^{-\kappa x_d} (\mathbf{x} - \mathbf{x}_K) R(x_1, \dots, x_{d-1}) \quad \text{for all } \gamma \subset \partial K$$

Notation:

$$R = g_K - \nabla u_h \cdot \mathbf{n}_K$$

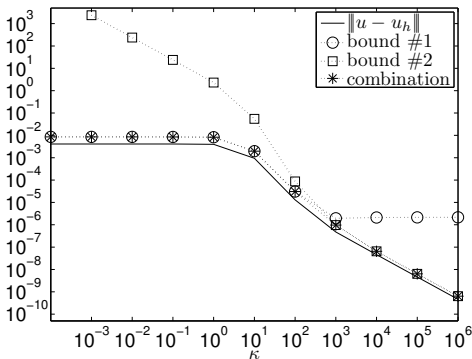
$\mathbf{x}_K \dots$ incentre of K

$\rho_K \dots$ inradius of K



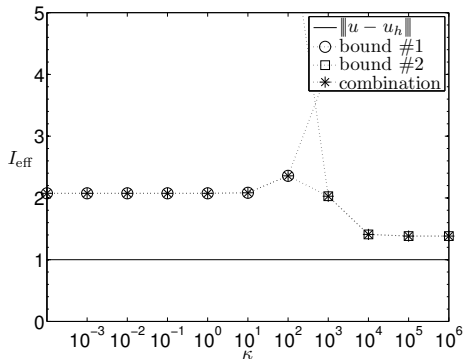
Example (cube)

Error estimators



$N_{\text{DOF}} = 29791$ $h = 0.03125$

Effectivity indices



$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$

Trace constants

Neumann oscillations:

$$\text{osc}_{\partial\Omega\cap\partial K}(g_N) = \min\{C_T^K, C_T^{K,\kappa}\} \|g_N - \Pi_\gamma^K g_N\|_{\partial\Omega\cap\partial K}$$

Trace theorems:

$$(1) \quad \|v - \bar{v}\|_{0,\partial K} \leq C_T^K \|\nabla v\|_{0,K} \quad \forall v \in H^1(K) \quad \bar{v} = \frac{1}{|\partial K|} \int_{\partial K} v \, dx$$

$$(2) \quad \|v\|_{0,\partial K} \leq C_T^{K,\kappa} (\|\nabla v\|_{0,K}^2 + \kappa^2 \|v\|_{0,K}^2)^{1/2} \quad \forall v \in H^1(K)$$

Connection to eigenvalues:

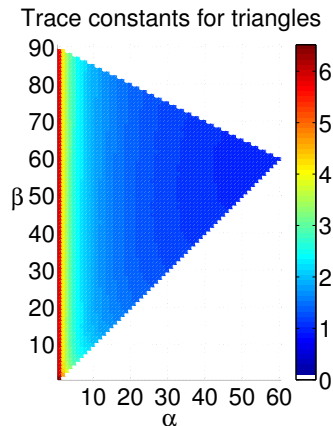
- ▶ $C_T^K = \lambda_2^{-1/2}$, where λ_2 is the smallest positive eigenvalue:
 $u_i \in H^1(\Omega) : (\nabla u_i, \nabla v)_\Omega = \lambda_i (u_i, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$
- ▶ $C_T^{K,\kappa} = \lambda_1^{-1/2}$, where λ_1 is the smallest eigenvalue:
 $u_i \in H^1(\Omega) : (\nabla u_i, \nabla v)_\Omega + \kappa^2 (u_i, v)_\Omega = \lambda_i (u_i, v)_{\partial\Omega} \quad \forall v \in H^1(\Omega)$

Two-sided bounds: [Šebestová, Vejchodský, preprint 2013]

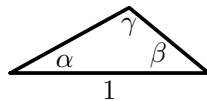
Trace constant (1) for triangles

$$\|v - \bar{v}\|_{0,\partial K} \leq C_T^K \|\nabla v\|_{0,K} \quad \forall v \in H^1(K) \quad \bar{v} = \frac{1}{|\partial K|} \int_{\partial K} v \, dx$$

Scaling: $C_T^K = h_K^{1/2} C_{T,h=1}^K$



- ▶ Upper bound
- ▶ Error 1%

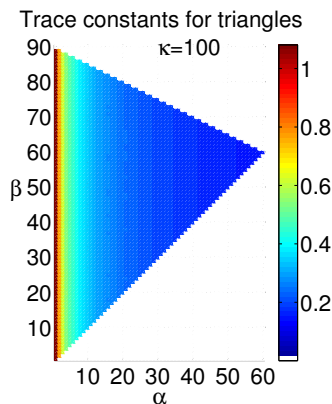
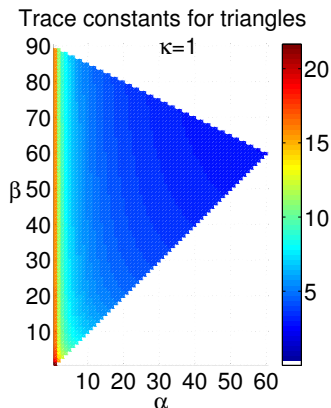


$$\alpha \leq \beta \leq \gamma$$

Trace constant (2) for triangles

$$\|v\|_{0,\partial K} \leq C_T^{K,\kappa} (\|\nabla v\|_{0,K}^2 + \kappa^2 \|v\|_{0,K}^2)^{1/2} \quad \forall v \in H^1(K)$$

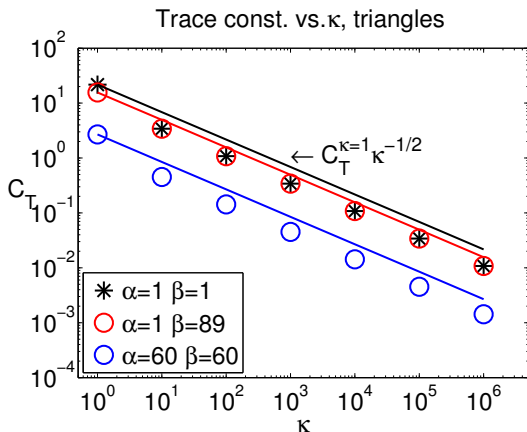
Scaling: $C_T^{K,\kappa} \leq \max\{1, \kappa^{-1}\} h_K^{1/2} C_T^{K,\kappa=1}$



Trace constant (2) for triangles

$$\|v\|_{0,\partial K} \leq C_T^{K,\kappa} (\|\nabla v\|_{0,K}^2 + \kappa^2 \|v\|_{0,K}^2)^{1/2} \quad \forall v \in H^1(K)$$

Numerical tests: $C_T^{K,\kappa} \leq \kappa^{-1/2} h_K^{1/2} C_T^{K,\kappa=1}$ for $\kappa > 1$



- ▶ Fast flux reconstruction
- ▶ Guaranteed upper bound on error
- ▶ Robust for all values of κ
- ▶ Arbitrary dimension
- ▶ Neumann b.c. require bounds for trace constants

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Thank you for your attention

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