INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

On the choice of functions spaces in the limit analysis for masonry bodies

Massimiliano Lucchesi
Miroslav Šilhavý
Nicola Zani

Preprint No. 11-2012

PRAHA 2012

ON THE CHOICE OF FUNCTIONS SPACES IN THE LIMIT ANALYSIS FOR MASONRY BODIES

MASSIMILIANO LUCCHESI, MIROSLAV ŠILHAVÝ, AND NICOLA ZANI

Abstract. The kinematic and static problems of limit analysis of no–tension bodies are formulated. The kinematic problem involves the infimum of kinematically admissible multipliers, and the static problem the supremum of statically admissible multipliers. The central question of the paper is under which conditions these two numbers coincide. This involves choices of function spaces for the competitor displacements and competitor stresses. A whole ordered scale of these spaces is presented. The mentioned problems are formulated as convex variational problems considered by Ekeland and Temam. The static problem is unconditionally shown to be the dual problem (in the sense of the mentioned reference) of the kinematic problem. A *necessary and sufficient condition,* the normality, guarantees that the kinematic and static problems give the same result. The normality is not always satisfied, as examples show (one of which is presented here). The qualification hypothesis of Ekeland and Temam, *sufficient* for the equality of the static and kinematic problems, is never satisfied in the spaces of admissible displacements of bounded deformation or of functions integrable together with the gradient in the power p, $1 \le p < \infty$. In the cases of lipschitzian displacements and of smooth displacements, the qualification hypothesis is equivalent to simple conditions that can be satisfied in the case of the pure traction problem. However, it is shown that then the space of admissible stresses must be enlarged to contain stressfields represented by finitely or countably additive tensor valued measures.

1. Introduction

No–tension (masonry–like) materials [3], [11], [5], [7], [14] cannot support all stresses: only negative semidefinite stresses are possible. Therefore, bodies made of no–tension materials cannot support all loads, certain loads lead to the collapse of the body. The goal of limit analysis is to determine the limit load, i.e., the largest possible load prior to collapse. It is customary to assume that the loads depend affinely on a scalar parameter λ, the loading multiplier, as described below in Section 2A, and the problem reduces to determining the collapse multiplier, i.e., the value of λ corresponding to the limit load. Limit analysis is traditionally based on the static and kinematic theorems, which determine the limit load as the supremum of statically admissible multipliers and the infimum of kinematically admissible multipliers, respectively. The traditional definition identifies the collapse multiplier as one with the collapse mechanism (called strong mechanism below in Section 2C). Under this assumption the supremum of statically admissible multipliers and the infimum of kinematically admissible multipliers are the same and coincide with the collapse multiplier. The reader is referred to [6] for the proofs of the static and kinematic theorems under this definition. There is no definition of the collapse multiplier in the present paper since the strong mechanism need not exist (cf. Example 7.6, below), and

Key words and phrases. Limit analysis, static and kinematic theorems, duality of displacements and forces.

This research was supported by the Regione of Toscana (project “Tools for modeling and assessing the structural behavior of ancient constructions: the NOSA-ITACA code”, PAR FAS 2007-2013). The research of M. Šilhavý was also supported by RVO: 67985840. The supports are gratefully acknowledged.

the mentioned supremum and infimum can be different, depending of the choice of the function spaces, as discussed below (cf. Example 7.12).

In this paper we call the definitions of the infimum and of the supremum mentioned above the kinematic and static problems of limit analysis of no–tension bodies. The kinematic problem involves the choice of the function space for admissible displacements. The static problem involves the choice of the function space for admissible stressfields. The central question of the paper is under which conditions these two numbers coincide. Various choices of the spaces are discussed in detail: for the space of displacements we consider the subspaces satisfying the null boundary condition on the fixed part of the boundary of (a) functions of bounded deformation, (b) the Sobolev spaces $W^{1,p}$, $1 \le p < \infty$, (c) the Sobolev space $W^{1,\infty}$ of lipschitzian displacements, and (d) the space C^1 of all smooth displacements on the closure of the body. Corresponding to these choices we are led to stress spaces which consist, respectively, of (a') the space of continuous functions, (b') the space L^q of stressfields that are integrable with the power q where q is the Hölder conjugate exponent of p, (c') the stressfields represented by finitely additive measures on the body that are absolutely continuous with respect to the Lebesgue measure, and (d') the stressfields represented by countably additive Borel measures on the closure of the body.

The mentioned problems are formulated as convex variational problems considered by Ekeland and Temam [8, Chapter III]. Following the similar application of the duality theory to the deformation (Hencky) theory of plasticity in Temam & Strang [26] and Temam [25], the static problem of the limit analysis is unconditionally shown to be the dual problem (in the sense of [8]) of the kinematic problem. The theory provides a *necessary and sufficient condition* for the primal and dual variational problems giving the same result, the *normality.* This is particularized to the static and kinematic problems of no–tension bodies. The condition can be applied with any choice of function spaces, and different choices lead to different results. Another condition, called *qualification hypothesis* in [8, Chapter III, Remark 2.4], provides a *sufficient condition.* However, normality is difficult to verify. The qualification hypothesis even cannot be satisfied, with some function spaces, by the no–tension material, at variance with the deformation theory of plasticity. This failure occurs in the spaces of admissible displacements as in (a) and (b), no matter how tame the loads. An example is presented in which the static and kinematic problems give different results. The necessary and sufficient condition and the qualification hypothesis can be satisfied with the choices (c) and (d). However, then the space of admissible stresses must be enlarged to contain measures as mentioned above.

Another application of the duality theory, different from the one employed in [25], is used to derive a simple condition for the static admissibility of a given multiplier.

In the introductions to the subsequent chapters we present brief outlines of the presented material. We also refer to the summary in Chapter 8 for a more detailed discussion of the results of the paper.

Throughout, we use the conventions for vectors and second order tensors identical with those in [12]. Thus Lin denotes the set of all second order tensors on $\mathbb{R}^n$, i.e., linear transformations from $\mathbb{R}^n$ into itself, Sym is the subspace of symmetric tensors, Sym^+ the set of all positive semidefinite elements of Sym; additionally Sym^- is the set of all negative semidefinite elements of Sym. The scalar product of $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ is defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin . If $\boldsymbol{A}$, $\boldsymbol{B} \in \mathrm{Sym}$, we write $\boldsymbol{A} \ge \boldsymbol{B}$ to say that $\boldsymbol{A} - \boldsymbol{B} \in \mathrm{Sym}^+$. We denote by $\mathbf{1}$ the unit tensor in Sym .

2. Abstract setting of the limit analysis for no–tension materials

This chapter starts in Section 2A with an informal introduction to the no–tension body and applied loads, i.e., surface tractions and body forces. The following section formulates the problem abstractly, with the energy of the loads as linear functionals on displacements and stresses as linear functionals on strains. Section 2C gives definitions of kinematically and statically admissible multipliers, of the infima and suprema thereof, of the collapse mechanism, and strong collapse mechanism. Section 2D reviews the duality theory of Ekeland and Temam. It presents the normality condition, which is equivalent to the equality of the results of the primal and dual problems, and a sufficient condition, the qualification hypothesis. Section 2E finally particularizes Section 2D to the kinematic and static problems of limit analysis of no–tension bodies. Proposition 2.5 gives a necessary condition for the equality of the results of the kinematic and static problems (i.e., for the normality) and Proposition 2.6 a sufficient condition (i.e., the qualification hypothesis). Finally Proposition 2.8 gives a necessary and sufficient condition for a multiplier to be statically admissible.

2A. **Loads and potential energy.** Let Ω be a reference configuration of a continuous body made of a no–tension material; it is assumed that Ω is a bounded connected open set in $\mathbb{R}^n$ (typically $n = 2$ or $n = 3$) with Lipschitz boundary $\partial\Omega$ in the sense of [8, Chapter X, Section 2.2], of outer normal $\boldsymbol{n}$. The body is fixed on an area measurable subset $\mathcal{D}$ of $\partial\Omega$ while on $\mathcal{S} := \partial\Omega \setminus \mathcal{D}$ the body is subjected to surface tractions depending on the loading multiplier as specified below. We consider displacements $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ from a Banach space of displacements V. Several choices of V are given below, and in all these choices it is meaningful to speak about the values of $\boldsymbol{v}$ on the boundary $\partial\Omega$, and in particular it is meaningful to require that

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \mathcal{D}, \tag{2-1}$$

either in the classical or in some generalized sense. We denote by W the set of all displacements from V which satisfy Condition (2-1).

We assume that the body is subjected to loads consisting of a surface traction on $\mathcal{S}$ and a body force in Ω; both the surface traction and the body force depend affinely on a real parameter λ called the *loading multiplier.* Thus if $\lambda \in \mathbb{R}$, the surface tractions $\boldsymbol{s}(\lambda) : \mathcal{S} \to \mathbb{R}^n$ and the body force $\boldsymbol{b}(\lambda) : \Omega \to \mathbb{R}^n$ are given by

$$\boldsymbol{s}(\lambda) = \boldsymbol{s}_\circ + \lambda\bar{\boldsymbol{s}}, \quad \boldsymbol{b}(\lambda) = \boldsymbol{b}_\circ + \lambda\bar{\boldsymbol{b}},$$

where $\boldsymbol{s}_\circ$ and $\bar{\boldsymbol{s}}$ are vector valued functions on $\mathcal{S}$ and $\boldsymbol{b}_\circ$ and $\bar{\boldsymbol{b}}$ are vector valued functions on Ω. The functions $\boldsymbol{s}_\circ$, $\bar{\boldsymbol{s}}$, $\boldsymbol{b}_\circ$, $\bar{\boldsymbol{b}}$ have to belong to appropriate spaces to make the discussion that follows meaningful. We call the pair $(\boldsymbol{s}(\lambda), \boldsymbol{b}(\lambda))$ the *loads corresponding to* λ, the pair $(\boldsymbol{s}_\circ, \boldsymbol{b}_\circ)$ the *permanent loads* and the pair $(\bar{\boldsymbol{s}}, \bar{\boldsymbol{b}})$ the *variable loads.* We define the *potential energy* $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$ of the loads corresponding to λ on a displacement $\boldsymbol{v}$ by

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = \langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle + \lambda \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \tag{2-2}$$

where

$$\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{b}_\circ \cdot \boldsymbol{v} \, d\mathcal{L}^n + \int_\mathcal{S} \boldsymbol{s}_\circ \cdot \boldsymbol{v} \, d\mathcal{H}^{n-1}, \tag{2-3}$$

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = \int_\Omega \bar{\boldsymbol{b}} \cdot \boldsymbol{v} \, d\mathcal{L}^n + \int_\mathcal{S} \bar{\boldsymbol{s}} \cdot \boldsymbol{v} \, d\mathcal{H}^{n-1}, \tag{2-4}$$

$\boldsymbol{v} \in W$, with $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ the volume and area measures, i.e., the Lebesgue measure and the $n - 1$ dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. Here the integrals are interpreted either classically or in an appropriate generalized sense to be precisely specified below.

We further define the infinitesimal strain tensor $\hat{\boldsymbol{E}}(\boldsymbol{v})$ of the displacement $\boldsymbol{v} \in W$ by

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \tfrac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}), \tag{2-5}$$

which will be either a function defined for almost all points of Ω, taking the values in the space Sym, or a Sym valued measure. Throughout, 'almost all,' 'almost every' and 'almost everywhere' always means with respect to the Lebesgue measure. It is then meaningful to consider the space W^+ of all displacements $\boldsymbol{v} \in W$ for which $\hat{\boldsymbol{E}}(\boldsymbol{v})$ is either a positive semidefinite tensor for almost every point of Ω or a measure taking values in the set of positive semidefinite tensors. This is a convex cone in W, i.e., we have the following implications:

$$\boldsymbol{v}, \boldsymbol{w} \in W^+ \quad \Rightarrow \quad \boldsymbol{v} + \boldsymbol{w} \in W^+,$$

$$\boldsymbol{v} \in W^+ \text{ and } t \in \mathbb{R}, t \geq 0 \quad \Rightarrow \quad t\boldsymbol{v} \in W^+.$$

We assume that for the selected space W, the strain tensors $\hat{\boldsymbol{E}}(\boldsymbol{v}) : \Omega \to \mathrm{Sym}$ belong to some Banach space Y, which will form either an appropriately chosen class of functions $\boldsymbol{F} : \Omega \to \mathrm{Sym}$, possibly defined only almost everywhere in Ω, or the space of Sym valued measures. We furthermore denote by Y^+ the convex cone of all $\boldsymbol{F} \in Y$ such that either $\boldsymbol{F}$ is positive semidefinite almost everywhere or a measure with values in the space of positive semidefinite tensors. Thus if $\boldsymbol{v} \in W^+$, then $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$.

Next, we consider the stressfields $\boldsymbol{T}$. In the classical cases, this will be a function from Ω to Sym, possibly defined only almost everywhere. We shall also consider stressfields $\boldsymbol{T}$ represented by a more general object, viz., a finitely additive or countably additive measure to be specified in the subsequent formal treatment. We denote by Y^* the linear space of all stressfields, which we assume to form a closed subspace of the Banach space Y' of all continuous linear functionals on Y, i.e., the dual of Y. We denote by $(\boldsymbol{T}, \boldsymbol{F})$ the pairing between elements $\boldsymbol{T} \in Y^*$ and elements $\boldsymbol{F} \in Y$, i.e., the value of the linear functional $\boldsymbol{T}$ on an element $\boldsymbol{F}$. In the classical case we have

$$(\boldsymbol{T}, \boldsymbol{F}) = \int_\Omega \boldsymbol{T} \cdot \boldsymbol{F} \, d\mathcal{L}^n; \tag{2-6}$$

the integral has to be interpreted in a generalized sense in the more general cases of Y and Y^*. The stressfields in the masonry bodies take negative semidefinite values. If $\boldsymbol{T} : \Omega \to \mathrm{Sym}$ is a negative semidefinite classical function, we have

$$(\boldsymbol{T}, \boldsymbol{F}) \leq 0 \quad \text{for all} \quad \boldsymbol{F} \in Y^+. \tag{2-7}$$

In the classical case we denote by Y^{*-} the set of all stressfields on Ω which take negative semidefinite values almost everywhere on Ω. When the stressfield $\boldsymbol{T} \in Y^*$ is not represented by a function, we define Y^{*-} as the set of all elements of Y^* which satisfy (2-7). (In the classical case this leads to the requirement posed previously.) We say that a stressfield $\boldsymbol{T}$ is Y^* admissible if $\boldsymbol{T} \in Y^{*-}$.

2B. **Abstract setting of the problem.** We summarize the essential features of the discussion in the preceding section. We consider general objects

$$W,\ \boldsymbol{l},\ \boldsymbol{l}_\circ,\ Y,\ Y^*,\ \hat{\boldsymbol{E}},\ Y^+,\ Y^{*-} \tag{2-8}$$

of the following nature:

(i) W is a Banach space;
(ii) $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are elements in the dual W^* of W;
(iii) Y is a Banach space and Y^* is a closed subspace of the dual Y' of Y;
(iv) $\hat{\boldsymbol{E}}$ is a continuous linear transformation from W to Y;

(v) Y^+ and Y^{*-} are closed convex cones in Y and in the dual Y^* of Y, respectively, which are mutually dual in the sense that

$$\{\boldsymbol{F} \in Y : (\boldsymbol{T}, \boldsymbol{F}) \le 0 \text{ for every } \boldsymbol{T} \in Y^{*-}\} = Y^+,$$

$$\{\boldsymbol{T} \in Y^* : (\boldsymbol{T}, \boldsymbol{F}) \le 0 \text{ for every } \boldsymbol{F} \in Y^+\} = Y^{*-}.$$

We interpret W as the set of displacement fields over Ω satisfying the kinematical constraint (2-1), and $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ as the energy functionals of the permanent and variable loads. We then define $\boldsymbol{l}(\lambda)$ by (2-2) for any $\lambda \in \mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between W^* and W; i.e., $\langle \boldsymbol{m}, \boldsymbol{v} \rangle \in \mathbb{R}$ denotes the value of the linear functional $\boldsymbol{m} \in W^*$ on an element $\boldsymbol{v} \in W$. The transformation $\hat{\boldsymbol{E}}(\cdot)$ associates with any displacement $\boldsymbol{v} \in W$ an element $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y$, which we interpret as the strain field of $\boldsymbol{v}$. We define W^+ by

$$W^+ = \{\boldsymbol{v} \in W : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+\}$$

and view W^+ as the set of all displacements with positive semidefinite strain tensor over Ω. We denote by $(\cdot, \cdot)$ the dual pairing between Y^* and Y, i.e., $(\boldsymbol{T}, \boldsymbol{F}) \in \mathbb{R}$ denotes the value of the linear functional $\boldsymbol{T} \in Y^*$ on the element $\boldsymbol{F} \in Y$. The stressfields are interpreted as the elements of Y^*, and Y^{*-} is interpreted as the set of all negative semidefinite stressfields. We say that $\boldsymbol{T} \in Y^*$ is an *admissible stressfield* if $\boldsymbol{T} \in Y^{*-}$; to emphasize the space of stresses, we sometimes say that $\boldsymbol{T}$ is Y^* admissible. We say that a stressfield $\boldsymbol{T} \in Y^*$ *equilibrates the loads corresponding to* λ if

$$(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$$

for every $\boldsymbol{v} \in W$.

In Sections 2C and 2E, below, we use the abstract setting described above. In the subsequent chapters we make concrete choices of the objects introduced here.

2C. **Kinematically and statically admissible multipliers.** We say that a multiplier λ is *kinematically admissible* if

$$\text{there exists a } \ \boldsymbol{v} \in W^+ \ \text{ such that } \ \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1 \ \text{ and } \ \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = 0. \tag{2-9}$$

The last notion depends on the choice of the space W, and to emphasize this, we will sometimes say that λ is W kinematically admissible. We denote the set of all kinematically admissible multipliers by $\bar{\Lambda}$. If λ is given, we call the element $\boldsymbol{v}$ as in (2-9) a *mechanism corresponding to* λ.

We say that a multiplier $\lambda \in \mathbb{R}$ is *statically admissible* if there exists an admissible stressfield $\boldsymbol{T}$ which equilibrates the loads corresponding to λ. We shall sometimes say that λ is Y^* statically admissible. We denote by Λ the set of all statically admissible multipliers.

The sets $\bar{\Lambda}$ and Λ are intervals, possibly empty, singletons, bounded, unbounded, open, or semiopen. To see, e.g., that $\bar{\Lambda}$ is an interval, we note that if $\lambda, \mu \in \bar{\Lambda}$ and $s \ge 0, t \ge 0, s + t = 1$, then also $s\lambda + t\mu \in \bar{\Lambda}$ since if $\boldsymbol{v}, \boldsymbol{w} \in W^+$ denote mechanisms corresponding to λ and μ, respectively, then $s\boldsymbol{v} + t\boldsymbol{w} \in W^+$ is a mechanism corresponding to $s\lambda + t\mu \in \bar{\Lambda}$. That Λ is an interval is proved similarly.

The interval Λ is situated to the left of the interval $\bar{\Lambda}$ in the sense that

$$\text{if } \ \lambda \in \Lambda \ \text{ and } \ \mu \in \bar{\Lambda} \ \text{ then } \ \lambda \le \mu. \tag{2-10}$$

Hence the intersection of Λ and $\bar{\Lambda}$ can contain at most one point. To prove (2-10), we note that if μ is kinematically admissible and $\boldsymbol{v}$ is a mechanism corresponding to it, then $\mu = -\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle$ while if λ is statically admissible and $\boldsymbol{T} \in Y^{*-}$ a corresponding stressfield, then $0 \ge (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$ and hence $\lambda \le -\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle$, which gives (2-10).

Central to our considerations are the numbers (or the symbols ∞ and $-\infty$)

$$\bar{\lambda}_W := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is } W \text{ kinematically admissible}\} \tag{2-11}$$

and

$$\lambda_{Y^*} := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is } Y^* \text{ statically admissible}\}. \tag{2-12}$$

We call (2-11) the *kinematic problem* and (2-12) the *static problem.* We furthermore call $\bar{\lambda}_W$ the *critical multiplier of the kinematic problem* and λ_{Y^*} the *critical multiplier of the static problem.* Implication (2-10) gives

$$\lambda_{Y^*} \le \bar{\lambda}_W. \tag{2-13}$$

We examine conditions under which we have the equality in (2-13). Example 7.12, below, shows that under common choices of function spaces and under bounded and piecewise continuous loads we have the strict inequality sign. In the following section we treat the equality in (2-13) by an application of the duality theory of Ekeland & Temam [8].

We close this section with a simple sufficient condition for the equality in (2-13). We say that the multiplier $\lambda_c \in \mathbb{R}$ *admits a strong mechanism* if there exists a statically admissible stressfield $\boldsymbol{T}_c \in Y^{*-}$ corresponding to λ_c and a $\boldsymbol{v}_c \in W$ such that

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v}_c \rangle = 1 \quad \text{and} \quad (\boldsymbol{T} - \boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0 \quad \text{for every} \quad \boldsymbol{T} \in Y^{*-}.$$

Proposition 2.1. *If λ_c admits a strong mechanism $\boldsymbol{v}_c$, then*

$$\lambda_{Y^*} = \bar{\lambda}_W = \lambda_c \tag{2-14}$$

and $\boldsymbol{v}_c$ is a corresponding mechanism.

Proof. The definition requires that λ_c be statically admissible and hence

$$\lambda_c \le \lambda_{Y^*}. \tag{2-15}$$

We have $(\boldsymbol{T} - \boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0$ for every $\boldsymbol{T} \in Y^{*-}$; replacing $\boldsymbol{T}$ by $t\boldsymbol{T}$ where $t > 0$, we obtain another element of Y^{*-} and hence $(t\boldsymbol{T} - \boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0$; dividing by t and letting $t \to \infty$ we thus obtain $(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0$, which by the assumed duality of the cones Y^+ and Y^{*-} stated in the preceding section implies that $\hat{\boldsymbol{E}}(\boldsymbol{v}_c) \in Y^+$, and hence $\boldsymbol{v}_c \in W^+$. Setting $\boldsymbol{T} = \boldsymbol{0}$ in the inequality $(\boldsymbol{T} - \boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0$ we obtain $(\boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \ge 0$ and as also $\boldsymbol{T}_c \in Y^{*-}$ and $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in Y^+$, we have $(\boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) \le 0$ and hence $(\boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) = 0$. Since $\boldsymbol{T}_c$ balances the loads corresponding to λ_c, we have

$$0 = (\boldsymbol{T}_c, \hat{\boldsymbol{E}}(\boldsymbol{v}_c)) = \langle \boldsymbol{l}(\lambda_c), \boldsymbol{v}_c \rangle$$

and since $\langle \bar{\boldsymbol{l}}, \boldsymbol{v}_c \rangle = 1$ as part of the definition of the strong mechanism, we see that λ_c is kinematically admissible and $\boldsymbol{v}_c$ a mechanism corresponding to it. Thus

$$\bar{\lambda}_W \le \lambda_c. \tag{2-16}$$

Combining Inequalities (2-15), (2-16) with (2-13) we obtain (2-14). Then $\boldsymbol{v}_c$ is a corresponding mechanism. □

Remark 2.2. In [6, IV.1], a definition is given of collapse mechanism for the general case of a normal linear material (of which the no–tension material is a special case) which eventually leads to the properties embodied in the present definition of strong mechanism. Indeed, as a consequence of [6, Definition IV.1] and of the assumptions of the kinematic theorem in [6], the collapse mechanism in the sense of [6] is assumed to exist. If we denote it by $\bar{\boldsymbol{v}}_{\rm c}$, it satisfies

$$\langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_{\rm c}\rangle > 0,$$

$$(\boldsymbol{T} - \boldsymbol{T}_{\rm c}, \hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_{\rm c})) \le 0 \ \text{ for every } \ \boldsymbol{T} \in Y^{*-}.$$

(Cf. the text between Equations (42) and (43), and the second sentence after Equation (37), respectively.) It then follows that $\boldsymbol{v}_{\rm c} := \bar{\boldsymbol{v}}_{\rm c}/\langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_{\rm c}\rangle$ is a strong mechanism in the present sense. Thus [6, Definition IV.1] of collapse mechanism is more restrictive than the present definition of strong mechanism and hence Proposition 2.1 covers all cases treated in the version of the kinematic theorem in [6].

We consider the assumption of the existence of collapse mechanism as too restrictive. Indeed, in Example 7.6 (below) we present loads which satisfy (2-14) and yet there is no collapse mechanism in the sense of [6, Definition IV.1] or strong mechanism in the present sense. In the subsequent treatment we seek to prove (2-14) under more general assumptions.

2D. **Primal and dual variational problems of convex analysis.** We here outline the duality theory for convex variational problems developed in [8, Chater III].

Consider a variational problem [8, Chapter III, Remark 4.2]

$$\bar{J} = \inf\{C(\boldsymbol{v}) + D(\hat{\boldsymbol{E}}(\boldsymbol{v})) : \boldsymbol{v} \in W\} \tag{2-17}$$

where $C : W \to \mathbb{R} \cup \{\infty\}$, $D : Y \to \mathbb{R} \cup \{\infty\}$ are general convex functions on Banach spaces W, Y and $\hat{\boldsymbol{E}}(\cdot) : W \to Y$ is a general bounded linear transformation. We call (2-17) the *primal problem.*

The *dual problem* is defined by

$$J = \sup\{-C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) - D^*(\boldsymbol{T}) : \boldsymbol{T} \in Y^*\} \tag{2-18}$$

where W^* is the dual of W and Y^* is a closed subspace of the dual space Y' of Y, $C^* : W^* \to \mathbb{R}\cup\{\infty\}$, $D^* : Y^* \to \mathbb{R}\cup\{\infty\}$ are the convex conjugates of C, D, respectively, and $\hat{\boldsymbol{E}}^* : Y^* \to W^*$ is the adjoint transformation of $\hat{\boldsymbol{E}}$. The convex conjugate functions are defined by

$$\boldsymbol{C}^*(\boldsymbol{m}) = \sup\{\langle \boldsymbol{m}, \boldsymbol{v}\rangle - C(\boldsymbol{v}) : \boldsymbol{v} \in W\},$$

$$D^*(\boldsymbol{T}) = \sup\{(\boldsymbol{T}, \boldsymbol{F}) - D(\boldsymbol{F}) : \boldsymbol{F} \in Y\}$$

for each $\boldsymbol{m} \in W^*$, $\boldsymbol{T} \in Y^*$, and the adjoint $\hat{\boldsymbol{E}}^*$ is a linear transformation defined by the relation

$$(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \hat{\boldsymbol{E}}^*\boldsymbol{T}, \boldsymbol{v}\rangle$$

for each $\boldsymbol{v} \in W$, $\boldsymbol{T} \in Y^*$.

We assume that the functions C, D are proper, i.e., each of them is less than ∞ somewhere and bigger than $-\infty$ everywhere. One has generally

$$-\infty \le J \le \bar{J} \le \infty.$$

Let $H : Y \to \mathbb{R} \cup \{-\infty, \infty\}$ be defined by

$$H(\boldsymbol{F}) = \inf\{C(\boldsymbol{v}) + D(\hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F}) : \boldsymbol{v} \in W\}, \tag{2-19}$$

$\boldsymbol{F} \in Y$, so that $H(\mathbf{0}) = \bar{J}$. The function H is convex. The problem (2-17) is said to be *normal* if $H(\mathbf{0})$ is finite and H is lowersemicontinuous at $\mathbf{0}$.

Proposition 2.3. *The following conditions are equivalent:*
(i) *Problem* (2-17) *is normal;*
(ii) *one has*

$$J = \bar{J} \tag{2-20}$$

and this number is finite.

The problem (2-17) is said to satisfy the *qualification hypothesis* if there exists a $\boldsymbol{v}_\circ \in W$ such that

$$C(\boldsymbol{v}_\circ) < \infty, \; D(\hat{\boldsymbol{E}}(\boldsymbol{v}_\circ)) < \infty \quad \text{and} \quad D \quad \text{is continuous at} \quad \hat{\boldsymbol{E}}(\boldsymbol{v}_\circ). \tag{2-21}$$

See [8, Chapter III, Remark 2.4].

Proposition 2.4. *If Problem* (2-17) *satisfies the qualification hypothesis, then the following two assertions hold:*
(i) *we have* (2-20)*;*
(ii) *if the number $J = \bar{J}$ is finite, then the dual problem has a solution, i.e., there exists a $\boldsymbol{T} \in Y^*$ such that*

$$J = -C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) - D^*(\boldsymbol{T}).$$

We emphasize that the qualification hypothesis is sufficient for (2-20) but not necessary, the necessary and sufficient condition is the normality.

2E. **Primal and dual variational problems of limit analysis.** The definition (2-11) of $\bar{\lambda}_W$ is easily seen to be equivalent to

$$\bar{\lambda}_W = \inf\{-\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle : \boldsymbol{v} \in W^+, \, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\}. \tag{2-22}$$

Proposition 2.5. *Let $H : Y \to \mathbb{R} \cup \{-\infty, \infty\}$ be given by*

$$H(\boldsymbol{F}) = \inf\{-\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle : \boldsymbol{v} \in W, \, \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+, \, \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\}, \tag{2-23}$$

$\boldsymbol{F} \in Y$. Then
(i) *H is convex and nondecreasing in the sense that $H(\boldsymbol{F}) \le H(\boldsymbol{G})$ whenever $\boldsymbol{G} - \boldsymbol{F} \in Y^+$;*
(ii) *we have*

$$\lambda_{Y^*} = \bar{\lambda}_W \in \mathbb{R}$$

if and only if $H(\mathbf{0})$ is finite and H is lowersemicontinuous at $\mathbf{0}$.

Here the finiteness of $H(\mathbf{0})$ and the lowersemicontinuity of H at $\mathbf{0}$ is the normality condition for Problem (2-22).

Proof. Problem (2-22) can be rewritten as

$$\bar{\lambda}_W = \inf\{C(\boldsymbol{v}) + D(\hat{\boldsymbol{E}}(\boldsymbol{v})) : \boldsymbol{v} \in W\} \tag{2-24}$$

where

$$C : W \to \mathbb{R} \cup \{\infty\}, \quad D : Y \to \mathbb{R} \cup \{\infty\}$$

are the functions defined by

$$C(\boldsymbol{v}) = \begin{cases} -\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle & \text{if } \boldsymbol{v} \in W \text{ and } \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1, \\ \infty & \text{if } \boldsymbol{v} \in W \text{ and } \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle \ne 1, \end{cases} \tag{2-25}$$

$$D(\boldsymbol{F}) = \begin{cases} 0 & \text{if } \boldsymbol{F} \in Y^+ \,, \\ \infty & \text{if } \boldsymbol{F} \in Y \setminus Y^+. \end{cases} \tag{2-26}$$

The dual problem reads

$$\lambda_{Y^*} = \sup\{\lambda \in \mathbb{R} : \lambda \text{ is statically admissible }\}. \tag{2-27}$$

To prove the last statement, calculate C^* and D^*. If $\boldsymbol{m} \in W^*$, then

$$C^*(\boldsymbol{m}) = \sup\{\langle \boldsymbol{m} + \boldsymbol{l}_\circ, \boldsymbol{v}\rangle : \boldsymbol{v} \in W, \langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1\}$$

and since this supremum is finite if and only if $\boldsymbol{m} + \boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are parallel, say $\boldsymbol{m} + \boldsymbol{l}_\circ = -\lambda\bar{\boldsymbol{l}}$ for some $\lambda \in \mathbb{R}$, and then $\langle \boldsymbol{m} + \boldsymbol{l}_\circ, \boldsymbol{v}\rangle = -\lambda$, we have

$$C^*(\boldsymbol{m}) = \begin{cases} -\lambda & \text{if there exists a } \lambda \in \mathbb{R} \text{ such that} \\ & \langle \boldsymbol{m}, \boldsymbol{v}\rangle + \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle = 0 \text{ for every } \boldsymbol{v} \in W, \\ \infty & \text{otherwise,} \end{cases} \tag{2-28}$$

$\boldsymbol{m} \in W^*$. Letting $\boldsymbol{T} \in Y^*$, setting $\boldsymbol{m} = -\hat{\boldsymbol{E}}^*\boldsymbol{T}$ and noting that the finite regime in (2-28) occurs if and only if $\boldsymbol{T}$ equilibrates the loads corresponding to λ, in the sense that $\big(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})\big) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle$ for each $\boldsymbol{v} \in W$, we obtain

$$C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) = \begin{cases} -\lambda & \text{if } \boldsymbol{T} \text{ equilibrates the loads corresponding to } \lambda, \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, the definition (2-26) of D^* and the fact that Y^+ is a cone implies that

$$D^*(\boldsymbol{T}) = \begin{cases} 0 & \text{if } \boldsymbol{T} \in Y^{*-} \\ \infty & \text{if } \boldsymbol{T} \in Y^* \setminus Y^{*-} \end{cases}$$

$\boldsymbol{T} \in Y^*$. Thus the right hand side of the dual variational problem (2-18) is

$$-C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) - D^*(\boldsymbol{T}) = \begin{cases} \lambda & \text{if } \boldsymbol{T} \text{ equilibrates the loads corresponding to } \lambda \\ & \text{and } \boldsymbol{T} \in Y^{*-}, \\ -\infty & \text{otherwise,} \end{cases}$$

and (2-27) follows.

The function H of (2-19) is given by (2-23).

(i): H is convex (as stated generally in Section 2D). To prove the nondecreasing character of H, it suffices to note that if $\boldsymbol{G} - \boldsymbol{F} \in Y^+$ then

$$\{\boldsymbol{v} \in W,\, \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{G} \in Y^+,\, \langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1\} \subset \{\boldsymbol{v} \in W,\, \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+,\, \langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1\}.$$

(ii): This follows from Proposition 2.3. □

Proposition 2.6. *Assume that*

$$\left.\begin{array}{c} \textit{there exists a } \bar{\boldsymbol{v}}_\circ \textit{ in } W^+ \textit{ satisfying } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_\circ\rangle > 0 \\ \textit{such that } \hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_\circ) \textit{ is an interior point of } Y^+. \end{array}\right\} \tag{2-29}$$

Then

$$\lambda_{Y^*} = \bar{\lambda}_W; \tag{2-30}$$

if, additionally, the number $\lambda_{Y^*} = \bar{\lambda}_W$ *is finite, then* $\bar{\lambda}_W$ *is statically admissible.*

Condition (2-29) is the qualification hypothesis for Problem (2-22).

Proof. Consider Problem (2-24) with C and D given by (2-25) and (2-26). This problem satisfies the qualification hypothesis if and only if there exists a $\bar{\boldsymbol{v}}_\circ$ as in (2-29). Indeed, assume that the problem satisfies the qualification hypothesis. Then there exists a point $\bar{\boldsymbol{v}}_\circ$ such that (2-21) hold; the first of these three conditions and the definition of C gives that $\langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_\circ \rangle = 1$; the second of these conditions and the definition of D gives $\bar{\boldsymbol{v}}_\circ \in W^+$, and the third condition gives that D is finite in some neighborhood of $\hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_\circ)$. Then $\bar{\boldsymbol{v}}_\circ$ is as in (2-29). Conversely, if Condition (2-29) holds, then the point $\boldsymbol{v}_\circ := \bar{\boldsymbol{v}}_\circ / \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_\circ \rangle$ satisfies (2-21).

Then the assertion of the proposition follows from Proposition 2.4. □

Remark 2.7. Condition (2-29) of Proposition 2.6 is never satisfied if one uses displacements from the space BD of functions of bounded deformation or if one uses displacements from the Sobolev space $W^{1,p}$ of functions integrable together with the gradient in the power p where $1 \le p < \infty$. Indeed, we shall see that then the cone Y^+ has empty interior, see Chapters 3 and 4. The cone Y^+ has nonempty interior if one uses lipschitzian displacements or continuously differentiable displacements on the closure of Ω, see Chapters 5 and 6. However, we shall see that in the case of continuously differentiable displacements Condition (2-29) can be satisfied essentially only in the pure traction problem, when $\mathcal{D} = \emptyset$. Condition (2-29) is only sufficient for (2-30); a necessary and sufficient condition is provided by Proposition 2.5: the finiteness of $H(\mathbf{0})$ and lowersemicontinuity of H at $\mathbf{0}$. The last condition is difficult to verify in concrete cases. Only in the cases of lipschitzian displacements or of continuously differentiable displacements, the lowersemicontinuity of H at $\mathbf{0}$ reduces to verifying lower semicontinuity of a real function of real variable. The lowersemicontinuity frequently holds even when Condition (2-29) fails to hold, because in concrete cases of loads we often have the equality (2-30).

We now address the problem of statical admissibility of a given multiplier $\lambda \in \mathbb{R}$.

Proposition 2.8. *A multiplier λ is Y^* statically admissible if and only if there exists a $c < \infty$ such that*

$$\sup\{\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle : \boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+\} \le c|\boldsymbol{F}| \tag{2-31}$$

for every $\boldsymbol{F} \in Y$ where $|\cdot|$ denotes the norm on Y.

Condition (2-31) is the normality of the problem of static admissibility of a given λ. This condition will be employed in Example 7.6. Clearly (2-31) implies that

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0 \tag{2-32}$$

for every $\boldsymbol{v} \in W^+$. However, (2-32) does not suffice for the static admissibility of λ, see Example 7.12 and Remark 7.15, below. Inequality (2-31) says, roughly, that if we allow displacements with slightly negative strain, then $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$ can become positive, but not too much.

Proof. Consider the problem

$$\bar{I} = \inf\{-\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle : \boldsymbol{v} \in W^+\}. \tag{2-33}$$

This problem takes the form

$$\bar{I} = \inf\{C(\boldsymbol{v}) + D(\hat{\boldsymbol{E}}(\boldsymbol{v})) : \boldsymbol{v} \in W\} \tag{2-34}$$

with

$$C(\boldsymbol{v}) = -\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle,$$
$$D(\boldsymbol{F}) = \begin{cases} 0 & \text{if } \boldsymbol{F} \in Y^+, \\ \infty & \text{otherwise,} \end{cases}$$

$\boldsymbol{v} \in W$, $\boldsymbol{F} \in Y$, and with $\hat{\boldsymbol{E}}(\cdot)$ the small strain mapping. Problem (2-34) reads explicitly

$$\bar{I} = \begin{cases} 0 & \text{if } \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0 \text{ for every } \boldsymbol{v} \in W^+, \\ -\infty & \text{otherwise.} \end{cases} \tag{2-35}$$

Indeed, if $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for every $\boldsymbol{v} \in W^+$ then the infimum in (2-33) is taken over the set of nonnegative numbers and thus $\bar{I} \ge 0$; on the other hand, setting $\boldsymbol{v} = \boldsymbol{0}$ in (2-33) we obtain $\bar{I} \le 0$ and thus we have the first regime in (2-35). In the second regime we have $\langle \boldsymbol{l}(\lambda), \bar{\boldsymbol{v}} \rangle > 0$ for some $\bar{\boldsymbol{v}} \in W^+$; setting $\boldsymbol{v} = s\bar{\boldsymbol{v}}$ where $s > 0$ in (2-33) we obtain

$$\bar{I} \le -s\langle \boldsymbol{l}(\lambda), \bar{\boldsymbol{v}} \rangle;$$

as this must be satisfied for all $s > 0$; we have the value asserted by the second regime in (2-35).

Determine the dual of (2-34). We have

$$C^*(\boldsymbol{m}) = \begin{cases} 0 & \text{if } \boldsymbol{m} = -\boldsymbol{l}(\lambda), \\ \infty & \text{otherwise,} \end{cases} \tag{2-36}$$

$\boldsymbol{m} \in W^*$, and

$$D^*(\boldsymbol{T}) = \begin{cases} 0 & \text{if } \boldsymbol{T} \in Y^{*-}, \\ \infty & \text{otherwise,} \end{cases}$$

$\boldsymbol{T} \in Y^*$. Setting $\boldsymbol{m} = -\hat{\boldsymbol{E}}^*\boldsymbol{T}$ in (2-36), we obtain

$$C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) = \begin{cases} 0 & \text{if } \hat{\boldsymbol{E}}^*\boldsymbol{T} = \boldsymbol{l}(\lambda), \\ \infty & \text{otherwise,} \end{cases}$$

$\boldsymbol{T} \in Y^*$. Then

$$-C^*(-\hat{\boldsymbol{E}}^*\boldsymbol{T}) - D^*(\boldsymbol{T}) = \begin{cases} 0 & \text{if } \hat{\boldsymbol{E}}^*\boldsymbol{T} = \boldsymbol{l}(\lambda) \text{ and } \boldsymbol{T} \in Y^{*-}, \\ -\infty & \text{otherwise,} \end{cases}$$

$\boldsymbol{T} \in Y^*$. Noting that the conditions $\hat{\boldsymbol{E}}^*\boldsymbol{T} = \boldsymbol{l}(\lambda)$ and $\boldsymbol{T} \in Y^{*-}$ mean exactly that λ is Y^* statically admissible, we have the following: the dual problem of (2-34) reads

$$I = \begin{cases} 0 & \text{if } \lambda \text{ is } Y^* \text{ statically admissible,} \\ -\infty & \text{otherwise.} \end{cases} \tag{2-37}$$

The function $H : Y \to \mathbb{R} \cup \{-\infty, \infty\}$ of (2-19) corresponding to Problem (2-34) is given by

$$H(\boldsymbol{F}) = \inf\{-\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle : \boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+\};$$

the function H has the following properties:

(i) H is convex;
(ii) H is nondecreasing in the sense that $H(\boldsymbol{F}) \le H(\boldsymbol{G})$ whenever $\boldsymbol{G} - \boldsymbol{F} \in Y^+$;
(iii) $H(\boldsymbol{F}) \le 0$ if $-\boldsymbol{F} \in Y^+$;
(iv) $H(s\boldsymbol{F}) = sH(\boldsymbol{F})$ for every $\boldsymbol{F} \in Y$ and $s > 0$;
(v) either $H(\boldsymbol{0}) = 0$ or $H(\boldsymbol{0}) = -\infty$.

Indeed, (i) follows from the convexity of the general H stated in Section 2D. (ii) follows from the fact that if $\boldsymbol{G} - \boldsymbol{F} \in Y^+$ then

$$\{\boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{G} \in Y^+\} \subset \{\boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+\}.$$

(iii) follows from the fact that if $-\boldsymbol{F} \in Y^+$ then

$$\boldsymbol{0} \in \{\boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+\}.$$

(iv) follows from the equation

$$\{\boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - s\boldsymbol{F} \in Y^+\} = s\{\boldsymbol{v} \in W,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) - \boldsymbol{F} \in Y^+\}.$$

(v) is proved by noting that $H(\mathbf{0}) \le 0$ by (iii) and $H(\mathbf{0}) = sH(\mathbf{0})$ for every $s > 0$ by (iv).

By definition, Problem (2-34) is normal if and only if $H(\mathbf{0}) = 0$ and H is lowersemicontinuous at $\mathbf{0}$. By Proposition 2.3 the normality is equivalent to $I = \bar{I} \in \mathbb{R}$ and this in turn is equivalent to the statical admissibility of λ. Thus, to prove that (2-31) is equivalent to the statical admissibility of λ, we have to show that the conditions $H(\mathbf{0}) = 0$ and H is lowersemicontinuous at $\mathbf{0}$ are equivalent to (2-31).

Indeed, let $H(\mathbf{0}) = 0$ and let H be lowersemicontinuous at $\mathbf{0}$. Then H is bounded from below on the unit ball in Y in the sense that there exists a $c > 0$ such that

$$H(\boldsymbol{F}) \ge -c \ \text{ for all } \boldsymbol{F} \in Y \text{ with } |\boldsymbol{F}| \le 1.$$

The positive homogeneity of H asserted in (iv) then implies

$$H(\boldsymbol{F}) \ge -c|\boldsymbol{F}| \ \text{ for all } \boldsymbol{F} \in Y \tag{2-38}$$

and the definition of H gives (2-31). Conversely, if (2-31) holds then we have (2-38) and this in turn implies that $H(\mathbf{0}) \ge 0$. As by (v) we have $H(\mathbf{0}) \le 0$, this implies $H(\mathbf{0}) = 0$, and (2-38) gives that H is lowersemicontinuous at $\mathbf{0}$. □

Proposition 2.9. *Let $\lambda \in \mathbb{R}$. If there exists a $\boldsymbol{v}_\circ \in W^+$ such that $\hat{\boldsymbol{E}}(\boldsymbol{v}_\circ)$ is an interior point of Y^+ then λ is Y^* statically admissible if and only if $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for all $\boldsymbol{v} \in W^+$.*

The hypothesis of this proposition is the qualification condition for the problem of the static admissibility of λ.

Proof. Problem (2-33) satisfies the qualification hypothesis if and only if there exists a $\boldsymbol{v}_\circ \in W^+$ such that $\hat{\boldsymbol{E}}(\boldsymbol{v}_\circ)$ is an interior point of Y^+. This is proved in the same way as in Proposition 2.6. Under the qualification hypothesis we have the equality $I = \bar{I}$ by Proposition 2.4 and the explicit forms (2-35) and (2-37) of the primal and dual problems show that λ is Y^* statically admissible if and only if $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for all $\boldsymbol{v} \in W^+$. □

3. Displacements in BD and continuous stresses

This chapter presents the kinematic and static problems with the choice of displacements of bounded deformation and with the choice of continuous tensorfields as the space of stresses. The qualification hypothesis as a sufficient condition for the equality of the results of the kinematic and static problems can never be satisfied with this choice.

3A. **Displacements, loads, and energies in the setting of BD.** We define the objects (2-8) as follows. We put

$$W = W_{BD} := \{\boldsymbol{v} \in BD(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ in the sense of trace on } \mathcal{D}\},$$

$$Y = \mathcal{M}(\Omega, \mathrm{Sym}),$$

so that $\hat{\boldsymbol{E}}(\cdot)$, defined by (2-5), is a bounded linear transformation from W_{BD} to $\mathcal{M}(\Omega, \mathrm{Sym})$. (We refer to Chapter 9, below, for the outline of the notation for the functions spaces employed here and in the subsequent treatment.) Furthermore, we put

$$Y^+ = \mathcal{M}(\Omega, \mathrm{Sym}^+) := \{\boldsymbol{F} \in \mathcal{M}(\Omega, \mathrm{Sym}) : \boldsymbol{F}(A) \ge \mathbf{0} \text{ for every Borel subset of } \Omega\},$$

$$W^+ = W^+_{BD} := \{\boldsymbol{v} \in W_{BD} : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in \mathcal{M}(\Omega, \mathrm{Sym}^+)\}.$$

Finally, we set

$$Y^* = C^0(\operatorname{cl}\Omega, \mathrm{Sym})$$

and the duality pairing is given by

$$(\boldsymbol{T}, \boldsymbol{F}) = \int_\Omega \boldsymbol{T} \cdot d\boldsymbol{F}$$

for every $\boldsymbol{T} \in C^0(\operatorname{cl}\Omega, \mathrm{Sym})$ and every $\boldsymbol{F} \in \mathcal{M}(\Omega, \mathrm{Sym})$. The cone Y^{*-} is given by

$$Y^{*-} = C^0(\operatorname{cl}\Omega, \mathrm{Sym}^-) = \{\boldsymbol{T} \in C^0(\operatorname{cl}\Omega, \mathrm{Sym}) : \boldsymbol{T} \le \boldsymbol{0} \text{ on } \Omega\}.$$

Remark 3.1. It is possible to introduce a duality between stresses and strains under different hypotheses on these objects. Namely the duality theory in [4, Section 3] gives the following result (see also [2] and [13]): if $\boldsymbol{T}$ is a stressfield in the set

$$S := \{\boldsymbol{T} \in L^\infty(\Omega, \mathrm{Sym}) : \operatorname{div}\boldsymbol{T} \in L^n(\Omega, \mathbb{R}^n)\}$$

and $\boldsymbol{v} \in BD(\Omega)$ then there exists a measure $[\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})] \in \mathcal{M}(\Omega, \mathbb{R})$ which behaves well under the weak convergence of $\boldsymbol{T}$ and such that $[\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})] = \boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v})$ if $\boldsymbol{T}$ is continuous on $\operatorname{cl}\Omega$. The measure thus plays the role of the product $\boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}))$ if $\boldsymbol{T}$ is not continuous on $\operatorname{cl}\Omega$ and the value $[\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})](\Omega)$ the role of $\int_\Omega \boldsymbol{T} \cdot d\hat{\boldsymbol{E}}(\boldsymbol{v})$. We can define the duality pairing between any $\boldsymbol{T} \in S$ and $\boldsymbol{F}$ in the space

$$Z := \{\boldsymbol{F} = \hat{\boldsymbol{E}}(\boldsymbol{v}) + \boldsymbol{G} : \boldsymbol{v} \in BD(\Omega),\ \boldsymbol{G} \in L^1(\Omega, \mathrm{Sym})\}$$

by

$$(\boldsymbol{T}, \boldsymbol{F}) = [\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})](\Omega) + \int_\Omega \boldsymbol{T} \cdot \boldsymbol{G}\, d\mathcal{L}^n.$$

It is easily shown that the value of $(\boldsymbol{T}, \boldsymbol{F})$ is independent of the choice of $\boldsymbol{v}$ and $\boldsymbol{G}$. We do not follow this possibility here, as the results under this duality are analogous to the results presented below.

To ensure that the energies of the loads, interpreted as functionals of displacements, are in W^*, we assume that the loads are represented by functions

$$\boldsymbol{s}_\circ,\ \bar{\boldsymbol{s}} \in L^\infty(\mathcal{S}, \mathbb{R}^n), \quad \boldsymbol{b}_\circ,\ \bar{\boldsymbol{b}} \in L^n(\Omega, \mathbb{R}^n),$$

and define $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ classically by the integrals in (2-3) and (2-4).

3B. **Limit analysis in the *BD* setting.** We define $\bar\lambda_W$ and λ_{Y^*} by

$$\bar\lambda_W = \bar\lambda_{BD} := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is } W_{BD} \text{ kinematically admissible}\}, \tag{3-1}$$

$$\lambda_{Y^*} = \lambda_\circ := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is } C^0(\operatorname{cl}\Omega, \mathrm{Sym}) \text{ statically admissible}\}.$$

The sufficient condition of Proposition 2.6 for the equality $\lambda_\circ = \bar\lambda_{BD}$ (i.e., the qualification hypothesis) is never satisfied in the present case since $\mathcal{M}(\Omega, \mathrm{Sym}^+)$ has empty interior. To see the last, let $\boldsymbol{F}$ be any element of $\mathcal{M}(\Omega, \mathrm{Sym}^+)$, let $\Omega_j \subset \Omega$ be a decreasing sequence of Borel sets such that $\mathcal{L}^n(\Omega_j) > 0$ for all j and $\bigcap_{j=1}^\infty \Omega_j = \emptyset$, and let the sequence $\boldsymbol{F}_j$ be defined by

$$\boldsymbol{F}_j(A) = \boldsymbol{F}(A \cap (\Omega \setminus \Omega_j)) - \mathcal{L}^n(A \cap \Omega_j)\boldsymbol{1}$$

for every Borel subset A of $\mathbb{R}^n$. Then $\boldsymbol{F}_j \notin \mathcal{M}(\Omega, \mathrm{Sym}^+)$ for all j and $\boldsymbol{F}_j \to \boldsymbol{F}$ in $\mathcal{M}(\Omega, \mathrm{Sym})$ as $j \to \infty$. Thus every point of $\mathcal{M}(\Omega, \mathrm{Sym}^+)$ is on the boundary of $\mathcal{M}(\Omega, \mathrm{Sym}^+)$.

4. Displacements in $W^{1,p}$, $1 \le p < \infty$, and stresses in L^q, $\infty \ge q > 1$

In Section 4A we consider the Sobolev spaces with finite exponent as the space of displacements and the space of power integrable tensorfields as stresses. Section 4B defines the corresponding multipliers for the kinematic and static problems, and shows that the qualification hypothesis can never be satisfied with this choice. Finally, in Section 4C a density condition is formulated to guarantee that the infimum of the kinematic problem with the space of displacements of bounded deformation equals the infimum of the kinematic problem with the Sobolev space. Section 7B gives a sufficient condition for this density.

4A. **Displacements, loads, and energies in the setting of power integrable functions.** We assume that $1 \le p < \infty$ is a given number and denote by

$$q := \begin{cases} p/(p-1) & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

the Hölder conjugate exponent. We define the objects (2-8) as follows. We put

$$W = W_p := \{\boldsymbol{v} \in W^{1,p}(\Omega, \mathbb{R}^n) : \boldsymbol{v} = \boldsymbol{0} \text{ in the sense of trace on } \mathcal{D}\},$$

$$Y = L^p(\Omega, \mathrm{Sym}),$$

so that $\hat{\boldsymbol{E}}(\cdot)$, defined by (2-5), is a bounded linear transformation from W_p to $L^p(\Omega, \mathrm{Sym})$. Furthermore, we put

$$Y^+ = L^p(\Omega, \mathrm{Sym}^+) := \{\boldsymbol{F} \in Y : \boldsymbol{F} \ge \boldsymbol{0} \text{ almost everywhere on } \Omega\},$$

$$W^+ = W_p^+ := \{\boldsymbol{v} \in W_p : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^p(\Omega, \mathrm{Sym}^+)\}.$$

Finally, Y^* is set equal to the dual of Y, i.e.,

$$Y^* = L^q(\Omega, \mathrm{Sym})$$

and the duality pairing $(\boldsymbol{T}, \boldsymbol{F})$ is given by (2-6), as is well known. The cone Y^{*-} is given by

$$Y^{*-} = L^q(\Omega, \mathrm{Sym}^-) = \{\boldsymbol{T} \in L^q(\Omega, \mathrm{Sym}) : \boldsymbol{T} \le \boldsymbol{0} \text{ almost everywhere on } \Omega\}.$$

To ensure that the energies of the loads, interpreted as functionals of displacements, are in W^*, we distinguish the cases $p < n$, $p = n$ and $p > n$. If $p < n$, we assume that the loads are represented by functions

$$\boldsymbol{s}_\circ,\ \bar{\boldsymbol{s}} \in L^s(\mathcal{S}, \mathbb{R}^n), \quad \boldsymbol{b}_\circ,\ \bar{\boldsymbol{b}} \in L^t(\Omega, \mathbb{R}^n), \tag{4-1}$$

where

$$s = (n-1)p/n(p-1), \quad t = np/(np - n + p),$$

and define $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ classically by the integrals in (2-3) and (2-4). If $p = n$, we assume that the loads are represented by functions as in (4-1), with some s, t satisfying

$$1 < s \le \infty, \quad 1 < t \le \infty$$

and again define $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ classically by the integrals in (2-3) and (2-4). The Sobolev imbedding theorem [1, Theorem 4.12, Case C] and the trace theorem [1, Theorem 5.36] imply that these definitions are well posed. If $p > n$, then the elements of W_p represent continuous functions on the closure $\mathrm{cl}\,\Omega$ of Ω. In this case the loads can be more general. Namely, the surface tractions can be represented by vector valued measures $\mathbf{s}_\circ$ and $\bar{\mathbf{s}}$ in $\mathcal{M}(\mathcal{S}, \mathbb{R}^n)$ and the body forces by vector valued measures $\mathbf{b}_\circ$ and $\bar{\mathbf{b}}$ in $\mathcal{M}(\Omega, \mathbb{R}^n)$. For this we assume that $\mathcal{S}$ is a Borel set. Then $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are given by

$$\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{v} \cdot d\mathbf{b}_\circ + \int_{\mathcal{S}} \boldsymbol{v} \cdot d\mathbf{s}_\circ, \tag{4-2}$$

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{v} \cdot d\bar{\mathbf{b}} + \int_{\mathcal{S}} \boldsymbol{v} \cdot d\bar{\mathbf{s}}, \tag{4-3}$$

for each $\boldsymbol{v} \in W_p$. This formalism includes concentrated loads. The distributed case, when the loads are represented by ordinary functions, is included by setting

$$\mathbf{s}_\circ = \boldsymbol{s}_\circ \mathcal{H}^{n-1} \llcorner \mathcal{S}, \quad \bar{\mathbf{s}} = \bar{\boldsymbol{s}} \mathcal{H}^{n-1} \llcorner \mathcal{S}, \quad \mathbf{b}_\circ = \boldsymbol{b}_\circ \mathcal{L}^n \llcorner \Omega, \quad \bar{\mathbf{b}} = \bar{\boldsymbol{b}} \mathcal{L}^n \llcorner \Omega,$$

where now the functions can belong to the spaces as follows:

$$\boldsymbol{s}_\circ,\ \bar{\boldsymbol{s}} \in L^1(\mathcal{S}, \mathbb{R}^n), \quad \boldsymbol{b}_\circ,\ \bar{\boldsymbol{b}} \in L^1(\Omega, \mathbb{R}^n)$$

and the notation of Chapter 9, below, has been employed. Then (4-2) and (4-3) reduce to (2-3) and (2-4).

4B. **Limit analysis in the setting of power integrable functions.** We define $\bar{\lambda}_W$ and λ_{Y^*} by

$$\bar{\lambda}_W = \bar{\lambda}_p := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is } W_p \text{ kinematically admissible}\}, \tag{4-4}$$
$$\lambda_{Y^*} = \lambda_q := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is } L^q(\Omega, \mathrm{Sym}) \text{ statically admissible}\}.$$

The sufficient condition of Proposition 2.6 for the equality $\lambda_q = \bar{\lambda}_p$ (i.e., the qualification hypothesis) is never satisfied in the present case, since $L^p(\Omega, \mathrm{Sym}^+)$ has empty interior for all $p \in [1, \infty)$. To see the last, let $\boldsymbol{F}$ be any element of $L^p(\Omega, \mathrm{Sym}^+)$, let $\Omega_j \subset \Omega$ be a decreasing sequence of measurable sets such that $\mathcal{L}^n(\Omega_j) > 0$ for all j and $\bigcap_{j=1}^\infty \Omega_j = \emptyset$, and let the sequence $\boldsymbol{F}_j$ be defined by

$$\boldsymbol{F}_j = \begin{cases} \boldsymbol{F} & \text{on } \Omega \setminus \Omega_j, \\ -\mathbf{1} & \text{on } \Omega_j. \end{cases}$$

Then $\boldsymbol{F}_j \notin L^p(\Omega, \mathrm{Sym}^+)$ for all j and $\boldsymbol{F}_j \to \boldsymbol{F}$ in $L^p(\Omega, \mathrm{Sym})$ as $j \to \infty$. Thus every point of $L^p(\Omega, \mathrm{Sym}^+)$ is on the boundary of $L^p(\Omega, \mathrm{Sym}^+)$.

Remark 4.1. Example 7.12 (below) shows that under very tame loads of a panel in the plane one can have $\lambda_q < \bar{\lambda}_p$.

Remark 4.2. The choice of W_p and $L^q(\Omega, \mathrm{Sym})$ with $p = q = 2$ plays a special role. The square integrable admissible equilibrating stressfields have a dynamical motivation in terms of the behavior of processes of masonry bodies with dissipation for large times [20]: if $\lambda < \lambda_2$, then under the loads corresponding to λ, the processes starting from arbitrary initial data stabilize and converge to the set of equilibrium states; on the other hand, if $\lambda > \lambda_2$, the processes blow up in the sense of norms, i.e., the collapse occurs. Further, in [15] it was shown that the existence of admissible square integrable stressfields balancing the loads is a necessary and sufficient condition for the total energy of a masonry body to be bounded from below. In [18] the equilibrium of panels subjected both to distributed loads and concentrated forces is studied, and equilibrated tensor valued measures are determined. Then, by using an integration procedure for parametric measures, equilibrated stress fields that are represented by integrable functions are explicitly determined. Note also that $L^2(\Omega, \mathrm{Sym})$ is also the space of stressfields employed in [6].

4C. **Comparison of the critical multipliers in the $\boldsymbol{BD}$ and $\boldsymbol{W^{1,p}}$ settings.** Let $1 \le p < \infty$, and let $q \in (1, \infty]$ be the Hölder conjugate exponent, and assume that the energies $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are continuous functionals on W_{BD}. (See the conditions in Chapter 3.) Since $W_p \subset W_{BD}$ and the inclusion is continuous, the restriction of $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ to W_p are continuous linear functionals on W_p as well (we denote these restrictions by the original symbols). Then in general we have

$$\bar{\lambda}_{BD} \le \bar{\lambda}_p$$

because the infimum in the definition of $\bar{\lambda}_p$ is taken over a smaller set than in the definition of $\bar{\lambda}_{BD}$. Similarly, we have

$$\lambda_\circ \le \lambda_q,$$

since the supremum for λ_q is taken over a larger set than that for $\lambda_\circ$. Under the condition that

$$\left.\begin{array}{c} \text{for each } \bar{\boldsymbol{v}} \in W^+_{BD} \text{ such that } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}} \rangle > 0 \\ \text{there exists a sequence } \bar{\boldsymbol{v}}_j \in W^+_p \text{ such that } \bar{\boldsymbol{v}}_j \to \bar{\boldsymbol{v}} \text{ in } L^{n/(n-1)}(\Omega, \mathbb{R}^n), \\ \text{and } \bar{\boldsymbol{v}}_j \to \bar{\boldsymbol{v}} \text{ in } L^1(\partial\Omega, \mathbb{R}^n) \end{array}\right\} \tag{4-5}$$

we have

$$\bar{\lambda}_{BD} = \bar{\lambda}_p.$$

Indeed, one takes the infimum in (4-4) over a dense subset of the set in the infimum in (3-1). In particular, if Condition (4-5) holds for every $p \in [1, \infty)$, then $\bar{\lambda}_p$ is independent of p and equal to $\bar{\lambda}_{BD}$.

It does not seem that there exists a relatively easily verifiable condition to guarantee the equality $\lambda_q = \lambda_\circ$. In Example 7D, below, the function $q \mapsto \lambda_q$ is not constant and hence the equality $\lambda_q = \lambda_\circ$ cannot hold for all $q \in (1, \infty]$.

5. Lipschitzian displacements and finitely additive measures representing stresses

Sections 5A and 5B formulate the kinematic and static problems for lipschitzian displacements and for stresses modeled as finitely additive bounded measures that are absolutely continuous with respect to the Lebesgue measure. A necessary and sufficient condition of Chapter 2 is particularized to the present choice of spaces in Theorem 5.1 and results in an examination of a real valued function of the real variable. The qualification hypothesis is derived in Theorem 5.2. Theorem 5.3 gives a very simple necessary and sufficient condition for the static admissibility of a general multiplier. Finally, Section 5C gives a density condition for the equality of the kinematic multipliers in the lipschitzian and Sobolev spaces settings.

5A. **Lipschitzian displacements and the representation of stresses.** We define the objects (2-8) as follows. We put

$$W = W_\infty := \{\boldsymbol{v} \in W^{1,\infty}(\Omega, \mathbb{R}^n) : \boldsymbol{v} = \boldsymbol{0} \text{ in the classical sense on } \mathcal{D}\},$$

$$Y = L^\infty(\Omega, \mathrm{Sym}),$$

so that $\hat{\boldsymbol{E}}(\cdot)$, defined by (2-5), is a bounded linear transformation from W to Y. Furthermore, we put

$$Y^+ = L^\infty(\Omega, \mathrm{Sym}^+) := \{\boldsymbol{F} \in Y : \boldsymbol{F} \ge \boldsymbol{0} \text{ almost everywhere on } \Omega\},$$

$$W^+ = W^+_\infty := \{\boldsymbol{v} \in W_\infty : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^\infty(\Omega, \mathrm{Sym}^+)\}.$$

Under our assumption on Ω, the elements of $W^{1,\infty}(\Omega, \mathbb{R}^n)$ are represented by lipschitzian functions on the closure $\operatorname{cl}\Omega$ of Ω, i.e., by functions $\boldsymbol{v} : \operatorname{cl}\Omega \to \mathbb{R}^n$ satisfying

$$|\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})| \le k|\boldsymbol{x} - \boldsymbol{y}|$$

for all $\boldsymbol{x}$, $\boldsymbol{y} \in \operatorname{cl}\Omega$ and some k, see [8, Chapter X, Section 2.2].

The loads can be represented by measures, as in the case $p > n$ in the preceding chapter, and $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are given by (4-2) and (4-3), respectively.

We denote by $Y^* = X_{\text{ba}}$ the dual of $L^\infty(\Omega, \text{Sym})$. We say that $\boldsymbol{T} \in X_{\text{ba}}$ is negative semidefinite if $(\boldsymbol{T}, \boldsymbol{F}) \le 0$ for each $\boldsymbol{F} \in L^\infty(\Omega, \text{Sym}^+)$ and denote by X_{ba}^- the set of all negative semidefinite $\boldsymbol{T} \in X_{\text{ba}}$. We interpret X_{ba}^- as the set of admissible stressfields. The elements $\boldsymbol{T}$ of X_{ba} are in general no longer representable by ordinary functions. Rather, the space X_{ba} is isomorphic to the space $\text{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \text{Sym})$ of bounded finitely additive Sym valued measures that are absolutely continuous with respect to the Lebesgue measure. Thus to each element $\boldsymbol{T} \in X_{\text{ba}}$ there exists a unique element $\mathfrak{T} \in \text{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \text{Sym})$ such that we have

$$(\boldsymbol{T}, \boldsymbol{F}) = \int_\Omega \boldsymbol{F} \cdot d\mathfrak{T} \tag{5-1}$$

for each $\boldsymbol{F} \in L^\infty(\Omega, \text{Sym})$, and conversely. We refer to [10, Sections 1.3.2 and Theorem 2.44] for details in the scalar case (in particular to the definition of the integral in (5-1)) and to an outline of the tensorial case in Chapter 9, below. An important subset of $\text{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \text{Sym})$ consists of measures of the form

$$\mathfrak{T} = \boldsymbol{T}\mathcal{L}^n \llcorner \Omega$$

where $\boldsymbol{T} \in L^1(\Omega, \text{Sym})$. In this case the measure $\mathfrak{T}$ is actually countably additive.

5B. **Limit analysis in the setting of lipschitzian displacements.** We define $\bar{\lambda}_W$ and λ_{Y^*} by

$$\bar{\lambda}_W = \bar{\lambda}_\infty := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is } W_\infty \text{ kinematically admissible}\}, \tag{5-2}$$

$$\lambda_{Y^*} = \lambda_{\text{ba}} := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is } X_{\text{ba}} \text{ statically admissible}\}.$$

Theorem 5.1. *Let $h : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ be defined by*

$$h(t) = \inf\{-\langle \boldsymbol{l}_\circ, \boldsymbol{v}\rangle : \boldsymbol{v} \in W_\infty,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) \ge t\boldsymbol{1} \text{ almost everywhere on } \Omega,\ \langle \bar{\boldsymbol{l}}, \boldsymbol{v}\rangle = 1\},$$

$t \in \mathbb{R}$. Then

(i) *h is nondecreasing, convex and $h(0) = \bar{\lambda}_\infty$.*

(ii) *$h(0)$ is finite and $\lim_{\substack{t\to 0,\\ t<0}} h(t) = h(0)$ if and only if $\lambda_{\text{ba}} = \bar{\lambda}_\infty \in \mathbb{R}$.*

Proof. The function H defined generally in (2-23) provides $h(t) = H(t\boldsymbol{1})$. The convexity and the nondecreasing character of H, asserted by Proposition 2.5(i) gives the same properties of h. Furthermore, clearly $h(0) = H(\boldsymbol{0}) = \bar{\lambda}_\infty$, which completes the proof of (i).

(ii): By Proposition 2.5(ii) then $\lambda_{\text{ba}} = \bar{\lambda}_\infty \in \mathbb{R}$ if and only if $H(\boldsymbol{0}) = h(0)$ is finite and H is lowersemicontinuous at $\boldsymbol{0}$. If the last holds, then $h(0) \in \mathbb{R}$ and h is lowersemicontinuous at 0 and as h is nondecreasing, this in turn implies that

$$\lim_{\substack{t\to 0,\\ t<0}} h(t) = h(0). \tag{5-3}$$

Conversely, let $h(0)$ be finite and let (5-3) hold. If $|\boldsymbol{F}|$ denotes the L^∞ norm of a general $\boldsymbol{F} \in L^\infty(\Omega, \mathrm{Sym})$ then $-|\boldsymbol{F}|\mathbf{1} \le \boldsymbol{F}$ and hence the monotonicity of H implies

$$h(-|\boldsymbol{F}|) \equiv H(-|\boldsymbol{F}|\mathbf{1}) \le H(\boldsymbol{F})$$

and thus

$$H(\mathbf{0}) \equiv h(0) = \lim_{\boldsymbol{F}\to\mathbf{0}} h(-|\boldsymbol{F}|) \equiv \lim_{\boldsymbol{F}\to\mathbf{0}} H(-|\boldsymbol{F}|\mathbf{1}) \le \liminf_{\boldsymbol{F}\to\mathbf{0}} H(\boldsymbol{F}).$$

Thus H is lowersemicontinuous at $\mathbf{0}$. □

Theorem 5.2. *Assume that*

$$\left.\begin{array}{c} \text{there exists a } \bar{\boldsymbol{v}}_\circ \text{ in } W_\infty^+ \text{ satisfying } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_\circ \rangle > 0 \\ \text{such that } \hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_\circ) \ge \alpha\mathbf{1} \text{ for some } \alpha > 0 \text{ and almost every point of } \Omega. \end{array}\right\} \tag{5-4}$$

Then $\lambda_{\rm ba} = \bar\lambda_\infty$; if additionally $\lambda_{\rm ba} = \bar\lambda_\infty \in \mathbb{R}$ then $\bar\lambda_\infty$ is statically admissible, i.e., there exists a bounded finitely additive negative semidefinite measure $\mathfrak{T}$ that is absolutely continuous with respect to $\mathcal{L}^n$ such that

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathfrak{T} = \langle \boldsymbol{l}(\bar\lambda_\infty), \boldsymbol{v} \rangle$$

for all $\boldsymbol{v} \in W_\infty$.

Paroni [21], dealing with inextensible nets with slack, and applying the duality theory of Ekeland & Temam, obtained balancing stresses represented by bounded finitely additive measures that are ‘almost absolutely continuous’ with respect to the Lebesgue measure (see the cited paper for a precise statement).

Proof. If Condition (5-4) holds, then $\hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_\circ)$ is an interior point of $L^\infty(\Omega, \mathrm{Sym}^+)$. Indeed, the interior of $L^\infty(\Omega, \mathrm{Sym}^+)$ consists of all $\boldsymbol{F} \in L^\infty(\Omega, \mathrm{Sym})$ which satisfy $\boldsymbol{F}(\boldsymbol{x}) \ge \alpha\mathbf{1}$ for some positive α and almost every point $\boldsymbol{x}$ of Ω. To see the sufficiency of this condition, note that if $\boldsymbol{G} \in L^\infty(\Omega, \mathrm{Sym})$ satisfies $|\boldsymbol{F} - \boldsymbol{G}| < \alpha/2$, where $|\cdot|$ is the L^∞ norm on $L^\infty(\Omega, \mathrm{Sym})$, then $\boldsymbol{G}(\boldsymbol{x}) \ge \frac{1}{2}\alpha\mathbf{1}$ for almost every $\boldsymbol{x} \in \Omega$. The necessity is proved similarly.

Thus $\bar{\boldsymbol{v}}_\circ$ satisfies Condition (2-29) and Proposition 2.6 implies the present proposition. □

Next we consider the statical admissibility of a general multiplier $\lambda \in \mathbb{R}$.

Theorem 5.3. *A multiplier $\lambda \in \mathbb{R}$ is $X_{\rm ba}$ statically admissible if and only if*

$$\sup\{\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle : \boldsymbol{v} \in W_\infty,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) \ge -\mathbf{1} \text{ almost everywhere on } \Omega\} < \infty. \tag{5-5}$$

The proof will show that Condition (5-5) implies that $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for every $\boldsymbol{v} \in W_\infty^+$.

Proof. Assume that λ is $X_{\rm ba}$ statically admissible and denote by $\boldsymbol{T} \in X_{\rm ba}^-$ an admissible equilibrating stressfield. If $\boldsymbol{v} \in W_\infty$ is such that $\hat{\boldsymbol{E}}(\boldsymbol{v}) \ge -\mathbf{1}$ almost everywhere on Ω then

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) \le (\boldsymbol{T}, -\mathbf{1}) = -(\boldsymbol{T}, \mathbf{1})$$

and thus the value of the supremum in (5-5) is less than or equal to $-(\boldsymbol{T}, \mathbf{1})$. This completes the proof of the direct implication.

To prove the converse implication, let (5-5) hold and prove that then also (2-31) holds. Prove first that (5-5) implies that

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0 \tag{5-6}$$

for every $\boldsymbol{v} \in W_\infty^+$. Indeed, (5-5) asserts that there exists a $c \in \mathbb{R}$ such that

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle < c \tag{5-7}$$

for every $\boldsymbol{v} \in W_\infty$ with $\hat{\boldsymbol{E}}(\boldsymbol{v}) \geq -\mathbf{1}$. Assume that $\boldsymbol{v} \in W_\infty^+$. Then for every $t > 0$ we have $\hat{\boldsymbol{E}}(t\boldsymbol{v}) \geq -\mathbf{1}$ and thus (5-7) gives

$$\langle \boldsymbol{l}(\lambda), t\boldsymbol{v} \rangle < c.$$

Fixing $\boldsymbol{v}$, dividing by $t > 0$ and letting $t \to \infty$ we obtain the desired conclusion $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \leq 0$.

Let us now prove that (2-31) holds. Let $\boldsymbol{F} \in L^\infty(\Omega, \mathrm{Sym})$, $\boldsymbol{F} \neq \mathbf{0}$, and let $\boldsymbol{v} \in W_\infty$ satisfy $\hat{\boldsymbol{E}}(\boldsymbol{v}) \geq \boldsymbol{F}$ almost everywhere on Ω. Then $\hat{\boldsymbol{E}}(\boldsymbol{v}/|\boldsymbol{F}|) \geq -\mathbf{1}$ almost everywhere on Ω, where $|\boldsymbol{F}|$ denotes the L^∞ norm of $\boldsymbol{F}$. Thus (5-5) implies

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \leq c|\boldsymbol{F}|,$$

i.e., Condition (2-31). If $\boldsymbol{F} = \mathbf{0}$, this argument does not hold but then $\hat{\boldsymbol{E}}(\boldsymbol{v}) \geq \mathbf{0}$ almost everywhere on Ω and (2-31) holds again by (5-6). □

5C. **Comparison of the critical multipliers in the lipschitzian and p integrable settings.** Let $1 \leq p < \infty$, and let $q \in (1, \infty]$ be the Hölder conjugate exponent, and assume that the energies $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are continuous functionals on W_p. (See the conditions in Chapter 4.) Since $W_\infty \subset W_p$ and the inclusion is continuous, the restriction of $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ to W_∞ are continuous linear functionals on W_∞ as well. Then in general we have

$$\bar{\lambda}_p \leq \bar{\lambda}_\infty$$

because the infimum in the definition of $\bar{\lambda}_\infty$ is taken over a smaller set than in the definition of $\bar{\lambda}_p$. Similarly, we have

$$\lambda_q \leq \lambda_{\mathrm{ba}},$$

since the supremum for λ_{ba} is taken over a larger set than that for λ_q. Under the condition that

$$\left.\begin{array}{r}\text{for each } \bar{\boldsymbol{v}} \in W_p^+ \text{ such that } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}} \rangle > 0 \\ \text{there exists a sequence } \bar{\boldsymbol{v}}_j \in W_\infty^+ \text{ such that } \bar{\boldsymbol{v}}_j \to \bar{\boldsymbol{v}} \text{ in } W_p,\end{array}\right\} \tag{5-8}$$

we have

$$\bar{\lambda}_p = \bar{\lambda}_\infty.$$

Indeed, one takes the infimum in (5-2) over a dense subset of the set in the infimum in (4-4). In particular, if Condition (5-8) holds for every $p \in [1, \infty)$, then $\bar{\lambda}_p$ is independent of p and equal to $\bar{\lambda}_\infty$.

As in the preceding choices of function spaces, it does not seem that there exists a condition to guarantee the equality $\lambda_q = \lambda_{\mathrm{ba}}$.

6. Smooth displacements and countably additive measures representing stresses

Section 6A and 6B formulate the kinematic and static problems within the context of smooth displacements and stressfields modeled as countably additive measures. The qualification hypothesis specialized in Theorem 6.1 is shown to hold only in the case of the pure traction problem. Theorem 6.3 provides a necessary and sufficient condition for the equality of the results of the kinematic and static problems and Theorem 6.4 a necessary and sufficient condition for the static admissibility of a general multiplier. Sections 6C and 6D provide sufficient conditions for the equality of the kinematic multiplier as defined here with those defined in the preceding chapters.

6A. **Smooth displacements and the representation of stresses.** We define the objects (2-8) as follows. We put

$$W = C_1 := \{\boldsymbol{v} \in C^1(\operatorname{cl}\Omega, \mathbb{R}^n) : \boldsymbol{v} = \mathbf{0} \text{ in the classical sense on } \mathcal{D}\},$$
$$Y = C^0(\operatorname{cl}\Omega, \mathrm{Sym}),$$

so that $\hat{\boldsymbol{E}}(\cdot)$, defined by (2-5), is a bounded linear transformation from C_1 to $C^0(\operatorname{cl}\Omega, \mathrm{Sym})$. Furthermore, we put

$$Y^+ = C^0(\operatorname{cl}\Omega, \mathrm{Sym}^+) := \{\boldsymbol{F} \in C^0(\operatorname{cl}\Omega, \mathrm{Sym}) : \boldsymbol{F} \geq \mathbf{0} \text{ on } \operatorname{cl}\Omega\},$$
$$W^+ = C_1^+ := \{\boldsymbol{v} \in C_1 : \hat{\boldsymbol{E}}(\boldsymbol{v}) \in C^0(\operatorname{cl}\Omega, \mathrm{Sym}^+)\}.$$

The loads can be represented by measures as in the case $p > n$ in Chapter 4; $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are given by (4-2) and (4-3), respectively.

We denote by $Y^* = X_{\mathcal{M}}$ the dual of $C^0(\operatorname{cl}\Omega, \mathrm{Sym})$. We say that $\boldsymbol{T} \in X_{\mathcal{M}}$ is negative semidefinite if $(\boldsymbol{T}, \boldsymbol{F}) \leq 0$ for each $\boldsymbol{F} \in C^0(\operatorname{cl}\Omega, \mathrm{Sym}^+)$, and denote by $X_{\mathcal{M}}^-$ the set of all negative semidefinite $\boldsymbol{T} \in Y_{\mathcal{M}}$. We interpret the elements $\boldsymbol{T}$ of $X_{\mathcal{M}}^-$ as admissible stressfields. The elements $\boldsymbol{T}$ of $X_{\mathcal{M}}$ are in general not representable by ordinary functions. Rather, the space $X_{\mathcal{M}}$ is isomorphic with the space $\mathcal{M}(\operatorname{cl}\Omega, \mathrm{Sym})$ of bounded countably additive Sym valued Borel measures on $\operatorname{cl}\Omega$. Thus for each element $\boldsymbol{T} \in X_{\mathcal{M}}$ there exists a unique element $\mathsf{T} \in \mathcal{M}(\operatorname{cl}\Omega, \mathrm{Sym})$ such that

$$(\boldsymbol{T}, \boldsymbol{F}) = \int_\Omega \boldsymbol{F} \cdot d\mathsf{T}$$

for each $\boldsymbol{F} \in C^0(\operatorname{cl}\Omega, \mathrm{Sym})$, and conversely. We refer to [10, Theorem 1.196] for details in the scalar case and to Chapter 9, below, for the tensorial case. Recalling Chapter 5, we note that neither of the sets $\mathrm{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \mathrm{Sym})$ and $\mathcal{M}(\operatorname{cl}\Omega, \mathrm{Sym})$ is a subset of the other, since the measures from $X_{\mathcal{M}}$ need not be absolutely continuous with respect to the Lebesgue measure and the measures from $\mathrm{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \mathrm{Sym})$ need not be countably additive. With general elements of $\mathcal{M}(\operatorname{cl}\Omega, \mathrm{Sym})$ there may be stresses concentrated on sets of dimension strictly less than n. An important subset of $\mathcal{M}(\operatorname{cl}\Omega, \mathrm{Sym})$ consists of measures of the form

$$\mathsf{T} = \boldsymbol{T}\mathcal{L}^n \llcorner \Omega$$

where $\boldsymbol{T} \in L^1(\Omega, \mathrm{Sym})$.

The set $X_{\mathcal{M}}^-$ is just the set of all Borel measures on $\operatorname{cl}\Omega$ which take negative semidefinite values on every Borel subset of $\operatorname{cl}\Omega$.

6B. **Limit analysis in the setting of smooth displacements.** We define $\bar{\lambda}_W$ and λ_{Y^*} by

$$\bar{\lambda}_W = \bar{\lambda}_\circ := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is } C_1 \text{ kinematically admissible}\},$$
$$\lambda_{Y^*} = \lambda_{\mathcal{M}} := \sup\{\lambda \in \mathbb{R} : \lambda \text{ is } X_{\mathcal{M}} \text{ statically admissible}\}.$$

Theorem 6.1. *Assume that*

$$\left.\begin{aligned} &\textit{there exists a } \bar{\boldsymbol{v}}_\circ \textit{ in } C_1^+ \textit{ satisfying } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}}_\circ \rangle > 0 \textit{ such that} \\ &\hat{\boldsymbol{E}}(\bar{\boldsymbol{v}}_\circ) \geq \alpha \mathbf{1} \textit{ for some } \alpha > 0 \textit{ and every point of } \operatorname{cl}\Omega. \end{aligned}\right\} \tag{6-1}$$

Then we have $\lambda_{\mathcal{M}} = \bar{\lambda}_\circ$; if, additionally, this number is finite, then $\bar{\lambda}_\circ$ is statically admissible, i.e., there exists a bounded countably additive negative semidefinite Borel measure T such that

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathsf{T} = \langle \boldsymbol{l}(\bar{\lambda}_\circ), \boldsymbol{v} \rangle$$

for all $\boldsymbol{v} \in C_1$.

Proof. The interior of $C^0(\operatorname{cl}\Omega, \operatorname{Sym}^+)$ consists of all $\boldsymbol{F} \in C_\circ$ such that there exists a positive α satisfying $\boldsymbol{F} \geq \alpha \mathbf{1}$ for all points of $\operatorname{cl}\Omega$. We thus see that the displacement $\bar{\boldsymbol{v}}_\circ$ as in (6-1) satisfies the hypothesis of Proposition 2.6. The same proposition then gives the assertions of the present proposition. □

Remark 6.2. Let $\boldsymbol{v} \in C_1$, and let $\boldsymbol{x}$ be a point in $\mathcal{D}$ and $\boldsymbol{t}$ a vector that is tangent to $\mathcal{D}$ in the sense that there is a smooth curve γ contained in $\mathcal{D}$ and containing $\boldsymbol{x}$ with the tangent vector $\boldsymbol{t}$ at $\boldsymbol{x}$. Differentiating the equation $\boldsymbol{v} = \mathbf{0}$ along γ at $\boldsymbol{x}$ we obtain $\nabla \boldsymbol{v}(\boldsymbol{x})\boldsymbol{t} = \mathbf{0}$. Thus $\hat{\boldsymbol{E}}(\boldsymbol{v})(\boldsymbol{x})\boldsymbol{t} \cdot \boldsymbol{t} = 0$ and Condition (6-1) cannot hold. This applies also to points of $\mathcal{D}$ at the corners or edges. Thus (6-1) can be effective essentially only in the case of the pure traction problem, when $\mathcal{D} = \emptyset$.

Negative semidefinite measures with nonzero singular part equilibrating the loads in no–tension materials were proposed in [16]. The general theory of stresses represented by Borel measures is given in [23].

The following two results are proved in essentially the same way as Theorems 5.1 and 5.3. The proofs are therefore omitted.

Theorem 6.3. *Let* $h : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ *be defined by*

$$h(t) = \inf\{-\langle \boldsymbol{l}_\circ, \boldsymbol{v} \rangle : \boldsymbol{v} \in C_1,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) \geq t\mathbf{1} \text{ on } \operatorname{cl}\Omega,\ \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 1\},$$

$t \in \mathbb{R}$. *Then*

(i) *h is nondecreasing, convex and* $h(0) = \bar{\lambda}_\circ$.

(ii) *$h(0)$ is finite and* $\lim_{\substack{t \to 0, \\ t < 0}} h(t) = h(0)$ *if and only if* $\lambda_{\mathcal{M}} = \bar{\lambda}_\circ \in \mathbb{R}$.

Theorem 6.4. *A multiplier $\lambda \in \mathbb{R}$ is $X_{\mathcal{M}}$ statically admissible if and only if*

$$\sup\{\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle : \boldsymbol{v} \in C_1,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) \geq -\mathbf{1} \text{ on } \Omega < \infty\} \tag{6-2}$$

Condition (6-2) implies that $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \leq 0$ for every $\boldsymbol{v} \in C_1$ but is stronger.

6C. **Comparison of the critical multipliers in the smooth and p integrable settings.** Let $1 \leq p < \infty$, and let $q \in (1, \infty]$ be the Hölder conjugate exponent, and assume that $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are continuous functionals on W_p and hence also on C_1. Then in general we have

$$\bar{\lambda}_p \leq \bar{\lambda}_\circ$$

because the infimum in the definition of $\bar{\lambda}_\circ$ is taken over a smaller set than in the definition of $\bar{\lambda}_p$. Similarly, we have

$$\lambda_q \leq \lambda_{\mathcal{M}}.$$

Under the condition that

$$\left.\begin{array}{c} \text{for each } \bar{\boldsymbol{v}} \in W_p^+ \text{ such that } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}} \rangle > 0 \\ \text{there exists a sequence } \bar{\boldsymbol{v}}_j \in C_1^+ \text{ such that and } \bar{\boldsymbol{v}}_j \to \bar{\boldsymbol{v}} \text{ in } W_p, \end{array}\right\} \tag{6-3}$$

we have

$$\bar{\lambda}_p = \bar{\lambda}_\circ.$$

This is completely analogous to Condition (5-8).

Papers [17] and [18] give conditions and examples under which the loads equilibrated by measures from a certain class can be also equilibrated by stressfields represented by ordinary functions from $L^\infty(\Omega, \mathrm{Sym}^+)$.

6D. **Comparison of the critical multipliers in the smooth and lipschitzian settings.** Assume that $\boldsymbol{l}_\circ$ and $\bar{\boldsymbol{l}}$ are continuous functionals on W_∞ and hence also on C_1. Then in general we have

$$\bar\lambda_\infty \le \bar\lambda_\circ$$

because the infimum in the definition of $\bar\lambda_\circ$ is taken over a smaller set than that in the definition of $\bar\lambda_\infty$. Under the condition that

$$\left.\begin{array}{c}\text{for each } \bar{\boldsymbol{v}} \in W_\infty^+ \text{ such that } \langle \bar{\boldsymbol{l}}, \bar{\boldsymbol{v}} \rangle > 0 \text{ there exists a sequence } \bar{\boldsymbol{v}}_j \in C_1^+ \\ \text{such that } \bar{\boldsymbol{v}}_j \to \bar{\boldsymbol{v}} \text{ in } L^\infty(\Omega, \mathbb{R}^n),\end{array}\right\} \tag{6-4}$$

we have

$$\bar\lambda_\infty = \bar\lambda_\circ.$$

This is completely analogous to Conditions (5-8) and (6-3).

We have

$$\lambda_{\mathrm{ba}} \le \lambda_{\mathcal{M}}$$

Indeed, each X_{ba} statically admissible multiplier is also $X_{\mathcal{M}}$ statically admissible, because if an admissible stressfield $\boldsymbol{T} \in X_{\mathrm{ba}}^-$ balances the loads corresponding to λ then the restriction $\tilde{\boldsymbol{T}}$ of $\boldsymbol{T}$ to $C^0(\mathrm{cl}\,\Omega, \mathrm{Sym})$ is $X_{\mathcal{M}}$ statically admissible and balances the same loads.

Remark 6.5. Returning to the functionals $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ from the preceding paragraph, we note that corresponding to $\boldsymbol{T}$ there exists a bounded finitely additive Sym valued measure $\mathfrak{T}$ on the class $\mathfrak{M}$ of all Lebesgue measurable subsets of Ω, absolutely continuous with respect to the Lebesgue measure, such that

$$(\boldsymbol{T}, \boldsymbol{F}) = \int_\Omega \boldsymbol{F} \cdot d\mathfrak{T}$$

for all $\boldsymbol{F} \in L^\infty(\Omega, \mathrm{Sym})$. It appears that generally $\mathfrak{T}$ need not be countably additive, despite the extra information of balancing. At the same time there exists a finite, countably additive measure T on the class $\mathfrak{B}$ of all Borel subsets of $\mathrm{cl}\,\Omega$, such that

$$(\tilde{\boldsymbol{T}}, \boldsymbol{F}) \equiv (\boldsymbol{T}, \boldsymbol{F}) = \int_{\mathrm{cl}\,\Omega} \boldsymbol{F} \cdot d\mathsf{T}$$

for all $\boldsymbol{F} \in C^0(\mathrm{cl}\,\Omega, \mathrm{Sym})$. It appears that generally T need not be absolutely continuous with respect to the Lebesgue measure. A natural question arises of what is the relationship between $\mathfrak{T}$ and T. Easy examples based on the extension of $\tilde{\boldsymbol{T}}$ by using the Hahn Banach theorem show that without the conditions of negative semidefiniteness and of balancing, there need not be any immediate relationship. However, with the two extra conditions just mentioned, the situation does not seem to be clear.

7. Monotonicity, density, and examples

Section 7A first shows that the displacements from the Sobolev spaces with positive semidefinite strain are monotone. Next, the same section shows that each displacement with positive semidefinite strain has to vanish, roughly speaking, on the interior of the convex hull of the set $\mathcal{D}$. Section 7B proves the density of the smooth displacements with positive semidefinite strain in the wider spaces of displacements considered above. Some of the results of these two sections are employed in Sections 7C and 7D which present two examples: a collapse without a corresponding mechanism, and loads for which the kinematic and static problems give different results.

7A. **Monotonicity and the convex hull.** If $\boldsymbol{v} \in BD(\Omega)$ (in particular, if $\boldsymbol{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$ where $1 \le p \le \infty$), we define the *precise representative* $\tilde{\boldsymbol{v}}$ of $\boldsymbol{v}$ on Ω by setting, for every $\boldsymbol{x} \in \operatorname{cl}\Omega$,

$$\tilde{\boldsymbol{v}}(\boldsymbol{x}) = \begin{cases} \lim\limits_{r\to 0} \dfrac{1}{\mathcal{L}^n(B(\boldsymbol{x},r)\cap\Omega)} \displaystyle\int_{B(\boldsymbol{x},r)\cap\Omega} \boldsymbol{v}\, d\mathcal{L}^n & \text{if the limit exists,} \\ \boldsymbol{0} & \text{otherwise,} \end{cases} \tag{7-1}$$

where $B(\boldsymbol{x}, r)$ is the open ball of center $\boldsymbol{x}$ and radius r. Denote by $G(\boldsymbol{v})$ the set of all points $\boldsymbol{x} \in \operatorname{cl}\Omega$ for which the limit in (7-1) exists and, moreover, if $\boldsymbol{x} \in \partial\Omega$, it satisfies

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(\boldsymbol{x},r)\cap\Omega)} \int_{B(\boldsymbol{x},r)\cap\Omega} |\boldsymbol{v} - \tilde{\boldsymbol{v}}(\boldsymbol{x})|\, d\mathcal{L}^n = 0. \tag{7-2}$$

If $\boldsymbol{v} \in BD(\Omega)$ then $\mathcal{L}^n(\Omega \setminus G(\boldsymbol{v})) = 0$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus G(\boldsymbol{v})) = 0$. The first assertion is the standard assertion about Lebesgue points (and actually holds for any $\boldsymbol{v} \in L^1(\Omega, \mathbb{R}^n)$) while for the second assertion, see the trace theorem in [25, Chapter II]. If $\boldsymbol{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$ where $1 \le p \le \infty$ then $\mathcal{H}^{n-1}(\operatorname{cl}\Omega \setminus G(\boldsymbol{v})) = 0$. Indeed, $\mathcal{H}^{n-1}(\Omega \setminus G(\boldsymbol{v})) = 0$ by [9, Theorem 1 in Section 4.8, Theorem 2 in Section 5.6.3, and Theorem 4 in Section 4.7.2] and $\mathcal{H}^{n-1}(\partial\Omega \setminus G(\boldsymbol{v})) = 0$ by [9, Definition and Remark, p. 133]. For every direction $\boldsymbol{t} \in \mathbb{S}^{n-1} := \{\boldsymbol{t} \in \mathbb{R}^n : |\boldsymbol{t}| = 1\}$, and for almost every line $\mathfrak{l}$ parallel to $\boldsymbol{t}$, $\tilde{\boldsymbol{v}}$ is absolutely continuous on $\mathfrak{s} := \mathfrak{l} \cap \operatorname{cl}\Omega$ (cf. [9, Theorem 2, Subsection 4.9.2] for lines parallel to the coordinate axes).

Remark 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with Lipschitz boundary and $\boldsymbol{v} \in BD(\Omega)$ (respectively, $\boldsymbol{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$ where $1 \le p \le \infty$), and let $\tilde{\boldsymbol{v}}$ be the precise representative. Then the following conditions are equivalent:*

(i) *we have $\hat{\boldsymbol{E}}(\boldsymbol{v})(B) \ge \boldsymbol{0}$ for every Borel subset B of Ω (respectively, $\hat{\boldsymbol{E}}(\boldsymbol{v}) \ge \boldsymbol{0}$ almost everywhere on Ω);*

(ii) *we have*

$$(\tilde{\boldsymbol{v}}(\boldsymbol{x}) - \tilde{\boldsymbol{v}}(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \ge 0 \tag{7-3}$$

for every $\boldsymbol{x}$, $\boldsymbol{y} \in G(\boldsymbol{v})$.

Note that Condition (ii) can be formulated even for nondifferentiable displacements.

Proof. Let σ be a smooth mollifier, i.e., a nonnegative class ∞ function on $\mathbb{R}^n$ with the support in the open ball with center $\boldsymbol{0}$ and radius 1, and $\int_{\mathbb{R}^n} \sigma\, d\mathcal{L}^n = 1$. For each $\epsilon > 0$, let $\boldsymbol{v}_\epsilon$ be the ϵ mollification of $\boldsymbol{v}$, i.e., a map $\boldsymbol{v}_\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ given by

$$\boldsymbol{v}_\epsilon(\boldsymbol{x}) = \epsilon^{-n} \int_\Omega \boldsymbol{v}(\boldsymbol{y}) \sigma((\boldsymbol{x} - \boldsymbol{y})/\epsilon)\, d\mathcal{L}^n(\boldsymbol{y})$$

for every $\boldsymbol{x}$ from the set

$$\Omega_\epsilon := \{\boldsymbol{x} \in \Omega : B(\boldsymbol{x}, \epsilon) \subset \Omega\}.$$

(i) $\Rightarrow$ (ii): The tensor $\hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon)$ is an ϵ mollification of $\hat{\boldsymbol{E}}(\boldsymbol{v})$ on Ω_ϵ (under either assumptions on $\boldsymbol{v}$) and hence $\hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon) \ge \boldsymbol{0}$ on Ω_ϵ. We have

$$\frac{d}{ds} \boldsymbol{v}_\epsilon(\boldsymbol{a} + s\boldsymbol{t}) \cdot \boldsymbol{t} = \nabla \boldsymbol{v}_\epsilon(\boldsymbol{a} + s\boldsymbol{t})\boldsymbol{t} \cdot \boldsymbol{t} = \hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon)(\boldsymbol{a} + s\boldsymbol{t})\boldsymbol{t} \cdot \boldsymbol{t} \ge 0$$

for every $\boldsymbol{a} \in \mathbb{R}^n$, every $s \in \mathbb{R}$, and every $\boldsymbol{t} \in \mathbb{S}^{n-1}$ for which $\boldsymbol{a}+s\boldsymbol{t} \in \operatorname{int}\Omega_\epsilon$. Thus the integration gives

$$(\boldsymbol{v}_\epsilon(\boldsymbol{x}) - \boldsymbol{v}_\epsilon(\boldsymbol{y})) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq 0$$

for every $\boldsymbol{x}$, $\boldsymbol{y} \in \operatorname{int}\Omega_\epsilon$. If $\boldsymbol{x}$, $\boldsymbol{y} \in \Omega$, then $\boldsymbol{x}$, $\boldsymbol{y} \in \operatorname{int}\Omega_\epsilon$ for all sufficiently small ϵ. We have $\boldsymbol{v}_\epsilon(\boldsymbol{x}) \to \tilde{\boldsymbol{v}}(\boldsymbol{x})$ as $\epsilon \to 0$ for every $\boldsymbol{x} \in G(\boldsymbol{v}) \cap \Omega$. This limit gives (7-3) for every $\boldsymbol{x}$, $\boldsymbol{y} \in G(\boldsymbol{v}) \cap \Omega$. Next assume that $\boldsymbol{x} \in G(\boldsymbol{v}) \cap \partial\Omega$ and $\boldsymbol{y} \in G(\boldsymbol{v}) \cap \Omega$. By the preceding case, we have

$$(\tilde{\boldsymbol{v}}(\boldsymbol{x}') - \tilde{\boldsymbol{v}}(\boldsymbol{y})) \cdot (\boldsymbol{x}' - \boldsymbol{y}) \geq 0$$

for every $\boldsymbol{x}' \in G(\boldsymbol{v}) \cap \Omega$. Hence, if $r > 0$, we have

$$\frac{1}{\mathcal{L}^n(B(\boldsymbol{x}, r) \cap \Omega)} \int_{B(\boldsymbol{x},r)\cap\Omega} (\tilde{\boldsymbol{v}}(\boldsymbol{x}') - \tilde{\boldsymbol{v}}(\boldsymbol{y})) \cdot (\boldsymbol{x}' - \boldsymbol{y}) \, d\mathcal{L}^n(\boldsymbol{x}') \geq 0$$

and the limit $r \to 0$ using (7-1) and (7-2) gives (7-3). Finally, if $\boldsymbol{x}$, $\boldsymbol{y} \in G(\boldsymbol{v}) \cap \partial\Omega$, we proceed in the same way as above to establish (7-3) generally.

(ii) ⇒ (i): Condition (ii) implies that if $\boldsymbol{t} \in \mathbb{S}^{n-1}$, and if $\mathfrak{l}$ is a line parallel to $\boldsymbol{t}$ then the function $\tilde{\boldsymbol{v}} \cdot \boldsymbol{t}$, defined on $\tilde{\mathfrak{l}} := \mathfrak{l} \cap G(\boldsymbol{v})$, is nondecreasing. By Fubini's theorem we have $\mathcal{H}^1((\mathfrak{l} \cap \Omega) \setminus \tilde{\mathfrak{l}}) = 0$ for $\mathcal{H}^{n-1}$ almost every line parallel to $\boldsymbol{t}$. Then $\boldsymbol{v}_\epsilon \cdot \boldsymbol{t}$ is nondecreasing on every closed line segment parallel to $\boldsymbol{t}$ in Ω_ϵ. Thus

$$0 \leq \frac{d}{ds} \boldsymbol{v}_\epsilon(\boldsymbol{a} + s\boldsymbol{t})\Big|_{s=0} \cdot \boldsymbol{t} = \hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon)(\boldsymbol{a})\boldsymbol{t} \cdot \boldsymbol{t}$$

for every point $\boldsymbol{a}$ of Ω_ϵ. We now let $\epsilon \to 0$ and obtain (i) under either assumption on $\boldsymbol{v}$. □

Remark 7.2. *Assume that $\Omega \subset \mathbb{R}^n$ is open and convex, let $\mathcal{D} \subset \partial\Omega$, denote by $\operatorname{int}\mathcal{D}$ the relative interior of $\mathcal{D}$ in $\partial\Omega$ and assume that $\operatorname{int}\mathcal{D}$ is a class 1 surface. Define the sets Z_i, $i = 1, \dots,$ by*

$$Z_1 = \{\boldsymbol{z} \in \Omega : \boldsymbol{z} = (1-t)\boldsymbol{x} + t\boldsymbol{y} \text{ for some } \boldsymbol{x}, \boldsymbol{y} \in \operatorname{int}\mathcal{D},\ \boldsymbol{x} \neq \boldsymbol{y},\ 0 < t < 1\}$$

and

$$Z_i = \{(1-t)\boldsymbol{x} + t\boldsymbol{y} : \boldsymbol{x}, \boldsymbol{y} \in Z_{i-1}, 0 \leq t \leq 1\}$$

if $i \geq 2$. Then (a) *the sequence Z_i is nondecreasing,* (b) *Z_i is open for every $i = 1, \dots,$* (c) *$Z_i \subset \Omega$ for every $i = 1, \dots,$ and* (d) *$Z_i = \operatorname{co} Z_1$ for every $i \geq n+1$.*

Here by the relative interior of $\mathcal{D}$ we mean the interior with respect to the relative topology on $\partial\Omega$, defined by intersections of $\partial\Omega$ with open subsets of $\mathbb{R}^n$.

$\operatorname{co} Z_1$ denotes the convex hull of Z_1.

Proof. The nondecreasing character of the sequence Z_i is immediate.

We shall prove (b) and (c) by induction on i.

We have $Z_1 \subset \Omega$. To prove that Z_1 is open, let $\boldsymbol{z} \in Z_1$ and $\boldsymbol{z} = (1-t)\boldsymbol{x} + t\boldsymbol{y}$ for some $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{int}\mathcal{D}$, $\boldsymbol{x} \neq \boldsymbol{y}$, $0 < t < 1$. Define the map $\Phi : \operatorname{int}\mathcal{D} \times (0,1) \to \Omega$ by

$$\Phi(\boldsymbol{y}', t') = (1-t')\boldsymbol{x} + t'\boldsymbol{y}', \tag{7-4}$$

$\boldsymbol{y}' \in \operatorname{int}\mathcal{D}$, $t' \in (0,1)$; hence

$$\Phi(\boldsymbol{y}, t) = \boldsymbol{z}.$$

Then the continuity implies that Φ maps some neighborhood $\mathcal{N}$ of $(\boldsymbol{y}, t)$ in $\operatorname{int}\mathcal{D} \times (0,1)$ into some neighborhood $\mathcal{M}$ of $\boldsymbol{z}$ that is contained in Ω. One then has $\mathcal{M} \subset Z_1$ by the definition of Z_1. We now employ the implicit function theorem to show that there exists a neighborhood $\mathcal{M}' \subset \mathcal{M}$ of $\boldsymbol{z}$ and a class 1 map $\Psi : \mathcal{M}' \to \mathcal{N}$ such that $\Phi(\Psi(\boldsymbol{z}')) = \boldsymbol{z}'$ for all $\boldsymbol{z}' \in \mathcal{M}'$. Indeed, the map Φ is continuously differentiable. To apply the implicit function theorem, we have to prove that the derivative $\mathrm{D}\,\Phi(\boldsymbol{y},t) : \mathrm{Tan}(\mathrm{int}\,\mathcal{D},\boldsymbol{y})\times\mathbb{R}\to\mathbb{R}^n$ is nonsingular, where $\mathrm{Tan}(\mathrm{int}\,\mathcal{D},\boldsymbol{y})$ is the tangent space to $\mathrm{int}\,\mathcal{D}$ at $\boldsymbol{y}$. If the local description of $\mathrm{int}\,\mathcal{D}$ near $\boldsymbol{y}$ is $\varphi(\boldsymbol{y}')=0$ where φ is a class 1 function with $\nabla\varphi(\boldsymbol{y})\neq\mathbf{0}$, then $\mathrm{Tan}(\mathrm{int}\,\mathcal{D},\boldsymbol{y})=\{\dot{\boldsymbol{y}}\in\mathbb{R}^n : \nabla\varphi(\boldsymbol{y})\cdot\dot{\boldsymbol{y}}=0\}$. The equation $\mathrm{D}\,\Phi(\boldsymbol{y},t)(\dot{\boldsymbol{y}},\dot{t})=\mathbf{0}$ reads

$$\dot{t}(\boldsymbol{y}-\boldsymbol{x})+t\dot{\boldsymbol{y}}=\mathbf{0}. \tag{7-5}$$

Multiplying scalarly by $\nabla\varphi(\boldsymbol{y})$ and using $\nabla\varphi(\boldsymbol{y})\cdot\dot{\boldsymbol{y}}=0$ we obtain

$$\nabla\varphi(\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x})\dot{t}=0. \tag{7-6}$$

Let us now show that

$$\nabla\varphi(\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x})\neq 0. \tag{7-7}$$

Indeed, the equation $\nabla\varphi(\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x})=0$ means that the vector $\boldsymbol{x}-\boldsymbol{y}$ belongs to the tangent space of $\partial\Omega$ at $\boldsymbol{y}$ or equivalently that $\boldsymbol{x}$ belongs to the affine space $\boldsymbol{y}+\mathrm{Tan}(\mathrm{int}\,\mathcal{D},\boldsymbol{y})$. Since Ω is convex and $\boldsymbol{x}\in\partial\Omega$, it follows that the line segment with endpoints $\boldsymbol{x}$ and $\boldsymbol{y}$ belongs to $\partial\Omega$ and hence in particular $\boldsymbol{z}\in\partial\Omega$. However, this is a contradiction as we assume that $\boldsymbol{z}\in\Omega$. This contradiction shows that we have (7-7). Equation (7-6) then implies $\dot{t}=0$ and (7-5) that $\dot{\boldsymbol{y}}=\mathbf{0}$. Thus $\mathrm{D}\,\Phi(\boldsymbol{y},t)$ is nonsingular and the implicit function theorem gives a class 1 map $\Psi:\mathcal{M}'\to\mathcal{N}$ such that $\Phi(\Psi(\boldsymbol{z}'))=\boldsymbol{z}'$ for all $\boldsymbol{z}'\in\mathcal{M}'$. This in turn means that Ψ maps some neighborhood of $(\boldsymbol{y},t)$ in $\mathrm{int}\,\mathcal{D}\times(0,1)$ onto some neighborhood of $\boldsymbol{z}$. Thus Z_1 is open.

Let $i\geq 2$. Then (c) follows from the induction hypothesis and the convexity of Ω. To prove (b), let $\boldsymbol{z}=(1-t)\boldsymbol{x}+t\boldsymbol{y}$ for some $\boldsymbol{x},\boldsymbol{y}\in Z_{i-1}$ and $0\leq t\leq 1$. To prove that there exist a neighborhood of $\boldsymbol{z}$ that is contained in Z_i, note that if $\boldsymbol{x}=\boldsymbol{y}$ or $t\in\{0,1\}$ then $\boldsymbol{z}\in Z_{i-1}$ and the induction hypothesis says that there exists a neighborhood of $\boldsymbol{z}$ that is contained in $Z_{i-1}\subset Z_i$. Thus we can assume that $\boldsymbol{x}\neq\boldsymbol{y}$ and $0<t<1$. Defining $\Gamma:Z_{i-1}\to\mathbb{R}^n$ by

$$\Gamma(\boldsymbol{y}')=(1-t)\boldsymbol{x}+t\boldsymbol{y}'$$

for every $\boldsymbol{y}'\in Z_{i-1}$ we find that Γ has the inverse Δ given by

$$\Delta(\boldsymbol{z}')=(\boldsymbol{z}'-(1-t)\boldsymbol{x})/t,$$

$\boldsymbol{z}'\in\mathbb{R}^n$. Hence Γ maps some neighborhood $\mathcal{N}\subset Z_{i-1}$ of $\boldsymbol{y}$ onto some neighborhood of $\boldsymbol{z}$ that is in Z_i. This completes the proof of (b) and (c).

(d) follows from Caratheodory's theorem on the convex hull [22, Theorem 17.1]. ☐

Proposition 7.3. *Let $1\leq p\leq\infty$, let Ω be convex, let $\mathcal{D}\subset\partial\Omega$ and assume that $\mathrm{int}\,\mathcal{D}$ is a class 1 surface. Let Z_i be the sequence defined in Remark 7.2. If $\boldsymbol{v}\in W_p^+$ then its precise representative $\tilde{\boldsymbol{v}}$ satisfies*

$$\tilde{\boldsymbol{v}}=\mathbf{0}\ \ \text{everywhere on } \mathrm{co}\,Z_1. \tag{7-8}$$

Proof. We shall prove that

$$\tilde{\boldsymbol{v}}=\mathbf{0}\ \ \text{everywhere on } Z_i \tag{7-9}$$

for every $i=1,\ldots$ by induction on i. Then (7-8) will follow from Remark 7.2(d).

Prove (7-9) for $i=1$. Note that if $\boldsymbol{x}\in\Omega\cap G(\boldsymbol{v})$ and if $\boldsymbol{y}\in\mathrm{int}\,\mathcal{D}$ then

$$\tilde{\boldsymbol{v}}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})\geq 0. \tag{7-10}$$

Inequality (7-10) follows from Remark 7.1 if $\boldsymbol{y}\in\mathrm{int}\,\mathcal{D}\cap G(\boldsymbol{v})$ since then $\tilde{\boldsymbol{v}}(\boldsymbol{y})=\mathbf{0}$. Finally, if $\boldsymbol{x}\in\Omega\cap G(\boldsymbol{v})$ and $\boldsymbol{y}$ in $\mathcal{D}$ is arbitrary, we make a limit in $\tilde{\boldsymbol{v}}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y}')\geq 0$ as $\boldsymbol{y}'\to\boldsymbol{y}$ and $\boldsymbol{y}'\in\mathrm{int}\,\mathcal{D}\cap G(\boldsymbol{v})$.

Let now $\boldsymbol{z} \in Z_1 \cap G(\boldsymbol{v})$ and write $\boldsymbol{z} = (1-t)\boldsymbol{x} + t\boldsymbol{y}$ for some $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{int} \mathcal{D}$, $\boldsymbol{x} \neq \boldsymbol{y}$, $0 < t < 1$. Then (7-10) implies

$$\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot (\boldsymbol{z} - \boldsymbol{x}) \geq 0 \quad \text{and} \quad \tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot (\boldsymbol{z} - \boldsymbol{y}) \geq 0.$$

This reads

$$t\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot (\boldsymbol{y} - \boldsymbol{x}) \geq 0, \quad \text{and} \quad (1-t)\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq 0,$$

and hence

$$\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot \boldsymbol{t} = 0 \tag{7-11}$$

where $\boldsymbol{t} = (\boldsymbol{y} - \boldsymbol{x})/|\boldsymbol{y} - \boldsymbol{x}|$. Define now a map Θ from $\mathbb{S}^{n-1}$ to $\partial\Omega \times \partial\Omega$ by the requirement that for each $\boldsymbol{t}' \in \mathbb{S}^{n-1}$ we put $\Theta(\boldsymbol{t}') = (\boldsymbol{x}', \boldsymbol{y}')$ where $\boldsymbol{x}'$, $\boldsymbol{y}' \in \partial\Omega$ are uniquely determined by the conditions that $\boldsymbol{t}' = (\boldsymbol{y}' - \boldsymbol{x}')/|\boldsymbol{y}' - \boldsymbol{x}'|$ and that the point $\boldsymbol{z}$ is on the (closed) line segment with endpoints $\boldsymbol{x}'$, $\boldsymbol{y}'$. Since $\operatorname{int} \mathcal{D} \times \operatorname{int} \mathcal{D}$ is a relatively open subset of $\partial\Omega \times \partial\Omega$, the continuity of Θ yields that there exists a neighborhood $\mathcal{N}$ of $\boldsymbol{t}$ in $\mathbb{S}^{n-1}$ such that Θ maps $\mathcal{N}$ onto some subset of $\operatorname{int} \mathcal{D} \times \operatorname{int} \mathcal{D}$. For all $\boldsymbol{t}' \in \mathcal{N}$ we have $\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot \boldsymbol{t}' = 0$, which is possible only if $\tilde{\boldsymbol{v}}(\boldsymbol{z}) = \boldsymbol{0}$. Thus $\tilde{\boldsymbol{v}} = \boldsymbol{0}$ on $Z_1 \cap G(\boldsymbol{v})$ and hence $\tilde{\boldsymbol{v}} = \boldsymbol{0}$ everywhere on Z_1.

Let $i \geq 2$. Let $\boldsymbol{z} \in Z_i$ and let $\boldsymbol{z} = (1-t)\boldsymbol{x} + t\boldsymbol{y}$ where $\boldsymbol{x}$, $\boldsymbol{y} \in Z_{i-1}$, $0 \leq t \leq 1$. To prove that $\tilde{\boldsymbol{v}}(\boldsymbol{z}) = \boldsymbol{0}$, we can assume that $\boldsymbol{x} \neq \boldsymbol{y}$ and $0 < t < 1$ since otherwise $\boldsymbol{z} \in Z_{i-1}$ and the equation $\tilde{\boldsymbol{v}}(\boldsymbol{z}) = \boldsymbol{0}$ follows from the induction hypothesis. Let $\boldsymbol{x}'$, $\boldsymbol{y}' \in Z_{i-1} \cap G(\boldsymbol{v})$ be such that $\boldsymbol{z} = (1-t)\boldsymbol{x}' + t\boldsymbol{y}'$. The induction hypothesis says that $\tilde{\boldsymbol{v}}(\boldsymbol{x}') = \tilde{\boldsymbol{v}}(\boldsymbol{y}') = \boldsymbol{0}$ and using this, we obtain $\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot \boldsymbol{t}' = 0$ where $\boldsymbol{t}' = (\boldsymbol{y}' - \boldsymbol{x}')/|\boldsymbol{y}' - \boldsymbol{x}'|$; varying $\boldsymbol{x}' \in Z_{i-1} \cap G(\boldsymbol{v})$ and $\boldsymbol{y}' \in Z_{i-1} \cap G(\boldsymbol{v})$ in such a way that $\boldsymbol{z} := (1-t)\boldsymbol{x}' + t\boldsymbol{y}'$ remains fixed, we obtain the validity of $\tilde{\boldsymbol{v}}(\boldsymbol{z}) \cdot \boldsymbol{t}' = 0$ for almost every $\boldsymbol{t}'$ from a nonempty open subset of $\mathbb{S}^{n-1}$, which is possible only if $\tilde{\boldsymbol{v}}(\boldsymbol{z}) = \boldsymbol{0}$. □

7B. **Density of monotone displacements satisfying boundary condition.** Let $M \subset C^1(\operatorname{cl}\Omega, \mathbb{R}^n)$ and $M^+ \subset M$ be defined by

$$M := \{\boldsymbol{v} \in C^1(\operatorname{cl}\Omega, \mathbb{R}^n) : \boldsymbol{v} = \boldsymbol{0} \text{ in a neighborhood of } \mathcal{D}\},$$

$$M^+ := \{\boldsymbol{v} \in M : \hat{\boldsymbol{E}}(\boldsymbol{v}) \geq \boldsymbol{0} \text{ on } \operatorname{cl}\Omega\}.$$

We here indicate situations when (α) M is dense in W_{BD} or in W_p, $1 \leq p \leq \infty$, in an appropriate sense, and (β) M^+ is dense in W_{BD}^+ or in W_p^+. The following result deals with Problem (α).

Proposition 7.4. *Suppose that $\mathcal{D}$ is relatively open in $\partial\Omega$. Let $1 \leq p \leq \infty$, and let $\boldsymbol{u} \in W_p$. Then there exists a sequence $\boldsymbol{u}_k \in M$ such that*

(i) *if $p < \infty$ then $\boldsymbol{u}_k \to \boldsymbol{u}$ in $W^{1,p}(\Omega, \mathbb{R}^n)$;*

(ii) *if $p = \infty$ then $\boldsymbol{u}_k \to \boldsymbol{u}$ uniformly on $\operatorname{cl}\Omega$, $\hat{\boldsymbol{E}}(\boldsymbol{u}_k) \to \hat{\boldsymbol{E}}(\boldsymbol{u})$ almost everywhere on Ω and $\sup_{k=1,\dots} |\nabla \boldsymbol{u}_k|_\infty < \infty$ where $|\cdot|_\infty$ is the L^∞ norm.*

This is proved in a way similar to [24, Theorem I.1 and I.3].

Problem (β) is more difficult. The standard results for the density of $C^\infty(\operatorname{cl}\Omega, \mathbb{R}^n)$ in $BD(\Omega)$ or in $W^{1,p}(\Omega, \mathbb{R}^n)$, employ constructions based on the multiplication of the function $\boldsymbol{v}$ by members θ_i of a suitable partition of unity. This operation does not preserve the positive semidefinite character of the strain tensor: One has

$$\hat{\boldsymbol{E}}(\theta_i \boldsymbol{v}) = \theta_i \hat{\boldsymbol{E}}(\boldsymbol{v}) + \tfrac{1}{2}(\boldsymbol{v} \otimes \nabla\theta_i + \nabla\theta_i \otimes \boldsymbol{v}),$$

and the expression in the bracket is not positive semidefinite in general. It is essentially only the homothetical extension that preserves positive semidefinitenes. We are therefore forced to impose strong hypotheses on Ω and $\mathcal{D}$ to be able to apply homothety to prove the density of M^+ in W_p^+.

Proposition 7.5. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set with Lipschitz boundary and let $\mathcal{D} \subset \partial\Omega$. Assume that one of the following three conditions is satisfied:*

(a) *the set $\mathcal{D}$ is nonempty, convex, is contained in some hyperplane H of normal $\boldsymbol{m}$, is open in H, and*

$$\Omega \subset \mathcal{C} := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \boldsymbol{a} + s\boldsymbol{m} \text{ for some } \boldsymbol{a} \in \mathcal{D} \text{ and some } s > 0\},$$

see Figure 1;

(b) $\operatorname{int}\mathcal{D}$ *is a class* 1 *surface and the set Z_1 defined in Remark 7.2 is nonempty.*

(c) $\mathcal{D} = \emptyset$.

Let $1 \le p \le \infty$. Let $\boldsymbol{u} \in W_p^+$. Then there exists a sequence $\boldsymbol{u}_k \in M^+$ such that we have Assertions (i) *and* (ii) *of Proposition 7.4.*

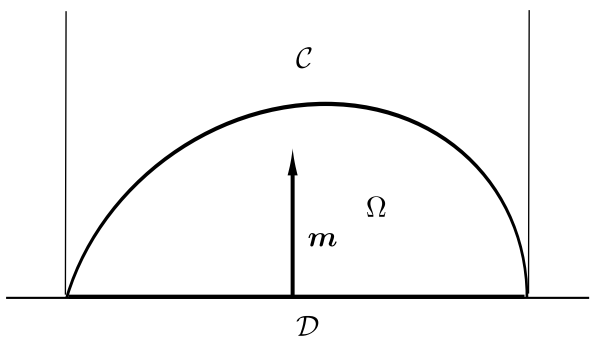

FIGURE 1.

Proof. Suppose that (a) holds. We can assume that $\mathbf{0} \in \mathcal{D}$. For each $\epsilon > 0$ let

$$\Omega_\epsilon := (1+\epsilon)\Omega \cup \{\boldsymbol{a} + s\boldsymbol{m} \in \mathbb{R}^n : \boldsymbol{a} \in (1+\epsilon)\mathcal{D}, -\epsilon < s \le 0\}.$$

We have $\operatorname{cl}\Omega \subset \Omega_\epsilon$ and Ω_ϵ is convex. Let $\boldsymbol{v}_\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ be defined by

$$\boldsymbol{v}_\epsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{u}(\boldsymbol{x}/(1+\epsilon)) & \text{if } \boldsymbol{x} \in (1+\epsilon)\Omega, \\ \mathbf{0} & \text{if } \boldsymbol{x} \in \Omega_\epsilon \setminus (1+\epsilon)\Omega. \end{cases}$$

Since $\boldsymbol{u} = \mathbf{0}$ on $\mathcal{D}$, we have $\boldsymbol{v}_\epsilon \in BD(\Omega_\epsilon)$ or $\boldsymbol{v}_\epsilon \in W^{1,p}(\Omega_\epsilon, \mathbb{R}^n)$ for each $\epsilon > 0$; moreover, $\hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon)$ is positive semidefinite. Let $\boldsymbol{w}_\epsilon$ be an $\epsilon/4$ mollification of $\boldsymbol{v}_\epsilon$, i.e., a map on $\Omega_{3\epsilon/4}$ with values in $\mathbb{R}^n$ given by

$$\boldsymbol{w}_\epsilon(\boldsymbol{y}) = (\epsilon/4)^{-n} \int_{\Omega_\epsilon} \boldsymbol{v}_\epsilon(\boldsymbol{x}) \sigma((\boldsymbol{y} - \boldsymbol{x})/(\epsilon/4))\, d\mathcal{L}^n(\boldsymbol{x})$$

for every $\boldsymbol{y} \in \operatorname{cl}\Omega_{3\epsilon/4}$. Here σ is a mollifier (see the proof of Remark 7.1). Then $\boldsymbol{w}_\epsilon \in C^\infty(\operatorname{cl}\Omega_{\epsilon/2}, \mathbb{R}^n)$, $\boldsymbol{w}_\epsilon(\boldsymbol{y}) = \mathbf{0}$ for every $\boldsymbol{y}$ from the set

$$\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} = \boldsymbol{a} + s\boldsymbol{m} : \boldsymbol{a} \in (1+\epsilon/2)\mathcal{D}, -\epsilon/2 < s < -\epsilon/4\}.$$

Moreover, $\hat{\boldsymbol{E}}(\boldsymbol{w}_\epsilon) \ge \mathbf{0}$ on $\operatorname{cl}\Omega_{3\epsilon/4}$ since $\hat{\boldsymbol{E}}(\boldsymbol{w}_\epsilon)$ is the $\epsilon/4$ mollification of $\hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon)$. Let finally $\boldsymbol{u}_\epsilon : \Omega \to \mathbb{R}^n$ be defined by

$$\boldsymbol{u}_\epsilon(\boldsymbol{x}) = \boldsymbol{w}_\epsilon(\boldsymbol{x} - \epsilon\boldsymbol{m})$$

for every $\boldsymbol{x} \in \Omega$. Then $\boldsymbol{u}_\epsilon \in M^+$ and an argument similar to that of [1, Proof of Proposition 3.22, pp. 69–70] shows that $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$ in the sense of Assertions (i)–(iii) of Proposition 7.4 as $\epsilon \to 0$. This completes the proof under Hypothesis (a).

Suppose that (b) takes place. We may suppose that $\mathbf{0} \in Z_1 \subset \operatorname{co} Z_1$. For each $\epsilon > 0$, let

$$\Omega_\epsilon := (1+\epsilon)\Omega$$

so that the convexity implies that $\operatorname{cl}\Omega \subset \Omega_\epsilon$. Let $\boldsymbol{v}_\epsilon : \Omega_\epsilon \to \mathbb{R}^n$ be defined by

$$\boldsymbol{v}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}/(1+\epsilon)), \quad \boldsymbol{x} \in \Omega_\epsilon.$$

Since $\boldsymbol{u} = \mathbf{0}$ on $\operatorname{co} Z_1$ by Proposition 7.3, we have

$$\boldsymbol{v}_\epsilon = \mathbf{0} \ \text{ on } \quad A_\epsilon := (1+\epsilon)\operatorname{co} Z_1.$$

Let $\boldsymbol{v}_\epsilon$ be the $\epsilon/4$ mollification of $\boldsymbol{v}_\epsilon$, so that

$$\boldsymbol{v}_\epsilon \in C^\infty(\Omega_{3\epsilon/4}, \mathbb{R}^n), \quad \hat{\boldsymbol{E}}(\boldsymbol{v}_\epsilon) \geq \mathbf{0} \ \text{ on } \ \Omega_{3\epsilon/4},$$

and

$$\boldsymbol{v}_\epsilon = \mathbf{0} \ \text{ on } \quad A_{3\epsilon/4}.$$

Then the restriction $\boldsymbol{u}_\epsilon$ of $\boldsymbol{v}_\epsilon$ to Ω satisfies $\boldsymbol{u}_\epsilon \in M^+$ and $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$ in the sense of Assertions (i) and (ii) of Proposition 7.4.

If (c) holds, one proceeds similarly, but the proof is easier, as no boundary condition has to be satisfied. □

7C. **Example of collapse without collapse mechanism.** In the example to be given here, the value of $\lambda = 1/4$ corresponds to the collapse, and yet there is no corresponding mechanism.

Example 7.6. *Let $\Omega = (0,1)^2$ and let $\mathcal{D} = [0,1] \times \{0\}$,*

$$\boldsymbol{s}_\circ(\boldsymbol{r}) = \begin{cases} -\boldsymbol{j} & \text{if } \boldsymbol{r} \in \mathcal{S}_t := (0,1) \times \{1\}, \\ \mathbf{0} & \text{if } \boldsymbol{r} \in \mathcal{S} \setminus \mathcal{S}_t, \end{cases}$$

$$\bar{\boldsymbol{s}}(\boldsymbol{r}) = \begin{cases} (4(x-1/2)^2 - 1)\boldsymbol{i} & \text{if } \boldsymbol{r} \in \mathcal{S}_t := (0,1) \times \{1\}, \\ \mathbf{0} & \text{if } \boldsymbol{r} \in \mathcal{S} \setminus \mathcal{S}_t, \end{cases}$$

$$\boldsymbol{b}_\circ = \bar{\boldsymbol{b}} = \mathbf{0} \ \text{ on } \quad \Omega,$$

see Figure 2. Then

(i) *we have*

$$\lambda_q = \bar{\lambda}_p = 1/4 \ \text{ for all } \ p, q \in [1, \infty];$$

(ii) *if $|\lambda| < 1/4$ then λ is $L^q(\Omega, \mathrm{Sym})$ statically admissible for any $q \in [1, \infty]$;*

(iii) *if $|\lambda| = 1/4$ then λ is $L^q(\Omega, \mathrm{Sym})$ statically admissible for any $q \in [1, 3)$;*

(iv) *if $q \in [1, \infty]$ then there is no $L^q(\Omega, \mathrm{Sym})$ mechanism corresponding to $\lambda = 1/4$.*

Here $\boldsymbol{i} = (1,0)$ and $\boldsymbol{j} = (0,1)$.

(a) We do not know if $\lambda = \pm 1/4$ is $L^q(\Omega, \mathrm{Sym})$ statically admissible for $q \geq 3$. (b) The case $p = q = 2$ has been treated in [15, Example 2.3]. It was shown that $\lambda_2 = \bar{\lambda}_2 = 1/4$, and an admissible stressfield $\boldsymbol{T}(\lambda)$ determined in [19] was shown to balance the loads for all $\lambda \in [-1/4, 1/4]$. The stressfield $\boldsymbol{T}(\lambda)$ was shown to be bounded if $|\lambda| < 1/4$ and to belong to $L^2(\Omega, \mathrm{Sym})$ if $|\lambda| = 1/4$. Furthermore, it was shown that no corresponding mechanism exists in W_2. Actually, $\boldsymbol{T}(1/4) \in L^q(\Omega, \mathrm{Sym})$ for all $q \in [1, 3)$, as a closer examination shows, and thus (ii), (iii) above follow from the construction in the cited references. Here we shall prove the existence of $\boldsymbol{T}$

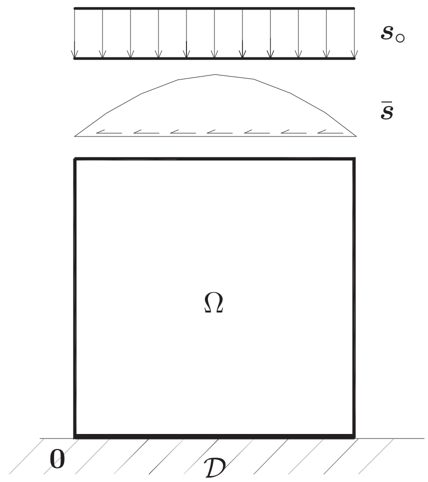

Figure 2.

balancing the loads for multipliers as in (ii), (iii), without giving an explicit formula for it, by using Proposition 2.8.

Lemma 7.7. *Let either* $0 \le \lambda < 1/4$ *and* $q \in [1, \infty]$ *or let* $\lambda = 1/4$ *and* $q \in [1, 3)$. *Then* λ *is* $L^q(\Omega, \mathrm{Sym})$ *statically admissible.*

Proof. We employ Proposition 2.8. Hence we seek to show that corresponding to λ and q there exists a constant c such that

$$\sup\{\langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle : \boldsymbol{v} \in W_p,\ \hat{\boldsymbol{E}}(\boldsymbol{v}) \ge \boldsymbol{F} \text{ almost everywhere on } \Omega\} \le\ c|\boldsymbol{F}|_p$$

for every $\boldsymbol{F} \in L^p(\Omega, \mathrm{Sym})$, where p is the Hölder conjugate exponent to q and $|\boldsymbol{F}|_p$ is the L^p norm of $\boldsymbol{F}$.

Thus let $\boldsymbol{F} \in L^p(\Omega, \mathrm{Sym})$ and let $\boldsymbol{v} \in W_p$ satisfy

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) \ge \boldsymbol{F} \ \text{ almost everywhere on } \Omega \tag{7-12}$$

and prove that

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle \le c|\boldsymbol{F}|_p. \tag{7-13}$$

Write $\boldsymbol{m}(\alpha) := \boldsymbol{s}(\lambda)(\alpha, 1)$ for every $\alpha \in [0, 1]$. For any $\alpha \in [0, 1]$ let

$$\mathfrak{l}(\alpha) := \{\boldsymbol{a}(\alpha) + z\boldsymbol{m}(\alpha) : z \in \mathbb{R}\}$$

be the line through the point $\boldsymbol{a}(\alpha) := (\alpha, 1) \in \mathcal{S}_t$ and of direction parallel to $\boldsymbol{m}(\alpha)$. The line $\mathfrak{l}(\alpha)$ always intersects the x axis because $\boldsymbol{m}(\alpha)$ is never horizontal, viz., at the point

$$\boldsymbol{b}(\alpha) = (\beta(\alpha), 0)$$

where

$$\beta(\alpha) = \alpha + \lambda(4(\alpha - 1/2)^2 - 1).$$

One has

$$\beta(0) = 0, \quad \beta(1) = 1.$$

If $0 \le \lambda \le 1/4$, then β is a nondecreasing function on the interval $[0, 1]$ thus

$$0 \le \beta(\alpha) \le 1 \ \text{ for every } \alpha \in [0, 1];$$

in other words, the line $\mathfrak{l}(\alpha)$ intersects the x axis at some point $\boldsymbol{b}(\alpha)$ of the base $\mathcal{D}$, where the panel is fixed. We now consider the segment $\mathfrak{s}(\alpha)=\mathfrak{l}(\alpha)\cap \operatorname{cl}\Omega$ with endpoint $\boldsymbol{a}(\alpha)$, $\boldsymbol{b}(\alpha)$. The precise representative $\tilde{\boldsymbol{v}}$ of $\boldsymbol{v}$ is absolutely continuous on the segment $\mathfrak{s}(\alpha)$ and $\boldsymbol{F}(\alpha)$ is defined for $\mathcal{H}^1$ almost every point of $\mathfrak{s}(\alpha)$ and $\boldsymbol{F}$ is $\mathcal{H}^1$ integrable on $\mathfrak{s}(\alpha)$. Equation (7-12) then gives

$$\frac{d}{dz}\boldsymbol{v}(\boldsymbol{a}(\alpha)+z\boldsymbol{t})\cdot\boldsymbol{t}=\hat{\boldsymbol{E}}(\boldsymbol{v})(\boldsymbol{a}(\alpha)+z\boldsymbol{t})\boldsymbol{t}\cdot\boldsymbol{t}\geq \boldsymbol{F}(\boldsymbol{a}(\alpha)+z\boldsymbol{t})\boldsymbol{t}\cdot\boldsymbol{t}$$

for $\mathcal{L}^1$ almost every $z\in[0,1]$, where we write $\boldsymbol{t}=\boldsymbol{m}(\alpha)$ for brevity. Thus integrating with respect to z over the interval $[0,1]$ and using $\boldsymbol{v}(\boldsymbol{a}(\alpha)+\boldsymbol{t})\equiv\boldsymbol{v}(\boldsymbol{b}(\alpha))=\boldsymbol{0}$, we find that

$$\boldsymbol{v}(\boldsymbol{a}(\alpha))\cdot\boldsymbol{t}\leq-\int_0^1\boldsymbol{F}(\boldsymbol{a}(\alpha)+z\boldsymbol{t})\boldsymbol{t}\cdot\boldsymbol{t}\,dz\leq\int_0^1|\boldsymbol{F}(\boldsymbol{a}(\alpha)+z\boldsymbol{t})||\boldsymbol{t}|^2\,dz.$$

We have $|\boldsymbol{t}|=|\boldsymbol{m}(\alpha)|\leq d<\infty$ for all $\boldsymbol{a}(\alpha)\in\mathcal{S}_t$ and thus we obtain that

$$\boldsymbol{v}(\boldsymbol{a}(\alpha))\cdot\boldsymbol{m}(\alpha)\leq d^2\int_0^1|\boldsymbol{F}(\boldsymbol{a}(\alpha)+z\boldsymbol{m}(\alpha))|\,dz \tag{7-14}$$

and hence, integrating (7-14) over $\mathcal{T}_t$ and using $\boldsymbol{m}(\alpha):=\boldsymbol{s}(\lambda)(\alpha,1)$ we obtain

$$\langle\boldsymbol{l}(\lambda),\boldsymbol{v}\rangle\leq d^2\int_0^1\int_0^1|\boldsymbol{F}(\boldsymbol{a}(\alpha)+z\boldsymbol{m}(\alpha))|\,dz\,d\alpha. \tag{7-15}$$

Let us now estimate the integral

$$I:=\int_0^1\int_0^1|\boldsymbol{F}((\alpha,1)+z\boldsymbol{m}(\alpha))|\,dz\,d\alpha.$$

Consider a change of variables Φ from (z,α) to $(x,y)=(\alpha,1)+z\boldsymbol{m}(\alpha))$, i.e.,

$$x=\alpha+z\lambda(4(\alpha-1/2)^2-1),\quad y=1-z.$$

The Jacobian of this transformation $J:=|\det \mathrm{D}\,\Phi|$ is

$$J=1+8\lambda\alpha z-4\lambda z.$$

Applying Hölder's inequality to the measure $J\,dz\,d\alpha$ we obtain

$$I=\int_0^1\int_0^1|\boldsymbol{F}((\alpha,1)+z\boldsymbol{m}(\alpha))|J^{-1}J\,\,dz\,d\alpha\leq K^{1/p}L^{1/q}$$

where

$$K=\int_0^1\int_0^1|\boldsymbol{F}((\alpha,1)+z\boldsymbol{m}(\alpha))|^pJ\,\,dz\,d\alpha\equiv\int_0^1\int_0^1|\boldsymbol{F}(x,y)|^p\,\,dx\,dy\equiv|\boldsymbol{F}|_p^p,$$

and

$$L=\int_0^1\int_0^1J^{1-q}\,\,dz\,d\alpha=\int_0^1\int_0^1(1+8\lambda\alpha z-4\lambda z)^{1-q}\,\,dz\,d\alpha.$$

We now distinguish two cases.

Case (a): Let $0\leq\lambda<1/4$. Then $1+8\lambda\alpha z-4\lambda z\geq 1-4\lambda$ and thus

$$I\leq(1-4\lambda)^{(1-q)/q}|\boldsymbol{F}|_p.$$

Inequality (7-15) then yields

$$\langle\boldsymbol{l}(\lambda),\boldsymbol{v}\rangle\leq d^2(1-4\lambda)^{(1-q)/q}|\boldsymbol{F}|_p$$

and thus we have (7-13) with $c=d^2(1-4\lambda)^{(1-q)/q}$. Consequently, the conclusion of Proposition 7.3 holds and thus λ is $L^q(\Omega,\mathrm{Sym})$ statically admissible for every $q\in[1,\infty]$.

Case (b): $\lambda = 1/4$. Then

$$L = \int_0^1 \int_0^1 (1 + 2z\alpha - z)^{1-q} \, dz \, d\alpha \tag{7-16}$$

and hence

$$\begin{aligned} L &= (2-q)^{-1} \textstyle\int_0^1 (2\alpha - 1)^{-1} (1 + 2z\alpha - z)^{2-q} \big|_{z=0}^{z=1} \, d\alpha \\ &= (2-q)^{-1} \textstyle\int_0^1 (2\alpha - 1)^{-1} ((2\alpha)^{2-q} - 1) \, d\alpha \end{aligned}$$

provided $q \neq 2$. Under this assumption, it is easily found that the integrand of the last integral has the only singularity at $\alpha = 0$, where it behaves as a constant multiple of α^{2-q}. The apparent singularity at $\alpha = 1/2$ does not occur as

$$\lim_{\alpha \to 1/2} (2\alpha - 1)^{-1} ((2\alpha)^{2-q} - 1) = 1 - q/2.$$

Thus, if $2 - q > -1$, i.e., if $1 \le q < 3$, and $q \neq 2$, the integral in (7-16) thus converges. It is easily found that the integral in (7-16) converges also if $q = 2$. Thus (7-15) yields

$$\langle \boldsymbol{l}(1/4), \boldsymbol{v} \rangle \le d^2 L^{1/q} |\boldsymbol{F}|_p$$

for every $\boldsymbol{F} \in L^p(\Omega, \mathrm{Sym})$ and we have (7-13) with $c = d^2 L^{1/q}$. □

Remark 7.8. Assertions (ii) and (iii) follow from Lemma 7.7 by noting that any $\lambda \in \mathbb{R}$ is $L^q(\Omega, \mathrm{Sym})$ statically admissible if and only if so also is $-\lambda$ (it suffices to change the orientation of the x axis). It also follows from (ii) (or from Lemma 7.7) that $\lambda_q \le 1/4$ for each $q \in [1, \infty]$.

Lemma 7.9. *For each $\lambda > 1/4$ and $p \in [1, \infty]$ there exists a W_p mechanism corresponding to λ.*

Proof. Let $p \in [1, \infty]$ be arbitrary. If $\omega : \mathbb{R} \to \mathbb{R}$ is any nonincreasing C^1 function vanishing on $(1, \infty)$ that does not vanish identically on $(0, 1)$ then $\boldsymbol{v} : \Omega \to \mathbb{R}^2$, given by

$$\boldsymbol{v}(\boldsymbol{r}) = \omega(x/y) \boldsymbol{r}^\perp, \tag{7-17}$$

$\boldsymbol{r} = (x, y) \in \Omega$, $\boldsymbol{r}^\perp := (-y, x)$ satisfies $\boldsymbol{v} \in W_p$ and $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^p(\Omega, \mathrm{Sym}^+)$. Indeed, one finds that $\boldsymbol{v} \in W^{1,p}(\Omega, \mathbb{R}^2)$ and since ω vanishes on $(1, \infty)$, $\boldsymbol{v}$ vanishes on

$$\Omega^- := \{\boldsymbol{r} \in \Omega : x/y > 1\}$$

and thus in particular on $\mathcal{D}$ (in the sense of trace). Hence $\boldsymbol{v} \in W_p$. Furthermore,

$$\hat{\boldsymbol{E}}(\boldsymbol{v})(\boldsymbol{r}) = -y^{-2} \omega'(x/y) \boldsymbol{r}^\perp \otimes \boldsymbol{r}^\perp$$

$\boldsymbol{r} \in \Omega$, and as $\omega' \le 0$ we have $\hat{\boldsymbol{E}}(\boldsymbol{v}) \in L^p(\Omega, \mathrm{Sym}^+)$. One has

$$\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle = -\int_{\mathcal{T}} \omega(x/y) x \, d\mathcal{H}^1(\boldsymbol{r}) = -\int_0^1 \omega(x) x \, dx$$

$$\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 4 \int_{\mathcal{T}} \omega(x/y) y x (1 - x) \, d\mathcal{H}^1(\boldsymbol{r}) = 4 \int_0^1 \omega(x) x (1 - x) \, dx;$$

noting that the last expression and the hypotheses on ω imply that $\langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle > 0$, we thus deduce that the value

$$\lambda = -\langle \boldsymbol{l}_0, \boldsymbol{v} \rangle / \langle \bar{\boldsymbol{l}}, \boldsymbol{v} \rangle = 4^{-1} \int_0^1 \omega(x) x \, dx \Big/ \int_0^1 \omega(x) x (1 - x) \, dx \tag{7-18}$$

is a kinematically admissible multiplier. Fixing $\epsilon \in (0,1)$ and taking a sequence of functions of the type of ω that converges to the function ω_ϵ given by

$$\omega_\epsilon(t) = \begin{cases} 1 & \text{if } t \le \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$t \in \mathbb{R}$, we deduce from (7-18) by evaluating the integrals that the value

$$\lambda = 1/4(1 - 2\epsilon/3)$$

is kinematically admissible. Varying $\epsilon \in (0,1)$ we obtain the interval $(1/4, 3/4)$. $\square$

Remark 7.10. It follows from Lemma 7.9 that $\bar{\lambda}_p \le 1/4$ for each $p \in [1,\infty]$. Combining $1/4 \le \bar{\lambda}_p \le \lambda_q \le 1/4$ we obtain (i).

Lemma 7.11. *If $\lambda = 1/4$ and $p \in [1,\infty]$ then there is no corresponding W_p mechanism.*

Proof. We prove that if $\boldsymbol{v} \in W_p^+$ and $\langle \boldsymbol{l}(1/4), \boldsymbol{v} \rangle = 0$ then $\boldsymbol{v} = \boldsymbol{0}$ almost everywhere on Ω. For any $\alpha \in [0,1]$ let $\boldsymbol{a}(\alpha) \in \mathcal{S}_t$ and $\boldsymbol{b}(\alpha) \in \mathcal{D}$ be as in the proof of Lemma 7.7. Then for $\mathcal{L}^1$ almost every $\alpha \in [0,1]$ we have

$$\big(\boldsymbol{v}(\boldsymbol{a}(\alpha)) - \boldsymbol{v}(\boldsymbol{b}(\alpha))\big) \cdot \big(\boldsymbol{a}(\alpha) - \boldsymbol{b}(\alpha)\big) \ge 0;$$

using that $\boldsymbol{v}(\boldsymbol{b}(\alpha)) = \boldsymbol{0}$, we obtain

$$\boldsymbol{v}(\boldsymbol{a}(\alpha)) \cdot \boldsymbol{s}(1/4)(\alpha, 1) \le 0.$$

Comparing this with

$$\langle \boldsymbol{l}(1/4), \boldsymbol{v} \rangle = \int_0^1 \boldsymbol{v} \cdot \boldsymbol{s}(1/4)(\alpha,1)\, d\alpha = 0$$

we obtain

$$\boldsymbol{v}(\boldsymbol{a}(\alpha)) \cdot \boldsymbol{s}(1/4)(\alpha, 1) = 0 \tag{7-19}$$

for $\mathcal{L}^1$ almost every $\alpha \in (0,1)$. Furthermore, for almost every $\alpha \in (0,1)$ we have

$$\big(\boldsymbol{v}(\boldsymbol{a}(\alpha)) - \boldsymbol{v}(\boldsymbol{0})\big) \cdot \boldsymbol{a}(\alpha) \ge 0, \quad \big(\boldsymbol{v}(\boldsymbol{a}(\alpha)) - \boldsymbol{v}(\boldsymbol{i})\big) \cdot \big(\boldsymbol{a}(\alpha) - \boldsymbol{i}\big) \ge 0,$$

which reduces to

$$\boldsymbol{v}(\boldsymbol{a}(\alpha)) \cdot \boldsymbol{a}(\alpha) \ge 0, \quad \boldsymbol{v}(\boldsymbol{a}(\alpha)) \cdot \big(\boldsymbol{a}(\alpha) - \boldsymbol{i}\big) \ge 0 \tag{7-20}$$

for $\mathcal{L}^1$ almost every $\alpha \in (0,1)$. It is now easily seen that Conditions (7-19) and (7-20) imply that $\boldsymbol{v}(\boldsymbol{a}(\alpha)) = \boldsymbol{0}$ for $\mathcal{L}^1$ almost every $\alpha \in (0,1)$. We thus have $\boldsymbol{v} = \boldsymbol{0}$ on $\mathcal{S}_t$ and on $\mathcal{D}$ and thus Proposition 7.3 implies that $\boldsymbol{v} = \boldsymbol{0}$ on $\operatorname{co} Z_1$ which is Ω. $\square$

7D. **Example of the violation of the kinematic theorem.** We here give an example in which the supremum of statically admissible multipliers depends dramatically on the choice of function spaces.

Example 7.12. *Let* $\Omega = (0,1)^2$, $\mathcal{D} = (0,1) \times \{0\}$, $\mathcal{S} = \partial\Omega \setminus \mathcal{D}$,

$$\boldsymbol{s}(\lambda)(\boldsymbol{r}) = \begin{cases} (1-\lambda)\boldsymbol{r} & \textit{if } \boldsymbol{r} \in \mathcal{S}_t := (0,1) \times \{1\}, \\ (\lambda/2 - 1)(\boldsymbol{i} + \boldsymbol{j}) & \textit{if } \boldsymbol{r} \in \mathcal{S}_r := \{1\} \times (0,1), \\ \boldsymbol{0} & \textit{if } \boldsymbol{r} \in \mathcal{S}_l := \{0\} \times (0,1) \end{cases}$$

$$\boldsymbol{b}(\lambda) = \boldsymbol{0} \quad \text{on} \quad \Omega,$$

$\lambda \in \mathbb{R}$. *Then*

$$\bar{\lambda}_p = 2 \quad \text{for all} \quad p \in [1, \infty] \tag{7-21}$$

$$\lambda_q = \begin{cases} 2 & \textit{if } q \in [1,2), \\ 1 & \textit{if } q \in [2,\infty]. \end{cases} \tag{7-22}$$

See Figure 3.

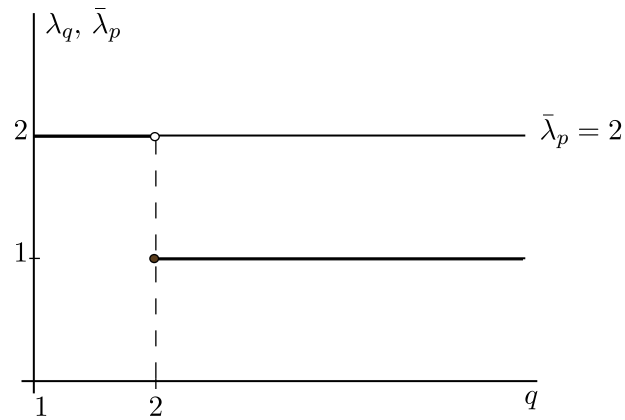

FIGURE 3.

Lemma 7.13. *For any* $\lambda \in \mathbb{R}$, *let* $\boldsymbol{T}(\lambda) : \Omega \to \mathrm{Sym}$ *be defined by*

$$\boldsymbol{T}(\lambda)(\boldsymbol{r}) = \begin{cases} (1-\lambda)\boldsymbol{r} \otimes \boldsymbol{r}/y^3 & \textit{if } \boldsymbol{r} \in \Omega^+ := \{\boldsymbol{r} \in \Omega : y/x > 1\}, \\ (\lambda/2 - 1)(\boldsymbol{i} + \boldsymbol{j}) \otimes (\boldsymbol{i} + \boldsymbol{j}) & \textit{if } \boldsymbol{r} \in \Omega^- := \{\boldsymbol{r} \in \Omega : y/x < 1\}. \end{cases}$$

Then $\boldsymbol{T}(\lambda)$ *equilibrates the loads in the sense that*

$$(\boldsymbol{T}(\lambda), \boldsymbol{v}) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \tag{7-23}$$

for each $\boldsymbol{v}$ *from the set*

$$M := \{\boldsymbol{v} \in C^1(\mathrm{cl}\,\Omega, \mathbb{R}^2) : \boldsymbol{v} = \boldsymbol{0} \text{ near } \mathcal{D}\};$$

moreover,

$$\left.\begin{array}{ll} \boldsymbol{T}(\lambda) \in L^q(\Omega, \mathrm{Sym}) & \text{for all } q \in [1,2) \text{ and all } \lambda \in \mathbb{R}, \\ \boldsymbol{T}(1) \in L^q(\Omega, \mathrm{Sym}) & \text{for all } q \in [1,\infty], \\ \boldsymbol{T}(\lambda) \notin L^q(\Omega^+, \mathrm{Sym}) & \text{for all } q \in [2,\infty] \text{ and all } \lambda \in \mathbb{R},\ \lambda \neq 1, \end{array}\right\} \tag{7-24}$$

and

$$\boldsymbol{T}(\lambda) \le \boldsymbol{0} \quad \mathcal{L}^2 \text{ almost everywhere on } \Omega \text{ if and only if } 1 \le \lambda \le 2. \tag{7-25}$$

Proof. One finds that (a) $\boldsymbol{T}(\lambda)$ can be continuously extended to $\partial\Omega^+ \setminus \{\mathbf{0}\}$ and also to $\partial\Omega^-$, (b) $\boldsymbol{T}(\lambda)$ satisfies the boundary condition

$$\boldsymbol{T}(\lambda)\boldsymbol{n} = \boldsymbol{s}(\lambda) \quad \text{on} \quad \mathcal{S} \setminus \{\mathbf{0}\},$$

in the classical sense, (c) $\boldsymbol{T}(\lambda)$ is discontinuous across the segment $\mathfrak{l} := \{\boldsymbol{r} \in \Omega : y = x\}$, but the normal component of $\boldsymbol{T}(\lambda)$ is continuous across $\mathfrak{l}$, (d) $\boldsymbol{T}(\lambda)$ is of class ∞ on Ω^+ and Ω^- with the classical divergence vanishing on each of these two regions:

$$\operatorname{div} \boldsymbol{T}(\lambda) = \mathbf{0} \quad \text{on} \quad \Omega^{\pm}.$$

If $\boldsymbol{v} \in M$ then there exists an $\epsilon > 0$ such that $\boldsymbol{v}$ vanishes on the line segment $\{\boldsymbol{r} \in \Omega : y = \epsilon\}$. Applying of the classical divergence theorem to $\Omega^+_\epsilon = \{\boldsymbol{r} \in \Omega^+ : y > \epsilon\}$ and $\Omega^-_\epsilon = \{\boldsymbol{r} \in \Omega^- : y > \epsilon\}$ separately (using of the properties (a)–(d)), and adding the results gives (7-23).

To prove the properties (7-24), we note that $\boldsymbol{T}(\lambda)$ is bounded on Ω^- while $\boldsymbol{T}(\lambda)$ has a singularity on Ω^+ at $\mathbf{0}$. Thus we have $\boldsymbol{T}(\lambda) \in L^q(\Omega, \mathrm{Sym})$ if and only if $\boldsymbol{T}(\lambda) \in L^q(\Omega^+, \mathrm{Sym})$. We have

$$|1 - \lambda|/y \le |\boldsymbol{T}(\lambda)(\boldsymbol{r})| \le \sqrt{2}|1 - \lambda|/y$$

for each $\boldsymbol{r} \in \Omega^+$. Thus

$$|1 - \lambda|^q I_q \le \int_{\Omega^+} |\boldsymbol{T}(\lambda)|^q \, d\mathcal{L}^2 \le (\sqrt{2})^q |1 - \lambda|^q I_q$$

where

$$I_q = \int_{\Omega^+} y^{-q} \, d\mathcal{L}^2.$$

One has $I_q < \infty$ if and only if $q \in [1, 2)$ and (7-24) follows.

Assertion (7-25) is immediate. □

Lemma 7.14. *All λ satisfying $1 \le \lambda < 2$ are W_p kinematically inadmissible for all $p \in [1, \infty]$, and every $\lambda \in (2, 7)$ is W_p kinematically admissible. In particular, we have* (7-21)*.*

Proof. Let $1 \le \lambda < 2$. By (7-23) and (7-25) we have

$$0 \ge (\boldsymbol{T}(\lambda), \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$$

for all $\boldsymbol{v} \in M^+ := \{\boldsymbol{v} \in M : \hat{\boldsymbol{E}}(\boldsymbol{v}) \ge \mathbf{0} \text{ on } \operatorname{cl} \Omega\}$. The continuity of the loads on W_p for every $p \in [1, \infty]$ and the density of M^+ in W_p^+ (Proposition 7.5(a)) then imply that the inequality $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ can be extended to W_p^+, i.e., we have $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for all $\boldsymbol{v} \in W_p^+$, and all $\lambda \in [1, 2)$. This implies that no $\lambda \in [1, 2)$ is kinematically admissible, as we now show. Indeed, one easily verifies from the definition that if $\lambda \in \mathbb{R}$ is kinematically admissible then for every $\mu > \lambda$ there exists a $\boldsymbol{v} \in W_p^+$ such that $\langle \boldsymbol{l}(\mu), \boldsymbol{v} \rangle > 0$. Thus the hypothesis that some $\lambda \in [1, 2)$ is kinematically admissible would imply that for every $\mu > \lambda$ there is a $\boldsymbol{v} \in W_p^+$ such that $\langle \boldsymbol{l}(\mu), \boldsymbol{v} \rangle > 0$. However, we have shown that $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \le 0$ for all $\boldsymbol{v} \in W_p^+$, and all $\lambda \in [1, 2)$; in particular $\langle \boldsymbol{l}(\mu), \boldsymbol{v} \rangle \le 0$ for all $\boldsymbol{v} \in W_p^+$, and all $\mu \in (\lambda, 2)$. Thus λ cannot be kinematically admissible.

Next we prove that every $\lambda \in (2, 7)$ is W_p kinematically admissible for every $p \in [1, \infty]$ by exhibiting a mechanism corresponding to λ which is in all W_p. Let s be a number satisfying $0 < s < 1$, put

$$\boldsymbol{m} = (1, s),$$

denote by Ω_s the set

$$\Omega_s = \{\boldsymbol{r} : y \ge 1 - sx\},$$

and define $\boldsymbol{v}_s$ by

$$\boldsymbol{v}_s(\boldsymbol{r}) = \begin{cases} (\boldsymbol{r}\cdot\boldsymbol{m}-1)\boldsymbol{m} & \text{if } \boldsymbol{r}\in\Omega_s, \\ \boldsymbol{0} & \text{if } \boldsymbol{r}\in\Omega\setminus\Omega_s. \end{cases}$$

Then $\boldsymbol{v}_s \in W_p^+$ for all $p\in[1,\infty]$ with

$$\hat{\boldsymbol{E}}(\boldsymbol{v}_s) = \begin{cases} \boldsymbol{m}\otimes\boldsymbol{m} & \text{on } \Omega_s, \\ \boldsymbol{0} & \text{on } \Omega\setminus\Omega_s. \end{cases}$$

Calculations give

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v}_s\rangle = ((-2s^3-3s^2+3+3s)\lambda + 2s^3+3s^2-6s-6)/6$$
$$\langle \boldsymbol{l}_\circ, \boldsymbol{v}_s\rangle = (-2s^3-3s^2+3+3s)/6$$

If $s\in(0,1)$ then one finds that

$$\langle \boldsymbol{l}(\lambda_s), \boldsymbol{v}_s\rangle = 0$$

for

$$\lambda_s = (2s^3+3s^2-6s-6)/(2s^3+3s^2-3s-3).$$

The function $s\mapsto\lambda_s$ is increasing and maps the interval $(0,1)$ onto the interval $(2,7)$, as one easily finds. Moreover,

$$\langle \boldsymbol{l}_\circ, \boldsymbol{v}_s\rangle > 0$$

for all $s\in(0,1)$. It therefore follows that for every $\lambda\in(2,7)$ there exist a s_λ such that $\boldsymbol{v}_{s_\lambda}$ is a mechanism corresponding to λ. This proves that every $\lambda\in(2,7)$ is kinematically admissible. □

Remark 7.15. The proof of Lemma 7.14 shows that if $1\le\lambda\le 2$ and $p\in[1,\infty]$ then $\langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle \le 0$ for each $\boldsymbol{v}\in W_p^+$. Despite of this, for $q\in[2,\infty]$ the multiplier λ is not statically admissible, which shows that the condition that $\langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle \le 0$ for each $\boldsymbol{v}\in W_p^+$ is only necessary, but not sufficient for the static admissibility. Cf. the discussion following Proposition 2.8.

Lemma 7.16. *A* $\lambda\in\mathbb{R}$ *is* $L^q(\Omega,\mathrm{Sym})$ *statically admissible if and only if*

$$\lambda\in\Lambda_q := \begin{cases} [1,2] & \textit{if } 1\le p<2, \\ \{1\} & \textit{if } 2\le p\le\infty. \end{cases}$$

In particular, we have (7-22).

Proof. All $\lambda<1$ are $L^q(\Omega,\mathrm{Sym})$ statically inadmissible for all $q\in[1,\infty]$. Indeed, if they were to be statically admissible, then we would have $\boldsymbol{s}(\lambda)\cdot\boldsymbol{n}\le 0$ on $\mathcal{S}_t$ by [15, Proposition 2.1(i)], where $\boldsymbol{n}$ is the outer normal to Ω on $\mathcal{S}_t$, while we have $\boldsymbol{s}(\lambda)\cdot\boldsymbol{n}>0$ everywhere on $\mathcal{S}_t$ for $\lambda<1$.

Further, $\lambda=1$ is $L^q(\Omega,\mathrm{Sym})$ statically admissible for all $q\in[1,\infty]$. Indeed, the stressfield $\boldsymbol{T}(1)$ is bounded on Ω, is admissible, and equilibrates the loads $\boldsymbol{l}(1)$.

If $1\le q<2$, then every $\lambda\in[1,2]$ is $L^q(\Omega,\mathrm{Sym})$ statically admissible. Indeed, the stressfield $\boldsymbol{T}(\lambda)$ is admissible, is in $L^q(\Omega,\mathrm{Sym})$, and equilibrates the loads in the sense that (7-23) holds for every $\boldsymbol{v}\in W_p$ where p is the Hölder conjugate of q. This is proved by using (7-23) for $\boldsymbol{v}\in M$ and applying the density of M in W_p (Proposition 7.4) and the continuity of loads on W_p.

If $2\le q\le\infty$, then every $\lambda\in(1,2)$ is $L^q(\Omega,\mathrm{Sym})$ statically inadmissible. Let $1<\lambda\le 2$ and prove that there is no admissible stressfield in $L^q(\Omega,\mathrm{Sym})$ equilibrating the loads $\boldsymbol{l}(\lambda)$. Assume, on the contrary, that $\boldsymbol{T}$ is an admissible stressfield equilibrating the loads. If $\omega:\mathbb{R}\to\mathbb{R}$ is any nonincreasing C^1 function vanishing on $(1,\infty)$ and with $\omega'<0$ on $(0,1)$, let $\boldsymbol{v}:\Omega\to\mathbb{R}^2$, defined by (7-17). As in the proof of Lemma 7.9, we have $\boldsymbol{v}\in W_p^+$. Furthermore, one finds that

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle = 0.$$

From $\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = (\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = 0$ and $\boldsymbol{T} \leq \boldsymbol{0}$, $\hat{\boldsymbol{E}}(\boldsymbol{v}) \geq \boldsymbol{0}$ on Ω we derive that $\boldsymbol{T} \cdot \hat{\boldsymbol{E}}(\boldsymbol{v}) = 0$ for $\mathcal{L}^2$ a.e. point of Ω. We have $\omega'(x/y) \neq 0$ for every point of Ω^+. Then $\boldsymbol{T}(\boldsymbol{r}) \cdot (\boldsymbol{r}^\perp \otimes \boldsymbol{r}^\perp) = 0$ for a.e. point of Ω^+ and by [15, Remark 7.3] then $\boldsymbol{T}(\boldsymbol{r})$ must be proportional to $\boldsymbol{r} \otimes \boldsymbol{r}$ and hence we write

$$\boldsymbol{T}(\boldsymbol{r}) = \eta(\boldsymbol{r})\boldsymbol{r} \otimes \boldsymbol{r}/y^3$$

for $\mathcal{L}^2$ a.e. $\boldsymbol{r} = (x, y) \in \Omega^+$ where $\eta : \Omega^+ \to \mathbb{R}$ is a $\mathcal{L}^2$ measurable function.

As in the proof of Example in [15, Proof of Example 2.4] we deduce from $\operatorname{div} \boldsymbol{T} = \boldsymbol{0}$ in Ω^+ and $\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}(\lambda)$ on $\mathcal{S}_t$ that

$$\boldsymbol{T}(\boldsymbol{r}) = \boldsymbol{T}(\lambda) \quad \text{on } \Omega^+ \ . \tag{7-26}$$

Indeed, the equation $\operatorname{div} \boldsymbol{T} = \boldsymbol{0}$ gives $\boldsymbol{r}(\boldsymbol{r} \cdot \nabla\eta) + 3\boldsymbol{r}\eta = \boldsymbol{0}$ which gives

$$\boldsymbol{r} \cdot \nabla\eta + 3\eta = 0.$$

The substitution $\eta = \hat{\eta}/y^3$ then provides

$$\boldsymbol{r} \cdot \nabla\hat{\eta} = 0.$$

Thus the directional derivative of $\hat{\eta}$ along any segment $\{\boldsymbol{r} \in \Omega^+ : \boldsymbol{r} = c\boldsymbol{d}, 0 < c < d\}$ is constant for any $\boldsymbol{d} \in \mathcal{S}_t$. The segment is completely characterized by the slope y/x and thus there exists a function $\tilde{\eta}$ on $(1, \infty)$ such that $\hat{\eta}(\boldsymbol{r}) = \tilde{\eta}(y/x)$ for every $\boldsymbol{r} \in \Omega^+$. Thus

$$\boldsymbol{T}(\lambda)(\boldsymbol{r}) = \tilde{\eta}(y/x)\boldsymbol{r} \otimes \boldsymbol{r}/y^3 \quad \text{on } \Omega^+.$$

The boundary condition $\boldsymbol{T}\boldsymbol{n} = \boldsymbol{s}(\lambda)$ on $\mathcal{S}_t$ then leads to $\tilde{\eta}(\boldsymbol{d}) = (\lambda - 1)$ for any $\boldsymbol{d} \in \mathcal{S}_t$ and we obtain finally (7-26). The argument above applies to the case when $\boldsymbol{T}$ is continuously differentiable. The general argument in case $\boldsymbol{T}$ is only measurable is give in [15, Proof of Example 2.4]. This part of the proof is omitted.

To complete the proof, we note that $\boldsymbol{T}(\lambda) \notin L^q(\Omega^+, \mathrm{Sym})$ and thus we arrive at a contradiction: Starting from arbitrary balancing stressfield in $L^q(\Omega, \mathrm{Sym})$ we obtain that $\boldsymbol{T} \notin L^q(\Omega, \mathrm{Sym})$. Thus λ is not $L^q(\Omega, \mathrm{Sym})$ statically admissible. □

8. Summary

(i) A decreasing continuous sequence of function spaces

$$W_{BD} \supset \cdots \supset W_{p_1} \supset W_{p_2} \supset \cdots \supset W_\infty \supset C_1$$

$(1 \leq p_1 < p_2 < \infty)$ has been presented. The kinematic problem with this decreasing sequence is more and more restrictive in the competitors space which results in apriori inequalities for the corresponding critical multipliers of the kinematic problems

$$\bar{\lambda}_{BD} \leq \bar{\lambda}_{p_1} \leq \bar{\lambda}_{p_2} \leq \cdots \leq \bar{\lambda}_\infty \leq \bar{\lambda}_\circ. \tag{8-1}$$

Density conditions have been given which guarantee that the sequence (8-1) is constant.

(ii) A continuous sequence of function spaces $C^0(\operatorname{cl}\Omega, \mathrm{Sym})$, $L^q(\Omega, \mathrm{Sym})$, X_{ba}, $X_{\mathcal{M}}$ for static problems has been given. This sequence is better wieved as *increasing* if the parameter q is *decreasing*:

$$C^0(\operatorname{cl}\Omega, \mathrm{Sym}) \subset \cdots \subset L^{q_1}(\Omega, \mathrm{Sym}) \subset L^{q_2}(\Omega, \mathrm{Sym}) \subset \cdots \subset X_{\mathrm{ba}} \subset X_{\mathcal{M}} \tag{8-2}$$

$(\infty \geq q_1 > q_2 > 1)$. (With this way of ordering of the set of all q's, we can consider q_1 and q_2 as the Hölder conjugates of p_1 and p_2 above, respectively.) The static problem with the sequence (8-2) admits wider and wider competitors space which results in apriori inequalities for the corresponding critical multipliers of the static problems

$$\lambda_\circ \leq \cdots \leq \lambda_{q_1} \leq \lambda_{q_2} \leq \cdots \leq \lambda_{\mathrm{ba}} \leq \lambda_{\mathcal{M}}. \tag{8-3}$$

In the case of the choice of the spaces W_∞ or C_1 in the kinematic problem, the set of all admissible stresses has to be enlarged to contain stresses represented by either finitely or countably additive tensor valued measures. No condition is currently available to the authors that would guarantee that the sequence (8-3) is constant. We also note in passing that all the multipliers in (8-1) and (8-3) coincide if

$$\lambda_\circ = \bar{\lambda}_\circ \tag{8-4}$$

that is, if the supremum of the statically admissible multipliers over the continuous stressfields coincides with the infimum of the kinematically admissible multipliers over the smooth displacement fields. In this case, the critical multipliers become independent of the choice of the function spaces. However, we do not know any condition guaranteeing (8-4).

(iii) A necessary and sufficient condition has been given under which the supremum of statically admissible multipliers equals the infimum of kinematically admissible multipliers. Two sufficient conditions for the last equality have been given. A simple sufficient condition involves the assumption of the existence of the strong mechanism. An example is given to show that this is not always satisfied, even with very regular loads.

(iv) An example has been given of loads in which the function $q \mapsto \lambda_q$ $(\infty \ge q > 1)$ has an increasing jump at $q = 2$ as q moves along $[1, \infty]$ *from right to left.* Moreover, $\lambda_q < \bar{\lambda}_p$ for $q \in [1, 2)$.

The above shows that the limit analysis problems are sensitive to the choice of function spaces.

9. Appendix. Notation for function spaces

We here describe briefly and somewhat informally the function spaces used in the main text. All spaces below are Banach spaces but we do not specify the corresponding norms.

If Z is a finite dimensional real inner product space and $1 \le p \le \infty$ then $L^p(\Omega, Z)$ is the set of all Lebesgue measurable functions $\beta : \Omega \to Z$ such that

$$\begin{cases} \displaystyle\int_\Omega |\beta|^p \, d\mathcal{L}^n < \infty & \text{if } p < \infty, \\ \operatorname{ess\,sup}\{|\beta(\boldsymbol{x})| : \boldsymbol{x} \in \Omega\} < \infty & \text{if } p = \infty. \end{cases}$$

The spaces $L^p(\mathcal{S}, Z)$ are defined analogously, with the Lebesgue measure replaced by the Hausdorff measure $\mathcal{H}^{n-1}$. See, e.g., [10, Chapter 2] for the scalar valued case. In the general case the space Z is isometrically isomorphic to $\mathbb{R}^k$ for a suitable k, and one employs the procedures of the scalar case to components. The same applies also to the other objects with values in Z, $\mathbb{R}^n$ or Sym to be considered below, without mentioning it. (The references we give below deal exclusively with the scalar case.)

If $1 \le p \le \infty$ then $W^{1,p}(\Omega, \mathbb{R}^n)$ denotes the Sobolev space of all $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ such that $\boldsymbol{v} \in L^p(\Omega, \mathbb{R}^n)$ and $\nabla \boldsymbol{v} \in L^p(\Omega, \mathrm{Lin})$. See e.g., [1, Chapter 3].

If Z is as above and A a Borel subset of $\operatorname{cl}\Omega$ then $\mathcal{M}(A, Z)$ denotes the space of all countably additive Z valued measures, i.e., functions $\mu : \mathfrak{B} \to Z$, where $\mathfrak{B}$ is the set of all Borel subsets of $\mathbb{R}^n$, such that

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mu(B_i)$$

for each disjoint sequence of Borel subsets of $\mathbb{R}^n$, and

$$\mu(B) = 0$$

if B is a Borel subset of $\mathbb{R}^n$ such that $A \cap B = \emptyset$. The elements $\mu \in \mathcal{M}(A, Z)$ are called Z valued countably additive measures on A. See, e.g., [10, Subsection 1.3.1]. Furthermore, if μ stands for the Lebesgue measure in $\mathbb{R}^n$ or for the $n-1$ dimensional Hausdorff measure and A is a μ measurable subset of $\mathbb{R}^n$, then $\mu \llcorner A$ denotes the restriction of μ to A, defined as the measure on $\mathbb{R}^n$ by

$$(\mu \llcorner A)(B) = \mu(A \cap B)$$

for every μ measurable set B. In this context, if β is a $\mu \llcorner A$ integrable function with values in Z then $\beta\mu \llcorner A$ denote the multiple of $\mu \llcorner A$ by β, defined as an element of $\mathcal{M}(A, Z)$, by

$$(\beta\mu \llcorner A)(B) = \int_{A \cap B} \beta \, d\mu$$

for all μ measurable sets B.

The space $\mathrm{ba}(\Omega, \mathfrak{M}, \mathcal{L}^n; \mathrm{Sym})$ of bounded finitely additive Sym valued measures that are absolutely continuous with respect to the Lebesgue measure is the set of all $\mathfrak{T} : \mathfrak{M} \to \mathrm{Sym}$, where $\mathfrak{M}$ is the system of all Lebesgue measurable subsets of $\mathbb{R}^n$, such that (a)

$$\mathfrak{T}(A \cup B) = \mathfrak{T}(A) + \mathfrak{T}(B)$$

for every A, $B \in \mathfrak{M}$ with $A \cap B = \emptyset$, (b)

$$\sup\{\sum_{i=1}^{j} |\mathfrak{T}(A_i)| : \{A_i\} \subset \mathfrak{M} \text{ is a finite partition of } \mathbb{R}^n\} < \infty,$$

(c) $\mathfrak{T}(A) = \mathbf{0}$ for each $A \in \mathfrak{M}$ with $\mathcal{L}^n(A) = 0$, (d) $\mathfrak{T}(A) = \mathbf{0}$ for each $A \in \mathfrak{M}$ with $A \cap \Omega = \emptyset$. See, e.g., [10, pp. 169–171].

$BD(\Omega)$ is the set of all $\boldsymbol{v} \in L^1(\Omega, \mathbb{R}^n)$ such that $\hat{\boldsymbol{E}}(\boldsymbol{v})$, interpreted as a distribution, is in $\mathcal{M}(\Omega, \mathrm{Sym})$. See [25, Chapter II].

$C^1(\mathrm{cl}\,\Omega, \mathbb{R}^n)$ is the set of all continuously differentiable functions $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ such that both $\boldsymbol{v}$ and $\nabla\boldsymbol{v}$ have continuous extensions from Ω to the closure $\mathrm{cl}\,\Omega$ of Ω. We often identify $\boldsymbol{v}$ and $\nabla\boldsymbol{v}$ with these extensions.

$C^0(\mathrm{cl}\,\Omega, \mathrm{Sym})$ is the space of all continuous functions $\boldsymbol{F} : \mathrm{cl}\,\Omega \to \mathrm{Sym}$. See, e.g., [10, p. 126].

Acknowledgment We thank the anonymous referee for many valuable remarks and suggestions.

References

[1] Adams, R. A.; Fournier, J. J. F., *Sobolev spaces (Second edition),* Academic Press, New York, 2003.

[2] Anzellotti, G., *Pairings between measures and bounded functions and compensated compactness,* Ann. Mat. Pura Appl. **135** (1983), 293–318.

[3] Anzellotti, G., *A class of convex non-corecive functionals and masonry-like materials,* Ann. Inst. Henri Poincare **2** (1985), 261–307.

[4] DalMaso, G.; Demyanov, A.; DeSimone, A., *Quasistatic Evolution Problems for Pressure-sensitive Plastic Materials,* Milan J. Math. **75** (2007), 117–134.

[5] Del Piero, G., *Constitutive equations and compatibility of the external loads for linear elastic masonry-like material materials,* Meccanica **24** (1989), 150–162.

[6] Del Piero, G., *Limit analysis and no-tension materials,* Int. J. Plasticity **14** (1998), 259–271.

[7] Di Pasquale, S., *New trends in the analysis of masonry structures,* Meccanica **27** (1992), 173–184.

[8] Ekeland, I.; Temam, R., *Convex analysis and variational problems,* SIAM, Philadelphia, 1999.

[9] Evans, L. C.; Gariepy, R. F., *Measure theory and fine properties of functions,* CRC Press, Inc., Boca Raton, 1992.

[10] Fonseca, I.; Leoni, G., *Modern Methods in the Calculus of Variations: L^p Spaces,* Springer, New York, 2007.

[11] Giaquinta, M.; Giusti, E., *Researches on the equilibrium of masonry structures,* Arch. Rational Mech. Anal. **88** (1985), 359–392.

[12] Gurtin, M. E., *An introduction to continuum mechanics,* Academic Press, Boston, 1981.
[13] Kohn, R.; Temam, R., *Dual spaces od stresses and strains, with applications to Hencky plasticity,* Appl. Math. Optim. **10** (1983), 1–35.
[14] Lucchesi, M.; Padovani, C.; Pasquinelli, G.; Zani, N., *Masonry Constructions: Mechanical Models and Numerical Applications,* Springer, Berlin, 2008.
[15] Lucchesi, M.; Padovani, C.; Šilhavý, M., *An energetic view on the limit analysis of normal bodies,* Quart. Appl. Math. **68** (2010), 713–746.
[16] Lucchesi, M.; Šilhavý, M.; Zani, N., *A new class of equilibrated stress fields for no–tension bodies,* Journal of Mechanics of Materials and Structures **1** (2006), 503–539.
[17] Lucchesi, M.; Šilhavý, M.; Zani, N., *Integration of measures and admissible stress fields for masonry bodies,* Journal of Mechanics of Materials and Structures **3** (2008), 675–696.
[18] Lucchesi, M.; Šilhavý, M.; Zani, N., *Integration of parametric measures and the statics of masonry panels,* Annals of Solid and Structural Mechanics **2** (2011), 33–44.
[19] Lucchesi, M.; Zani, N., *Some explicit solutions to equilibrium problem for masonry like bodies,* Structural Engineering and Mechanics **16** (2003), 295–316.
[20] Padovani, C.; Pasquinelli, G.; Šilhavý, M., *Processes in masonry bodies and the dynamical significance of collapse,* Math. Mech. Solids **13** (2008), 573–610.
[21] Paroni, R., *An existence theorem for inextensible nets with slack,* Mathematics and Mechanics of Solids **17** (2012), 460–472.
[22] Rockafellar, R. T., *Convex analysis,* Princeton University Press, Princeton, 1970.
[23] Šilhavý, M., *Cauchy's stress theorem for stresses represented by measures,* Continuum Mechanics and Thermodynamics **20** (2008), 75–96.
[24] Temam, R., *Navier-Stokes equations,* North-Holland, Amsterdam, 1977.
[25] Temam, R., *Problemes mathematiques en plasticite,* Gauthier–Villars, Paris, 1983.
[26] Temam, R.; Strang, G., *Duality and relaxation in the variational problems of plasticity,* J. Me canique **19** (1980), 493–527.

Dipartimento di Costruzioni e Restauro, Università di Firenze, Piazza Brunelleschi 6, 50121 Florence, Italy
E-mail address: massimiliano.lucchesi@unifi.it

Institute of Mathematics of the AS CR, Žitná 25, 115 67 Prague 1, Czech Republic
E-mail address: silhavy@math.cas.cz

Dipartimento di Costruzioni e Restauro, Università di Firenze, Piazza Brunelleschi 6, 50121 Florence, Italy
E-mail address: nicola.zani@unifi.it