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Abstract

We consider a simplified model arising in radiation hydrodynamics based on the incompressible Navier-Stokes-Fourier system describing the macroscopic fluid motion, and a transport equation modeling the propagation of radiative intensity. We establish global-in-time existence for the associated initial-boundary value problem in the framework of weak solutions.

Key words: Radiation hydrodynamics, incompressible Navier-Stokes-Fourier system, weak solution

1 Introduction

The paper concerns the incompressible heat conduction radiation fluid. We consider that the motion of the fluid is governed by the standard field equations of classical continuum fluid mechanics describing the evolution of the *velocity field* $\vec{u} = \vec{u}(t, x)$, and the *absolute temperature* $\vartheta = \vartheta(t, x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation, represented by its quanta - massless particles called *photons* traveling at the speed of light c - is incorporated in the *radiative intensity* $I = I(t, x, \vec{\omega}, \nu)$, depending on the direction vector $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. The collective effect of radiation is then

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expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I. In particular, the radiation energy E_R is given as

$$E_R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{1.1}$$

The time evolution of I is described by a transport equation with a source term depending on the absolute temperature, while the effect of radiation on the macroscopic motion of the fluid is represented by extra source terms in the momentum and energy equations evaluated in terms of I.

More specifically, the system of equations to be studied reads as follows:

Equation of continuity:

$$\mathrm{div}_x \vec{u} = 0 \text{ in } (0,T)\times\Omega; \tag{1.2}$$

Momentum equation:

$$\partial_t \vec{u} + \mathrm{div}_x(\vec{u}\otimes\vec{u}) + \nabla_x p = \mathrm{div}_x \mathbb{T} - \vec{S}_F \text{ in } (0,T)\times\Omega; \tag{1.3}$$

Energy balance equation:

$$\partial_t\left(\frac{1}{2}|\vec{u}|^2 + \vartheta\right) + \mathrm{div}_x\left(\left(\frac{1}{2}|\vec{u}|^2+\vartheta\right)\vec{u}\right) + \mathrm{div}_x\Big(p\vec{u} + \vec{q} - \mathbb{T}\vec{u}\Big) \tag{1.4}$$
$$= -S_E \text{ in } (0,T)\times\Omega;$$

Radiation transport equation:

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.5}$$

Furthermore, $\mathbb{T}$ is the viscous stress tensor determined by *Newton's rheological law*

$$\mathbb{T} = \mu\left(\nabla_x\vec{u} + \nabla_x^t\vec{u}\right), \tag{1.6}$$

where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ is effective function of the absolute temperature. Similarly, $\vec{q}$ is the heat flux given by *Fourier's law*

$$\vec{q} = -\kappa\nabla_x\vartheta, \tag{1.7}$$

with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

Finally,

$$S = S_{a,e} + S_s, \tag{1.8}$$

where

$$S_{a,e} = \sigma_a\Big(B(\nu,\vartheta) - I\Big),\ S_s = \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} - I\right), \tag{1.9}$$

and

$$S_E = \int_{\mathcal{S}^2} \int_0^\infty S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \ \vec{S}_F = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} S(\cdot, \nu, \vec{\omega}) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega}, \tag{1.10}$$

with the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geq 0$, and the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$. More restrictions on the structural properties of constitutive relations will be imposed in Section 2 below.

System (1.2 - 1.5) is supplemented with the boundary conditions:

slip condition, no-flux:

$$\vec{u} \cdot \vec{n}|_{\partial\Omega} = 0, \ (\mathbb{T}\vec{n})_\tau = 0, \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0; \tag{1.11}$$

Transparency:

$$I(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0, \tag{1.12}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

System (1.2 - 1.12) can be viewed as a simplified model in radiation hydrodynamics. Concerning physical motivation see works of Pomraning [32] and Mihalas and Weibel-Mihalas. Similar systems have been investigated more recently in astrophysics and laser applications (in the relativistic and inviscid case) by Lowrie, Morel and Hittinger [24], Buet and Després [3].

Our goal in the present paper is to show that the existence theory for the incompressible Navier-Stokes-Fourier system coupled with the radiation.

The paper is organized as follows.In section 2 we list the principal hypothesis, notation,formulation of the problem and state the main result. Section 3 we introduce the approximation scheme. Section 4 is devoted to uniform estimates. In last two sections we are passing to the limits with $_epsilon$ and η.

2 Hypotheses and main results

The transport coefficients μ and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1 \leq \mu(\vartheta) < c_2, \tag{2.1}$$

$$0 < c_1 \leq \kappa(\vartheta) \leq c_2 \tag{2.2}$$

for any $\vartheta \geq 0$.

Finally, we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 \leq \sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \leq c_1, \ 0 \leq \sigma_a(\nu, \vartheta) B(\nu, \vartheta) \leq c_2, \tag{2.3}$$

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta), \sigma_a(\nu, \vartheta) B(\nu, \vartheta) \leq h(\nu), \ h \in L^1(0, \infty), \tag{2.4}$$

and

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \leq c\vartheta \tag{2.5}$$

for all $\nu \geq 0$, $\vartheta \geq 0$. Relations (2.3 - 2.5) represent a rather crude "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν and low vales of the temperature ϑ. Note, however, that relations similar to (2.5) were derived by Ripoll et al. [33].

2.1 Notation

For arbitrary $r > 0$ by $L^r(\Omega)$ we denote the usual Lebesgue spaces equipped with the norm $\|.\|_r$. We denote $W^{1,r}(\Omega)$ the usual Sobolev spaces with the norm $\|.\|1, r$.

We define

$$\begin{array}{rcl} W^{1,r}_n & := & \{\vec{v}; \vec{v} \in W^{1,r}(\Omega)^d, \mathrm{tr}\vec{v} \cdot n = 0 \text{ on } \partial\Omega\} \\ W^{1,r}_{n,div} & := & \{\vec{v}; \vec{v} \in W^{1,r}_n(\Omega)^d;\ \mathrm{div}\ \vec{v} = 0\} \\ L^r_n & := & \{\vec{v} \in W^{1,r}_{n,div}\}. \end{array} \tag{2.6}$$

2.2 Weak formulation

In the weak formulation of the Navier-Stokes-Fourier system, the momentum equation (1.3) is replaced by

$$\int_0^T \int_\Omega (\vec{u} \cdot \partial_t \varphi + \vec{u} \otimes \vec{u} : \nabla_x \varphi + p\mathrm{div}_x \varphi)\ \mathrm{d}x\ \mathrm{d}t \tag{2.7}$$

$$= \int_0^T \int_\Omega \mathbb{T} : \nabla_x \varphi + \vec{S}_F \cdot \varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\vec{u})_0 \cdot \varphi(0, \cdot)\ \mathrm{d}x$$

for any $\varphi \in C^\infty_c([0,T) \times \Omega; \mathbb{R}^3)$. As the viscous stress contains first derivatives of the velocity $\vec{u}$, for (2.7) to make sense, the field $\vec{u}$ must belong to a certain Sobolev space with respect to the spatial variable. We will specify it in definitin.

As a matter of fact, the total energy balance (1.4) is not suitable for the weak formulation since, at least according to the recent state-of-art, the term $\mathbb{T}\vec{u}$ is only weakly lower semicontinuous in ∇_x. Following [12], we replace (1.4) by the internal energy equation

$$\partial_t e + \mathrm{div}_x(e\vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{T} : \nabla_x \vec{u} - p\mathrm{div}_x \vec{u} + \vec{u} \cdot \vec{S}_F - S_E. \tag{2.8}$$

Furthermore, dividing (2.8) on ϑ , we may rewrite (2.9) as the entropy equation

$$\partial_t s + \mathrm{div}_x(\varrho s\vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta}\right) = \frac{1}{\vartheta}\Big(\mathbb{T} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta}\Big) + \frac{1}{\vartheta}\Big(\vec{u} \cdot \vec{S}_F - S_E\Big). \tag{2.9}$$

Finally, similarly to [9], equation (2.9) is replaced in the weak formulation by an *inequality*, specifically,

$$\int_0^T \int_\Omega (s\partial_t \varphi + \vec{u} \cdot \nabla_x \varphi + \vec{q}\vartheta \cdot \nabla_x \varphi)\ \mathrm{d}x\ \mathrm{d}t \tag{2.10}$$

$$\leq - \int_\Omega s_0 \varphi(0,\cdot)\ \mathrm{d}x$$

$$- \int_0^T \int_\Omega \frac{1}{\vartheta}\left(\mathbb{T} : \nabla_x \vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega \frac{1}{\vartheta}\Big(\vec{u}\cdot\vec{S}_F - S_E\Big)\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, $\varphi \geq 0$.

Since replacing *equation* (1.4) by *inequality* (2.10) would certainly result in a formally under-determined problem, system (2.7), (2.10) must be supplemented with the total energy balance

$$\int_\Omega \left(\frac{1}{2}|\vec{u}|^2 + \vartheta + E_R\right)(\tau,\cdot)\ \mathrm{d}x \tag{2.11}$$

$$+ \int_0^\tau \int\int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0} \int_0^\infty \vec{\omega}\cdot\vec{n} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2}|\vec{u}_0|^2 + \vartheta_0 + E_{R,0}\right)\ \mathrm{d}x,$$

where E_R is given by (1.1), and

$$E_{R,0} = \int_{\mathcal{S}^2}\int_0^\infty I_0(\cdot,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu.$$

The transport equation (1.5) can be extended to the whole physical space $\mathbb{R}^3$ provided we set

$$\sigma_a(x,\nu,\vartheta) = 1_\Omega \sigma_a(\nu,\vartheta),\ \ \sigma_s(x,\nu,\vartheta) = 1_\Omega\sigma_s(\nu,\vartheta)$$

and take the initial distribution $I_0(x,\vec{\omega},\nu)$ to be zero for $x \in \mathbb{R}^3\backslash\Omega$. Accordingly, for any fixed $\vec{\omega}\in\mathcal{S}^2$, equation (1.5) can be viewed as a linear transport equation defined in $(0,T)\times\mathbb{R}^3$, with a right-hand side S. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$.

We define the functionals

$$\mathcal{E}(t,\varphi) = (\vartheta(t,.) + \frac{1}{2}|\vec{u}(t,.)|^2 + E_R, \varphi),\ \mathcal{E}_0 = (\vartheta_0 + \frac{1}{2}|\vec{u}_0(t,.)|^2 + E_{R,0},\varphi)$$

Definition 2.1 *We say that $(\vec{u},\vartheta,p,I)$ is a weak solution of problem (1.2 - 1.12) if*

$$\vartheta > 0 \textit{ for a.a. } (t,x)\times\Omega,\ \ I \geq 0 \textit{ a.a. in } (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty),$$

$$\vec{u} \in C(0,T;L^2_{weak}(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}_{n,div}(\Omega;\mathbb{R}^3)),$$

$$\vec{u}_t \in L^{\frac{5}{3}}(0,T;W^{-1,\frac{5}{3}}),$$

$$p \in L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega)),$$
$$\vartheta \in L^\infty(0,T;L^1(\Omega)) \cap L^\infty(0,T;W^{1,n}(\Omega)),\ \textit{for all } n \in [1,5/4),$$
$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),\ I(t,\cdot) \in L^\infty(0,T;L^1(\Omega\times\mathcal{S}^2\times(0,\infty)),$$

$$\mathcal{E}(t,\varphi) \in C([(0,T])\ \textit{and}\ \lim_{t\to 0+}\mathcal{E}(t,\varphi) = \mathcal{E}_0(\varphi),$$

$$\lim_{t\to 0+}\|\vec{u}(t)-\vec{u}_0\|_2 = 0,$$

$$\int_0^T \int_\Omega (\vec{u}\cdot\partial_t\varphi + \vec{u}\otimes\vec{u} : \nabla_x\varphi + p\mathrm{div}_x\varphi)\ \mathrm{d}x\ \mathrm{d}t \tag{2.12}$$
$$= \int_0^T \int_\Omega \mathbb{T} : \nabla_x\varphi + \vec{S}_F\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \vec{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in L^\infty(0,T;W^{1,\infty}_n)$,

$$\int_0^T \Big(-(\vartheta,\psi_t)-(\vec{u}\vartheta,\nabla\psi)+(k(\vartheta)\nabla\vartheta,\nabla\psi)-(\mu(\vartheta)|\mathbf{D}(\vec{u})|^2,\psi\Big)dt+\Big(\vec{u}\cdot\vec{S}_F - S_E,\psi\Big) \geq (\vartheta_0,\psi(0))$$

$$\textit{for all } \psi \in \mathcal{D}(-\infty,T;\mathcal{C}^\infty(\bar{\Omega})).$$

The total energy balance is satisfied

$$\int_\Omega \left(\frac{1}{2}|\vec{u}|^2 + \vartheta + E_R\right)(\tau,\cdot)\ \mathrm{d}x \tag{2.13}$$
$$+\int_0^\tau \int\int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0}\int_0^\infty \vec{\omega}\cdot\vec{n}I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t$$
$$= \int_\Omega \left(\frac{1}{2}|\vec{u}_0|^2 + \vartheta_0 + E_{R,0}\right)\ \mathrm{d}x,$$

and I *satisfied the integral identity (1.5).*

2.3 Main result

The main result of the present paper can be stated as follows.

Theorem 2.1 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain. Assume that the transport coefficients* μ, κ, σ_a, *and* σ_s *comply with (2.1 - 2.5). Then there exists a weak solution to (1.2 - 1.12).*

3 Approximation scheme

We consider the quasicompressible approximation

$$\begin{array}{rcll} \epsilon \Delta p & = & \text{div } \vec{u} & \text{in } \Omega \times (0,T), \\ \frac{\partial p}{\partial n} & = & 0 & \text{on } \partial\Omega \times (0,T), \\ \int_\Omega p dx & = & 0. & \end{array} \tag{3.1}$$

We regularized $\vec{u}$ by using the mollifying operators γ_η such that for any function $\vec{u} : R^3 \to R^3$ with compact support

- $\gamma_\eta(\vec{u}) \in C_c^\infty(R^3)$, supp $(\gamma_\eta(\vec{u})) \subset \text{supp}(\vec{u})$,
- if $\nabla \cdot \vec{u} = 0$ then $\nabla \cdot \gamma_\eta(\vec{u}) = 0$,
- for all $p \in [1, \infty)$, there exists a constant c such that for all η

$$\gamma_\eta(\vec{u})_{L^p(R^3)} \le c\|u\|_{L^p(R^3)},$$

$$\gamma_\eta(\vec{u})_{W^{1,p}(R^3)} \le c\|u\|_{W^{1,p}(R^3)}.$$

- if $u \in W^{1,p}(R^3)$ with $p \in [1, \infty)$, then

$$\gamma_\eta(\vec{u}) \to \vec{u} \text{ in } W^{1,p}(R^3) \text{ as } \eta \to 0.$$

Definition 2.2 *We say that $(\vec{u}^{\epsilon,\eta}, \vartheta^{\epsilon,\eta}, p^{\epsilon,\eta}, I^{\epsilon,\eta})$ is a weak solution of the (ϵ, η)-approximation of problem (1.2 - 1.12) if*

$$\vartheta > 0 \textit{ for a.a. } (t,x) \times \Omega, \ I \ge 0 \textit{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),$$

$$\vec{u} \in C(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W_n^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vec{u}_t \in L^2(0,T; W^{-1,2}),$$

$$p \in L^2(0,T; W^{1,2}(\Omega)),$$

$$\vartheta \in L^\infty(0,T; L^1(\Omega)),$$

$$\vartheta^{\frac{1+\lambda}{2}} \in L^2(0,T; W^{1,2}(\Omega)), \textit{ for all } \lambda \in (-1,0),$$

$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ I(t,\cdot) \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),$$

$$\lim_{t\to 0^+} \|\vec{u}(t) - \vec{u}_0\|_2 = 0 \textit{ and } \lim_{t\to 0^+} \|\vartheta(t) - \vartheta_0\|_1 = 0,$$

$$-\epsilon(\nabla p(t), \nabla\phi) = (\phi,\ div\ \vec{u}(t)),\ for\ all\ \phi \in W^{1,2}(\Omega)\ a.a.\ t \in (0,T), \tag{3.2}$$

$$\begin{aligned} &\int_0^T (<\vec{u}_{,t}, \varphi> - <\vec{u}_\eta \otimes \vec{u}, \nabla_x\varphi> + <\mathbb{T}, \nabla_x\varphi>)\,\mathrm{d}t = \\ &\int_0^T (p, \mathrm{div}_x\varphi) + (\vec{S}_F, \cdot\varphi)\ \mathrm{d}t - \int_\Omega \vec{u}_0 \cdot \varphi(0,\cdot)\ \mathrm{d}x \\ &for\ any\ \varphi \in L^2(0,T; W_n^{1,2}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int_0^T \Big(-(\vartheta, \psi_t) - (\vec{u}_\eta\vartheta, \nabla\psi) + (k(\vartheta)\nabla\vartheta, \nabla\psi) - (\mu(\vartheta)|\mathbf{D}(\vec{u})|^2, \psi\Big)dt + \Big(\vec{u}\cdot\vec{S}_F - S_E, \psi\Big) \\ &= (\vartheta_0, \psi(0)) \\ &for\ all\ \psi \in \mathcal{D}(-\infty, T; \mathcal{C}^\infty(\bar{\Omega})). \end{aligned} \tag{3.4}$$

I satisfied the integral identity (1.5).

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the transport coefficients μ, κ, σ_a, and σ_s comply with (2.1 - 2.5). Then there exists a weak solution to (ϵ,η)-approximation of problem (1.2 - 1.12).*

Proof:

We consider a linear mapping $\mathcal{F} : \mathbf{W_n^{1,2}} \to \mathbf{W}^{2,2}$ which for any $\vec{u} \in \mathbf{W_n^{1,2}}$ $\mathcal{F}(\vec{u}) = p \in W^{2,2}$ solve the problem (3.1). From the regularity theory for the Neumann problem the mapping is continuous.

Now, we define $\{\mathbf{w}_j\}_{j=1}^\infty$ be an orthogonal basis in $\mathbf{W}_n^{1,2}$ and orthonornal in $\mathbf{L^2}$. Then for given $S_F(J), S_E(J)$ we construct the Galerking approximation $\{\vec{u}^N\}$ with

$$\vec{u}^N := \sum_{i=1}^N \mathbf{c}_i^N(t)\mathbf{w}_j \tag{3.5}$$

and $p^N := \mathcal{F}(\vec{u}^N)$ where $\mathbf{c}_i^N := (c_1^{N,}..., c_N^N)$ solve the system of ordinary differential equation

$$\begin{aligned} &\tfrac{d}{dt}(\vec{u}^N, \mathbf{w}_j) - (\vec{u}_\eta^N \otimes \vec{u}^N, \nabla_x\mathbf{w}_j) + (\mathbb{T}(\vec{u}^N), \nabla_x\mathbf{w}_j) = \\ &-(\mathcal{F}(\vec{u}^N), \mathrm{div}_x\mathbf{w}_j) + (\vec{S}_F, \mathbf{w}_j)\ \mathrm{d}t, j = 1, ..., N. \end{aligned} \tag{3.6}$$

ϑ^N, I^N are determined through the system of equations

$$\begin{aligned} &\partial_t\vartheta^N + \mathrm{div}_x(\vartheta^N\vec{u}^N) + \mathrm{div}_x\vec{q} = \mathbb{T}(\vec{u}^N) : \nabla_x\vec{u}^N - \mathcal{F}(\vec{u}^N)\mathrm{div}_x\vec{u}^N + \vec{u}^N\cdot\vec{S}_F - S_E. \\ &\tfrac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S(\vartheta^N) \end{aligned} \tag{3.7}$$

We consider the following initial conditions

$$\vec{u}^N(.,0) = \vec{u}_0^N, \vartheta_0^N(0,.) = \vartheta_0^N, I_0^N(0,.) = I_0^N$$

where $\vec{u}_0^N := \sum_{j=1}^N c_0^N \mathbf{w}_j$ are the projections of $\vec{u}_0$ onto linear hulls of $\{\mathbf{w}_j\}_{j=1}^N$. With $\vartheta_0^N \in C^\infty$ such that

$$\inf{}_{n\in N, x\in\Omega} \vartheta_0^N > 0,\ \vartheta_0^N \to \vartheta_0 \text{ in } L^1 \tag{3.8}$$

By standard argument following the Galerkin method and classical parabolic equation we can show the uniform estimates independent on N and then pass to the limit in N.

4 Uniform bounds

We interest in the uniform (*a priori*) bounds for $(\vec{u}, \vartheta, p, I) = (\vec{u}^{\epsilon,\eta}, \vartheta^{\epsilon,\eta}, p^{\epsilon,\eta}, I^{\epsilon,\eta})$.

4.1 Estimates of the radiation intensity

At this stage we focus on the transport equation (1.5). Since the quantity I_ε is non-negative, we have

$$\frac{1}{c}\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon \leq \sigma_s(\nu,\vartheta_\varepsilon)B(\nu,\vartheta_\varepsilon) + \sigma_a(\nu,\vartheta_\varepsilon)\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} \tag{4.1}$$

as the coefficients σ_s, σ_a are non-negative. Moreover, making use of the "cut-off" hypothesis (2.3), we deduce a uniform bound

$$0 \leq I_\varepsilon(t,x,\nu,\vec{\omega}) \leq c(T)(1 + \sup_{x\in\Omega,\ \nu\geq 0, \vec{\omega}\in\mathcal{S}^2} I_{0,\varepsilon}) \leq c(T)(1+I_0) \text{ for any } t\in[0,T]. \tag{4.2}$$

Finally, hypothesis (2.4), together with (4.2), yield

$$\|S_{E,\varepsilon}\|_{L^\infty((0,T)\times\Omega)} + \|\vec{S}_{F,\varepsilon}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c, \tag{4.3}$$

4.2 Energy estimates

¿From (3.2-3.4) we obtain

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vec{u}\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{4.4}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vartheta\|_{L^1(\Omega)} \leq c, \tag{4.5}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\nabla\vartheta^{\frac{\lambda+1}{2}}\|_{L^2(\Omega)} \leq c, \tag{4.6}$$

Since the viscosity coefficients satisfy (2.1), we get

$$\int_0^T\int_\Omega \mathbb{T}_\varepsilon : \nabla_x\vec{u}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t \geq c_1\left\|\nabla_x\vec{u}_\varepsilon + \nabla_x^t\vec{u}_\varepsilon\right\|^2_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})}$$

$$\geq c_2\|\vec{u}_\varepsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

4.3 Pressure estimates

We consider the auxiliary Neumann problem for $h = h^{\epsilon,\eta}$:

$$\begin{array}{l}\Delta h = |p|^{\beta-2}p - \frac{1}{|\Omega|}\int_\Omega |p|^{\beta-2}p \text{ in } \Omega \\ \nabla h \cdot n = 0 \text{ on } \partial\Omega, \int_\Omega h = 0.\end{array} \tag{4.7}$$

Muliplying by ∇h the momentum equation we are getting by standard approach see [4]

$$\int_0^T \|p\|_{\frac{5}{3}}^{\frac{5}{3}} dt \leq C. \tag{4.8}$$

Moreover, from the momentum equation and the internal energy

$$\|\vec{u}_{,t}\|_{L^2(0,T;W_n^{-1,2})} + \|\theta_t\|_{L^1(0,T;W^{-1,q'})} \leq c\eta^{-1} \tag{4.9}$$

and

$$\|\vec{u}_{,t}\|_{L^{\frac{5}{3}}(0,T;W_n^{-1,\frac{5}{3}})} + \|\theta_t\|_{L^1(0,T;W^{-1,q'})} \leq c. \tag{4.10}$$

5 Limit in ϵ

¿From uniform estimates and using Aubin-Lions lemma we get (fixing η and passing with $\epsilon \to 0$)

- $\vec{u}^\epsilon_{,t} \to \vec{u}_{,t}$ weakly in $L^2(0,T;W_n^{-1,2})$
- $\vec{u}^\epsilon \to \vec{u}$ weakly in $L^2(0,T;W_n^{1,2})$
- $\vec{u}^\epsilon \to \vec{u}$ weakly * in $L^\infty(0,T;L^2(\Omega)^3)$
- $\vec{u}^\epsilon \to \vec{u}$ strongly in $L^q(0,T;L^q)$ for $q \in [1,10/3)$
- $\vartheta^\epsilon \to \vartheta$ weakly in $L^n(0,T;W^{1,n})$ for n $[1,5/4)$
- $\vartheta^\epsilon \to \vartheta$ strongly in $L^m(0,T;L^m)$ for $m \in [1,5/3)$
- $(\vartheta^\epsilon)^{\frac{\lambda+1}{2}} \to \vartheta^{\frac{\lambda+1}{2}}$ weakly in $L^2(0,T;W^{1,2})$ for $\lambda \in (-1,0)$
- $\mu(\vartheta^\epsilon)\mathbf{D}(\vec{u}^\epsilon) \to \mu(\vartheta)\mathbf{D}(\vec{u})$ weakly in $L^2(0,T;L^2)$
- $p^\epsilon \to p$ weakly in $L^2(0,T;L^2)$

$$\begin{array}{l}\left|\int_0^T(\text{ div } \vec{u},\phi)dt\right| = \lim_{\epsilon\to 0}\left|\int_0^T(\text{ div } \vec{u},^\epsilon \phi)dt\right| = \\ \\ \lim_{\epsilon\to 0}\left|\int_0^T\int_\Omega \nabla\phi\cdot\nabla p^\epsilon dt\right| \leq \\ \\ \leq \lim_{\epsilon\to 0}\sqrt{\epsilon}\Big(\int_Q |\nabla\phi|^2 dxdt\Big)^{\frac{1}{2}}\Big(\int_\Omega \epsilon|\nabla p^\epsilon|^2\Big)^{\frac{1}{2}} = 0.\end{array} \tag{5.1}$$

As in [4] we get

$$\int_Q \mu(\vartheta^\epsilon)|D(\vec{u}^\epsilon)|^2 \to \int_Q \mu(\vartheta)|D(\vec{u})|^2 \tag{5.2}$$

Our next goal is to establish convergence of the quantities

$$\vec{u}_\varepsilon \cdot \vec{S}_{F,\varepsilon} = \frac{1}{c}\vec{u}_\varepsilon \cdot \int_0^\infty \sigma_a(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2} \vec{\omega}\,(B(\nu,\vartheta_\varepsilon) - I_\varepsilon)\ \mathrm{d}\vec{\omega}\right)\ \mathrm{d}\nu$$

$$+\frac{1}{c}\vec{u}_\varepsilon \cdot \int_0^\infty \sigma_s(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2} \vec{\omega}\left(\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon\ \mathrm{d}\vec{\omega}\right) - I_\varepsilon\right)\ \mathrm{d}\vec{\omega}\right)\ \mathrm{d}\nu$$

and

$$S_{E,\varepsilon} = \frac{1}{c}\int_0^\infty \sigma_a(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2} (B(\nu,\vartheta_\varepsilon) - I_\varepsilon)\ \mathrm{d}\vec{\omega}\right)\ \mathrm{d}\nu.$$

Since $\vartheta \to \vartheta$ strongly in $L^m(0,T;L^m(\Omega))$ for $m \in [1,5/3)$, and

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

the desired result follows from compactness of the velocity averages over the sphere $\mathcal{S}^2$ established by Golse et al. [16, 17], see also Bournaveas and Perthame [2], and hypothesis (2.4). Specifically, we use the following result (see [16]):

Proposition 5.1 *Let $I \in L^q([0,T] \times R^{n+1} \times \mathcal{S}^2)$, $\partial_t I + \omega \cdot \nabla_x I \in L^q([0,T] \times R^{n+1} \times \mathcal{S}^2)$ for a certain $q > 1$. In addition, let $I_0 \equiv I(0,\cdot) \in L^\infty(R^{n+1} \times \mathcal{S}^2)$.*
Then

$$\tilde{I} \equiv \int_{\mathcal{S}^2} I(\cdot,\nu)\ \mathrm{d}\vec{\omega}$$

belongs to the space $W^{s,q}([0,T] \times R^{n+1})$ for any s, $0 < s < \inf\{1/q, 1-1/q\}$, and

$$\|\tilde{I}\|_{W^{s,q}} \le c(I_0)(\|I\|_{L^q} + \|\partial_t I + \omega \cdot \nabla I\|_{L^q}).$$

As the radiation intensity I_ε satisfies the transport equation (1.5), by virtue of the cut - off hypothesis (2.9)-(2.11) where S is bounded in $L^q \cap L^\infty([0,T) \times \Omega \times R^1 \times S^2)$, a direct application of Proposition 5.1 yields the desired conclusion

$$\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\nu)\ \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} I(\cdot,\nu)\ \mathrm{d}\vec{\omega} \text{ in } L^2((0,T)\times\Omega)$$

and

$$\int_{\mathcal{S}^2} \vec{\omega} I_\varepsilon(\cdot,\nu)\ \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} \vec{\omega} I(\cdot,\nu)\ \mathrm{d}\vec{\omega} \text{ in } L^2((0,T)\times\Omega)$$

for any fixed ν.

Consequently,

$$\vec{u}_\varepsilon \cdot S_{F,\varepsilon} \to \vec{u} \cdot F_s$$

and, similarly,

$$S_{E,\varepsilon} \to S_E$$

as required.

So, we can pass to the limit in the momentum equation and also in the internal energy equation.

6 Limit in η

We are passing with $\eta \to 0$. We already have $\text{div}\ \vec{u}_\eta = 0$. ¿From uniform estimates and using Aubin-lions lemma we get

- $\vec{u}^\eta_{,t} \to \vec{u}_{,t}$ weakly in $L^{\frac{5}{3}}(0,T;W_n^{-1,\frac{5}{3}})$
- $\vec{u}^\eta \to \vec{u}$ weakly in $L^2(0,T;W^{1,2}_{n,div})$
- $\vec{u}^\eta \to \vec{u}$ weakly * in $L^\infty(0,T;L^2(\Omega)^3)$
- $\vec{u}^\eta \to \vec{u}$ strongly in $L^n(0,T;L^n)$ for $n \in [1,10/3)$
- $\vartheta^\eta \to \vartheta$ weakly in $L^s(0,T;W^{1,s})$ for $s \in [1,5/4)$
- $\vartheta^\eta \to \vartheta$ strongly in $L^m(0,T;L^m)$ for $m \in [1,5/3)$
- $(\vartheta^\eta)^{\frac{\lambda+1}{2}} \to \vartheta^{\frac{\lambda+1}{2}}$ weakly in $L^2(0,T;W^{1,2})$ for $\lambda \in (-1,0)$
- $\mu(\vartheta^\eta)\mathbf{D}(\vec{u}^\eta) \to \mu(\vartheta)\mathbf{D}(\vec{u})$ weakly in $L^2(0,T;L^2)$
- $p^\eta \to p$ weakly in $L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}})$

Using lower semicontinuity in the L^2 norm implies that

$$\int_0^T \int_\Omega \mu(\vartheta^\eta)|D(\vec{u}^\eta|^2\psi dxdt \leq \liminf_{\eta\to 0^+} \int_0^T \int_\Omega \mu(\vartheta)|D(\vec{u})|^2\psi dxdt \tag{6.1}$$

Finally as in the previous section we get using [2] [16]

$$\vec{u}_\varepsilon \cdot S_F^\eta \to \vec{u} \cdot S_F$$

and, similarly,

$$S_E^\eta \to S_E$$

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