INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

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Preprint No. 41-2013

PRAHA 2013

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Contents

Abstract We consider the problem of motion of a rigid body in an incompressible viscous fluid, filling a bounded domain. This problem was studied by many authors. They considered classical non-slip boundary conditions, which gave them very paradoxical result of no collisions of the body with the boundary of the domain.

In this work we study the case where the Navier slip conditions are prescribed on the boundary of the body (instead of classical non-slip conditions). We prove for this model the global existence of weak solution, which permit collisions with the boundary of the domain.

Keywords *rigid body, global weak solution, collisions in finite time*

MSC: 35Q30

1 Introduction

The problem of motion of one or several rigid bodies in viscous fluids filling a bounded domain was studied by several authors see [7, 8], [20], [25]. They consider non-slip boundary condition, which give them very paradoxical result of no collisions with boundary, see work of Hesla [16], Hillairet [17]. This result was extended to the three dimensional situation by Hillairet, Takahashi [18]. The relation between the regularity of velocity fields given by motion of bodies and the regularity of boundaries, which can give us answer to the problem of existence and no existence of collision was done by Starovoitov [26]. From his theorem it was proved that in the case of very viscous fluids (for instance Non-Newtonian fluids) if the initial distance between bodies is positive then in a finite time the distance continuous to be positive (see [26]). Also we would like to mention work of Gerard-Varet, [19]. They considered the regularity of boundary $C^{1,\lambda}$ and on the dependency of λ they showed the collision of bodies or existence of strong solution. Aim of this paper is to study the case, when Navier boundary conditions are prescribed on the boundary of the body, and to show the global solvability result. To our knowledge there is a particular result done by Neustupa, Penel [23], [24], where they consider the Navier boundary conditions and the motion of body with the collision of the boundary of the domain is prescribed. Recently Gérard-Varet, Hillairet [12] has shown a local-in-time existence result: up to collisions.

We will investigate the motion of a rigid body inside of a viscous incompressible fluid. We assume the fluid and the body occupy a bounded domain $\Omega \subset \mathbb{R}^N$, ($N = 2$ or $N = 3$), such that the boundary $\partial\Omega \in C^2$. Let the body be an open simply-connected set $S_0 \subset \Omega$, having the boundary $\partial S_0 \in C^2$, at the initial time $t = 0$. The fluid fills the domain $F_0 = \Omega \backslash \overline{S_0}$ at $t = 0$.

The Cartesian coordinates $\mathbf{y}$ of points of the body at $t = 0$ are called Lagrangian coordinates. The motion of any material point $\mathbf{y} = (y_1, .., y_N)^T \in S_0$ is described by two functions

$$t \to \mathbf{q}(t) \in \mathbb{R}^N \quad \text{and} \quad t \mapsto \mathbb{Q}(t) \in SO(N) \qquad \text{for} \quad t \in [0, T],$$

where $\mathbf{q} = \mathbf{q}(t)$ is the position of the body mass center at a time t and $SO(N)$ is the rotation group in $\mathbb{R}^N$, i.e. the $\mathbb{Q} = \mathbb{Q}(t)$ is a matrix, satisfying $\mathbb{Q}(t)\mathbb{Q}(t)^T =$

$\mathbb{I}$, $\mathbb{Q}(0) = \mathbb{I}$ with $\mathbb{I}$ being the identity matrix. Therefore, the trajectories of all points of the body are described by a preserving orientation isometry

$$A(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for any} \quad \mathbf{y} \in S_0 \tag{1.1}$$

and at any time t the body occupies the set

$$S(t) = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = A(t, \mathbf{y}), \quad \mathbf{y} \in S_0\} = A(t, S_0). \tag{1.2}$$

The velocity of the body is a rigid function

$$\mathbf{u}_s = \mathbf{a}(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for all} \quad \mathbf{x} \in S(t), \tag{1.3}$$

where $\mathbf{a} = \mathbf{a}(t) \in \mathbb{R}^N$ is the translation velocity and $\mathbb{P} = \mathbb{P}(t)$ -the angular velocity. The velocity $\mathbf{u}_s$ has to be compatible with A in the sense

$$\frac{d\mathbf{q}}{dt} = \mathbf{a} \quad \text{and} \quad \frac{d\mathbb{Q}}{dt}\mathbb{Q}^T = \mathbb{P} \quad \text{in } [0, T]. \tag{1.4}$$

The angular velocity $\mathbb{P}$ is a skew-symmetric matrix, i.e. there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^N$, such that

$$\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^N. \tag{1.5}$$

The full system of equations modelling the motion of the body and of the fluid can be written as

$$\begin{aligned} m\frac{d\mathbf{a}}{dt} &= \frac{d}{dt}\int_{S(t)} \rho_s \mathbf{u}_s \, d\mathbf{x} = -\int_{\partial S(t)} P_f \mathbf{n} \, d\mathbf{x} + \int_{S(t)} \rho_s \mathbf{g} \, d\mathbf{x} \quad \text{ for } \mathbf{x} \in S(t), \\ \rho_s \frac{d(\mathbb{J}\boldsymbol{\omega})}{dt} &= \frac{d}{dt}\int_{S(t)} \rho_s (\mathbf{x} - \mathbf{q}(t)) \times \mathbf{u}_s \, d\mathbf{x} \\ &= -\int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times P_f \mathbf{n} \, d\mathbf{x} + \int_{S(t)} \rho_s (\mathbf{x} - \mathbf{q}(t)) \times \mathbf{g} \, d\mathbf{x}, \end{aligned} \tag{1.6}$$

$$\begin{aligned} \partial_t \rho_f + (\mathbf{u}_f \cdot \nabla)\rho_f = 0, \qquad \operatorname{div}\mathbf{u}_f = 0 \quad \text{for } \mathbf{x} \in F(t) = \Omega \backslash \overline{S(t)}, \\ \rho_f(\partial_t \mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f) = \operatorname{div}P + \rho_f \mathbf{g}. \end{aligned} \tag{1.7}$$

ρ_f and ρ_s are densities of the fluid and the body; $m = \int_{S(t)} \rho_s \, d\mathbf{x}-$ the mass of the body; P_f- the value of the stress tensor P of the fluid on $\partial S(t)$; $\mathbf{n}(\mathbf{x})-$ the unit *interior* normal at $\mathbf{x} \in \partial S(t)$, i.e. the vector $\mathbf{n}$ is directed inside of $S(t)$; $\mathbf{g}-$the external force;

$$\mathbb{J} = \int_{S(t)} (|\mathbf{x} - \mathbf{q}(t)|^2 \mathbb{I} - (\mathbf{x} - \mathbf{q}(t)) \otimes (\mathbf{x} - \mathbf{q}(t))) \, d\mathbf{x}$$

-the matrix of the inertia moments of the body $S(t)$ related to its mass center. In (1.7) $\mathbf{u}_f$ is the fluid velocity;

$$P = -pI + 2\mu_f \, \mathbb{D}\mathbf{u}_f \qquad \text{and} \qquad \mathbb{D}\mathbf{u}_f = \frac{1}{2}\left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T\right)$$

-the stress tensor and the deformation-rate tensor; p is the fluid pressure; $\mu_f > 0$ is the constant viscosity of the fluid. In the sequel we will define common velocity and density for the body and the fluid in the whole domain $\Omega_T = (0,T) \times \Omega$ as

$$(\mathbf{u}, \rho) = \begin{cases} (\mathbf{u}_s, \rho_s), & \mathbf{x} \in S(t); \\ (\mathbf{u}_f, \rho_f), & \mathbf{x} \in F(t). \end{cases}$$

In addition to the coupled system (1.6)-(1.7) the following initial-boundary conditions are prescribed

$$\begin{aligned} S &= S_0, \quad \rho = \rho_0(\mathbf{x}), \quad \mathbf{u} = \mathbf{u}_0 \quad \text{at } t = 0 \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.8}$$

and Navier's boundary conditions on $\partial S(t)$

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_s \cdot \mathbf{n}, \qquad (P_f \mathbf{n} + \gamma(\mathbf{u}_f - \mathbf{u}_s)) \cdot \boldsymbol{\tau} = 0. \tag{1.9}$$

Here $\mathrm{div}\mathbf{u}_0 = 0$ in $\mathcal{D}'(\Omega)$ and $\mathbb{D}\mathbf{u}_0 = 0$ in $\mathcal{D}'(S_0)$; $\mathbf{u}_f$ and $\mathbf{u}_s$ are the velocity values of the fluid and the body on $\partial S(t)$; the constant $\gamma > 0$ is the friction coefficient; $\boldsymbol{\tau}(\mathbf{x})$ is any tangent vector to $S(t)$ at $\mathbf{x} \in \partial S(t)$.

The outline of the article is as follows:

In the section 2 we give the weak formulation of our problem (1.6)-(1.9) and the solvability result (Theorem 2.1).

In the section 3 we introduce an approximated problem and describe the main steps of the proof of the solvability result.

The section 4 contains:

- the justification of the definition of weak solution (the subsection 4.1) and some technical results, mainly used to justify the approximation of the "jump" term on $\partial S(t)$ in the definition of weak solution, see (2.3) (the subsection 4.2);
- classical results for the transport equations (the subsection 4.3) and a compactness result of the convective term in the Navier-Stokes equations (the subsection 4.4).

2 Main result

Let us introduce some necessary notations to define the concept of weak solution for system (1.6)-(1.9). We define the space

$$V^{k,p}(\Omega) = \mathrm{closure}_{W^{k,p}(\Omega)}\{ \mathbf{v} \in \mathcal{D}(\Omega) \ : \quad \mathrm{div}\,\mathbf{v} = 0\}.$$

According to classical results, we have

$$\begin{aligned} V^{0,2}(\Omega) &= \{\mathbf{v} \in L^2(\Omega) : \mathrm{div}\mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega), \ \ \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{in } H^{-1/2}(\partial\Omega)\}, \\ V^{1,2}(\Omega) &= \{\mathbf{v} \in W_0^{1,2}(\Omega) : \mathrm{div}\mathbf{v} = 0 \ \text{ a.e. in } \Omega\}, \end{aligned}$$

where $\mathbf{n}$ is the unit normal to the boundary of Ω. Let $\mathcal{M}(\Omega)$ be the space of bounded Radon measures. We define the spaces of functions of bounded deformation as

$$LD^2(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega) : \mathbb{D}\mathbf{v} \in L^2(\Omega) \right\}, \quad BD(\Omega) = \left\{ \mathbf{v} \in L^1(\Omega) : \mathbb{D}\mathbf{v} \in \mathcal{M}(\Omega) \right\}$$

endowed by the norms

$$||\mathbf{v}||_{LD^2(\Omega)} = ||\mathbf{v}||_{L^2(\Omega)} + ||\mathbb{D}\mathbf{v}||_{L^2(\Omega)}, \qquad ||\mathbf{v}||_{BD(\Omega)} = ||\mathbf{v}||_{L^1(\Omega)} + |\mathbb{D}\mathbf{v}|(\Omega),$$

respectively. Also we consider the spaces

$$\begin{aligned} LD_0^2(\Omega) &= \left\{ \mathbf{v} \in LD^2(\Omega) : \mathbf{v} = 0 \quad \text{on } \partial\Omega \right\}, \\ BD_0(\Omega) &= \left\{ \mathbf{v} \in BD(\Omega) : \mathbf{v} = 0 \quad \text{on } \partial\Omega \right\}. \end{aligned}$$

Let us point that due to the Korn inequality

$$||\nabla\mathbf{v}||_{L^2(\Omega)} \leqslant \sqrt{2}||\mathbb{D}\mathbf{v}||_{L^2(\Omega)}, \qquad \forall \mathbf{v} \in LD_0^2(\Omega), \tag{2.1}$$

we have that $LD_0^2(\Omega) \cap V^{0,2}(\Omega)$ coincides with the space $V^{1,2}(\Omega)$.

Let S be an open simply-connected subset of Ω with the boundary $\partial S \in C^2$. We introduce the following spaces of vector functions

$$\begin{aligned} C^k(\overline{S}) &= \left\{ \mathbf{v} \in C^k(S) : D^\alpha \mathbf{v} \text{ has a continuous extension on } \overline{S},\ |\alpha| \leqslant k \right\}, \\ K(S) &= \left\{ \mathbf{v} \in V^{1,2}(\Omega) : \mathbb{D}\mathbf{v} = 0 \text{ a.e. on } S \right\}, \\ KB(S) &= \left\{ \mathbf{v} \in BD_0(\Omega) : \mathbb{D}\mathbf{v} \in L^2(\Omega \backslash \overline{S}), \quad \mathbb{D}\mathbf{v} = 0 \text{ a.e. on } S, \right. \\ &\qquad \left. \operatorname{div}\mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega) \right\}. \end{aligned}$$

The class of all characteristic functions of subsets of $\mathbb{R}^N$ is denoted by $Char(\mathbb{R}^N)$. In the sequel for any $\phi \in Char(\mathbb{R}^N)$, $S(\phi)$ is a set of points where $\phi = 1$.

Let us present the definition of weak solution of (1.6)-(1.9), based on Lemma 4.1, which is given in the subsection 4.1.

Definition 2.1 *The triple* $\{A, \rho, \mathbf{u}\}$ *is a weak solution of system* (1.6)-(1.9), *if the following three conditions are satisfied:*

1) The function $A(t,\cdot) : \Omega \to \Omega$ *is a preserving orientation isometry* (1.1), *which defines a time dependent set* $S(t)$ *by* (1.2). *The isometry* A *is compatible with* $\mathbf{u} = \mathbf{u}_s$ *on* $S(t)$ *in the sense of the equalities* (1.3)-(1.5)*;*

2) The function $\rho \in L^\infty(\Omega_T)$ *satisfies the integral equality*

$$\int_{\Omega_T} \rho(\xi_t + (\mathbf{u}\cdot\nabla)\xi)\,dtd\mathbf{x} = -\int_\Omega \rho_0\xi(0,\cdot)\,d\mathbf{x} \tag{2.2}$$

for any $\xi \in C^1(\Omega_T)$, $\xi(T,\cdot) = 0$;

3) The function $\mathbf{u} \in L^2(0,T; KB(S(t))) \cap L^\infty(0,T; V^{0,2}(\Omega))$ *satisfies the integral equality*

$$\begin{aligned} &\int_0^T \left\{ \int_{\Omega\backslash\partial S(t)} \rho\mathbf{u}\{\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi}\} - 2\mu_f\,\mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}d\mathbf{x} \right\} dt \\ &= -\int_\Omega \rho_0\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\,d\mathbf{x} + \int_0^T \left\{ \int_{\partial S(t)} \gamma(\mathbf{u}_s - \mathbf{u}_f)(\boldsymbol{\psi}_s - \boldsymbol{\psi}_f)\,d\mathbf{x} \right\} dt, \end{aligned} \tag{2.3}$$

which holds for any test function $\boldsymbol{\psi}$, *such that*

$$
\begin{aligned}
\boldsymbol{\psi} &\in L^{2(N-1)}(0,T;KB(S(t))), \\
\boldsymbol{\psi}_t &\in L^2(0,T;L^2(\Omega\backslash\partial S(t))), \quad \boldsymbol{\psi}(T,\cdot)=0.
\end{aligned}
\tag{2.4}
$$

Here $\mathbf{u}_s(t,\cdot)$, $\boldsymbol{\psi}_s(t,\cdot)$ *and* $\mathbf{u}_f(t,\cdot)$, $\boldsymbol{\psi}_f(t,\cdot)$ *are trace values of* $\mathbf{u}$, $\boldsymbol{\psi}$ *on* $\partial S(t)$ *from the "solid" side* $S(t)$ *and the "fluid" side* $F(t)$, *respectively.*

Our main result is the following theorem.

Theorem 2.1 *Let us assume that*

$$
\rho_0(\mathbf{x}) = \begin{cases} \rho_s(\mathbf{x}) \geqslant const > 0, & \mathbf{x}\in S_0; \\ \rho_f = const > 0, & \mathbf{x}\in F_0, \end{cases} \qquad \text{and} \quad \rho_s \in L^\infty(S_0),
$$

$$
\mathbf{u}_0 \in V^{0,2}(\Omega), \qquad \mathbf{g}\in L^2(\Omega_T). \tag{2.5}
$$

Then problem (1.6)-(1.9) *possesses a weak solution* $\{A,\rho,\mathbf{u}\}$, *such that the isometry* $A(t,\cdot)$ *is Lipschitz continuous with respect to* $t\in[0,T]$,

$$
\rho(t,\mathbf{x}) = \begin{cases} \rho_s(A^{-1}(t,\mathbf{x})), & \mathbf{x}\in S(t); \\ \rho_f = const, & \mathbf{x}\in F(t), \end{cases} \qquad \text{for a.e. } t\in(0,T), \tag{2.6}
$$

$\mathbf{u}\in C_{\text{weak}}(0,T;V^{0,2}(\Omega))$ *and the energy inequality*

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r)\ d\mathbf{x} + \int_0^r \left\{ \int_{\Omega\backslash\partial S(t)} 2\mu_f\,|\mathbb{D}\,\mathbf{u}|^2\ d\mathbf{x} + \int_{\partial S(t)} \gamma|\mathbf{u}_f-\mathbf{u}_s|^2\ d\mathbf{x}\right\} dt \\
\leqslant \frac{1}{2}\int_\Omega \rho_0|\mathbf{u}_0|^2\ d\mathbf{x} + \int_0^r\int_\Omega \rho\mathbf{g}\mathbf{u}\ dtd\mathbf{x}
\end{aligned}
\tag{2.7}
$$

holds for a.a. $r\in(0,T)$.

3 Approximate problem

First let us introduce some notations, which we use for the construction of an approximated problem to system (1.6)-(1.9).

Let us consider an open simply-connected set $S\subset\mathbb{R}^N$, having C^2-smooth boundary ∂S, and denote by

$$
d_S(\mathbf{x}) = \text{dist}[\mathbf{x},\,\mathbb{R}^N\setminus S] - \text{dist}[\mathbf{x},S], \qquad \forall \mathbf{x}\in\mathbb{R}^N,
$$

the *signed* distance to the boundary ∂S, where $\text{dist}[\mathbf{x},S]=\inf_{\mathbf{y}\in S}|\mathbf{x}-\mathbf{y}|$. For a given $\delta>0$ we define the $\delta-$ kernel of S and the $\delta-$ neighborhood of S by

$$
[S]_\delta = d_S^{-1}((\delta,+\infty)) \quad \text{and} \quad]S[_\delta = d_S^{-1}((-\delta,+\infty)). \tag{3.1}
$$

Let $\sigma \in C^\infty(\mathbb{R})$ be a positive even function with support in $(-1,1)$, such that $\int_{-1}^{1} \sigma(z)\, dz = 1$. Let $\sigma^\tau(\mathbf{x}) = \frac{1}{\tau^N}\sigma\left(\frac{|\mathbf{x}|}{\tau}\right)$ for any $\tau > 0$ be the regularization kernel for the convolution

$$\overline{f}^\tau(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{y})\ \sigma^\tau(\mathbf{x}-\mathbf{y})\ d\mathbf{y}, \quad \forall \mathbf{x} \in \mathbb{R}^N \tag{3.2}$$

of a function $f \in L^1(\Omega)$, which is extended by zero outside of Ω. In the following sections 3.1 - 3.3 the index τ will be omitted in the convolution $\overline{f}^\tau$ just for the convenience of reading.

3.1 Solvability result

Let us consider the characteristic functions $\varphi_{0,\delta}(\mathbf{x})$ and $\chi_{0,\delta}(\mathbf{x})$ of the sets $[S_0]_\delta$ and $U_\delta(\partial S_0) = S_0 \backslash \overline{[S_0]_\delta}$, defined in the whole space $\mathbb{R}^N$, respectively. We extend ρ_0 by the constant value ρ_f outside of Ω.

We study the following approximate problem to system (1.6)-(1.9), consisting from the linear transport equations

$$\partial_t \rho + \operatorname{div}(\rho \overline{\mathbf{u}}) = 0, \qquad \partial_t \varphi + \operatorname{div}(\varphi \overline{\mathbf{u}}) = 0, \qquad \partial_t \chi + \operatorname{div}(\chi \overline{\mathbf{u}}) = 0 \tag{3.1.1}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^N)$ with the initial data

$$\rho = \rho_{0,\varepsilon\delta} = (1-\chi_{0,\delta})\rho_0 + \varepsilon\chi_{0,\delta}, \quad \varphi = \varphi_{0,\delta}, \quad \chi = \chi_{0,\delta} \quad \text{at } t = 0 \tag{3.1.2}$$

and the momentum equation

$$\begin{aligned}\int_{\Omega_T} [\rho\mathbf{u}\partial_t\boldsymbol{\psi} \quad &+ \quad \rho\mathbf{u}\,(\overline{\mathbf{u}}\cdot\nabla)\,\boldsymbol{\psi} - \mu_\varepsilon \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}]\ dtd\mathbf{x} \\ &= \quad -\int_\Omega \rho_{0,\varepsilon\delta}\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\ d\mathbf{x},\end{aligned} \tag{3.1.3}$$

which is valid for any test function $\boldsymbol{\psi} \in L^2(0,T;V^{1,2}(\Omega)) \cap H^1(\Omega_T)$, such that $\boldsymbol{\psi}(T,\cdot) = 0$. The function $\mathbf{u}(t,\cdot)$ is extended by zero outside Ω for a.e. $t \in (0,T)$ in equations (3.1.1)-(3.1.3). Here

$$\mu_\varepsilon = \frac{1}{\varepsilon}\varphi + 2\mu_f\theta + \gamma_0\chi\int_\Omega \chi\ d\mathbf{x}, \ \ \theta = 1-\varphi-\chi \tag{3.1.4}$$

with the constants $\gamma_0 = \frac{\gamma}{|\partial S_0|}$, $|\partial S_0| = \int_{\partial S_0} 1 d\mathbf{x}$.

The $\varepsilon-$dependent of the "viscosity" μ_ε can be easily identified as the penalization, introduced by Hoffmann and Starovoitov [20], where the rigid bodies are replaced by the fluid of high viscosity becoming singular for $\varepsilon \to 0$. The third term in μ_ε is introduced to define a mixture region between the fluid and the body, which approximates the "jump" boundary term on $\partial S(t)$ in (2.3). The parameter τ controls the regularity of the velocity for the transport equations (3.1.1), even when a real velocity $\mathbf{u}$ of the motion of fluid and solid has a jump on $\partial S(t)$. In order to facilitate the analysis, we assume that $\tau \in (0, \tau_0)$, where $\tau_0 = \tau_0(S_0)$ is introduced in Lemma 4.2 for $S = S_0$.

For $\varepsilon > 0$, $\delta > 0$ and $\tau > \delta$ fixed we refer the following existence result that can be proved by means of the standard arguments: Method of characteristics, given in Lemma 4.6, Galerkin's method and Schauder's fixed point argument [1], [14].

Proposition 3.1 *Under the assumptions of Theorem 2.1 problem (3.1.1) - (3.1.3) possesses a weak solution*

$$\begin{aligned} \rho &\in L^\infty((0,T)\times \mathbb{R}^N), \qquad \varphi, \chi \in L^\infty(0,T; Char(\mathbb{R}^N)), \\ \mathbf{u} &\in C_{\text{weak}}(0,T; V^{0,2}(\Omega)) \cap L^2(0,T; V^{1,2}(\Omega)), \end{aligned} \tag{3.1.5}$$

satisfying the energy inequality

$$\begin{aligned} \frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r)\ d\mathbf{x} \quad &+ \quad \int_0^r \int_\Omega \mu_\varepsilon |\mathbb{D}\mathbf{u}|^2\ dt d\mathbf{x} \\ &\leqslant \frac{1}{2}\int_\Omega \rho_{0,\varepsilon\delta}|\mathbf{u}_0|^2\ d\mathbf{x} + \int_0^r\int_\Omega \rho \mathbf{g}\mathbf{u}\ dt d\mathbf{x} \end{aligned} \tag{3.1.6}$$

for a.a. $r \in (0,T)$ *and positive* $\rho, \varphi, \chi \in C(0,T;\ L^p_{loc}(\mathbb{R}^N))$, $\quad \forall p \in [1,\infty)$, *such that*

$$\begin{aligned} ||\rho(r,\cdot)||_{L^p(\mathbb{R}^N)} &= ||\rho_{0,\varepsilon\delta}||_{L^p(\mathbb{R}^N)}, \quad ||\varphi(r,\cdot)||_{L^p(\mathbb{R}^N)} = ||\varphi_{0,\delta}||_{L^p(\mathbb{R}^N)}, \\ ||\chi(r,\cdot)||_{L^p(\mathbb{R}^N)} &= ||\chi_{0,\delta}||_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{3.1.7}$$

3.2 Solidification. Viscous limit on $\varepsilon \to 0$

In this section we consider that δ and τ are fixed. Let us denote the solution of problem (3.1.1)-(3.1.3) by $\rho_\varepsilon,\ \varphi_\varepsilon,\ \chi_\varepsilon,\ \mathbf{u}_\varepsilon$ for the fixed δ, τ. Estimate (3.1.6) and (3.1.7) imply

$$\int_\Omega \rho_\varepsilon |\mathbf{u}_\varepsilon|^2\ d\mathbf{x},\ \int_{\Omega_T} \mu_\varepsilon |\mathbb{D}\mathbf{u}_\varepsilon|^2\ dt d\mathbf{x} < C \tag{3.2.1}$$

for some constant C independent of ε.

Due to the Korn inequality (2.1) we have the uniform boundedness of $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in $L^2(0,T;V^{1,2}(\Omega))$, hence, using (3.1.7) and passing to a suitable subsequence, we have

$$\begin{aligned} \mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} \qquad \text{weakly in } L^2(0,T;V^{1,2}(\Omega)), \\ \overline{\mathbf{u}_\varepsilon} &\rightharpoonup \overline{\mathbf{u}} \qquad \text{weakly in } L^2(0,T;W^{k,2}_{loc}(\mathbb{R}^N)),\ \ \forall k \geqslant 0, \\ \sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon &\rightharpoonup \sqrt{\rho}\mathbf{u} \qquad \text{*-weakly in } L^\infty(0,T;L^2(\Omega)), \\ \rho_\varepsilon,\ \varphi_\varepsilon,\ \chi_\varepsilon &\to \rho,\ \varphi,\ \chi \quad \text{in } C(0,T;\ L^p_{loc}(\mathbb{R}^N)), \quad \forall p \in [1,\infty) \end{aligned} \tag{3.2.2}$$

and for any fixed $\sigma > 0$ there exists $\varepsilon_0(\sigma) > 0$, such that

$$S_\varepsilon(t) \subset]S(t)[_\sigma \qquad \text{for all } \varepsilon < \varepsilon_0 \ \text{ and } \forall t \in [0,T], \tag{3.2.3}$$

where $S_\varepsilon(t) = \boldsymbol{\eta}_\varepsilon(t,[S_0]_\delta) \equiv S(\varphi_\varepsilon(t))$, $S(t) = \boldsymbol{\eta}(t,[S_0]_\delta) \equiv S(\varphi(t))$ and $\boldsymbol{\eta}_\varepsilon$, $\boldsymbol{\eta}$ solve

$$\begin{aligned} \frac{\partial}{\partial t}\boldsymbol{\eta}_\varepsilon(t,\mathbf{y}) &= \overline{\mathbf{u}_\varepsilon}(t,\boldsymbol{\eta}(t,\mathbf{y})), \qquad \boldsymbol{\eta}_\varepsilon(0,\mathbf{y}) = \mathbf{y}, \\ \frac{\partial}{\partial t}\boldsymbol{\eta}(t,\mathbf{y}) &= \overline{\mathbf{u}}(t,\boldsymbol{\eta}(t,\mathbf{y})), \qquad \boldsymbol{\eta}(0,\mathbf{y}) = \mathbf{y} \in \mathbb{R}^N. \end{aligned} \tag{3.2.4}$$

By (3.2.1), (3.2.2) and the low semi-continuity property of integral, we derive

$$0 \leqslant \int_{\Omega_T} \varphi|\mathbb{D}\mathbf{u}|^2 dtd\mathbf{x} \leqslant \liminf_{\epsilon\to 0} \int_{\Omega_T} \varphi_\epsilon|\mathbb{D}\mathbf{u}_\epsilon|^2 dtd\mathbf{x} \leqslant 0.$$

Hence $\varphi\mathbb{D}\mathbf{u} = 0$ a.e. in Ω_T. Accounting that $\mathbf{u} \in L^2(0,T;V^{1,2}(\Omega))$ and $\mathbf{u}(t,\cdot) = 0$ outside of Ω, $t \in (0,T)$, we conclude that there exists a rigid velocity $\mathbf{u}_s = \mathbf{k}(t) + \mathbb{P}(t)\mathbf{x} \equiv \mathbf{k}(t) + \boldsymbol{\omega}(t) \times \mathbf{x}$, such that

$$\mathbf{u} = \mathbf{u}_s \qquad \text{in } S(t), \quad \text{a.e. } t \in (0,T). \tag{3.2.5}$$

Since $\mathbf{u} = \mathbf{u}_s = \overline{\mathbf{u}_s}$ in $[S(t)]_\tau$ (τ-fixed) and $\mathbf{u} \in L^2(0,T;V^{1,2}(\Omega))$, we have $\mathbf{k},\boldsymbol{\omega} \in L^2(0,T)$. Therefore due to Lemma 4.6 there exists the unique solution $\boldsymbol{\eta} = \boldsymbol{\eta}(t,\cdot) : [S_0]_{\delta+\tau} \to [S(t)]_\tau$ of (3.2.4), being the isometry

$$\boldsymbol{\eta}(t,\mathbf{y}) \equiv \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for all } \mathbf{y} \in [S(t)]_\tau, \tag{3.2.6}$$

where the pair $\{\mathbf{q},\mathbb{Q}\}$ is the unique solution of the system

$$\begin{aligned} \frac{d\mathbf{q}}{dt} - \boldsymbol{\omega}(t) \times \mathbf{q} &= \mathbf{k}(t), \qquad \mathbf{q}(0) = \frac{1}{|[S_0]_\delta|}\int\limits_{[S_0]_\delta} \mathbf{y}\ d\mathbf{y}, \\ \frac{d\mathbb{Q}}{dt} &= \mathbb{P}\mathbb{Q}, \qquad \mathbb{Q}(0) = \mathbb{I}. \end{aligned} \tag{3.2.7}$$

By (3.2.6) and $\mathbb{P}^T = -\mathbb{P}$, it is easy to check that

$$\begin{aligned} \mathbf{q}(t) &= \frac{1}{|S(t)|}\int\limits_{S(t)} \mathbf{x}\ d\mathbf{x}, \\ \mathbb{Q}(t) &= \exp\Big(\int_0^t \mathbb{P}(s)ds\Big), \quad \text{such that } \mathbb{Q}^T\mathbb{Q} = \mathbb{I}. \end{aligned} \tag{3.2.8}$$

Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be the isometry, defined as

$$A(t,\mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for all } \mathbf{y} \in \mathbb{R}^N, \quad t \in [0,T], \tag{3.2.9}$$

therefore, accounting (3.2.5) and $\mathbf{u}(t,\cdot) = 0$ outside of Ω, we obtain

$$\begin{aligned} S(t) &\equiv S(\varphi(t)) = A(t,[S_0]_\delta) \subset \Omega \quad \text{for all } t \in [0,T], \\ \varphi(t,\mathbf{x}) &= \varphi_{0,\delta}(A^{-1}(t,\mathbf{x})) \quad \text{for a.e. } (t,\mathbf{x}) \in \Omega_T. \end{aligned} \tag{3.2.10}$$

By the same method as in [25] we can obtain the convergence

$$\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\rho}\mathbf{u} \qquad \text{strongly in } \ L^2(\Omega_T), \tag{3.2.11}$$

see also p. 1358-1361, *"5.2 Point-wise convergence of the velocities"* of [11]. We omit the proof of (3.2.11), since nowadays it is standard.

Let us choose in (3.1.3) for a fixed $\sigma > 0$ the test function $\boldsymbol{\psi} = \boldsymbol{\psi}_\sigma$, such that

$$\mathbb{D}\boldsymbol{\psi}_\sigma(t, \mathbf{x}) = 0 \qquad \text{for } \ t \in [0, T], \quad \mathbf{x} \in]S(t)[_\sigma,$$

that exists due to Proposition 4.3 in [25]. Due to (3.2.2)-(3.2.3), (3.2.11), we can take the limit $\varepsilon \to 0$ in (3.1.1)-(3.1.3) and then pass on $\sigma \to 0$. Therefore for the fixed $\tau > \delta > 0$ we derive the solvability of the system, which consists from the transport equations (3.1.1) with the initial data

$$\rho = \rho_{0,\delta} \equiv (1 - \chi_{0,\delta})\rho_0, \quad \varphi = \varphi_{0,\delta}, \quad \chi = \chi_{0,\delta} \quad \text{ at } t = 0 \tag{3.2.12}$$

and from the integral equality

$$\begin{aligned} \int_{\Omega_T} [\rho\mathbf{u}\partial_t\boldsymbol{\psi} \quad &+ \quad \rho\mathbf{u}\,(\overline{\mathbf{u}} \cdot \nabla)\,\boldsymbol{\psi} - \mu\mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}]\ dtd\mathbf{x} \\ &= \quad -\int_{\Omega} \rho_{0,\delta}\mathbf{u}_0\boldsymbol{\psi}(0, \cdot)\ d\mathbf{x}, \end{aligned} \tag{3.2.13}$$

which holds for any

$$\boldsymbol{\psi} \in L^2(0, T; K(S(t))) \cap H^1(\Omega_T), \quad \text{such that } \ \boldsymbol{\psi}(T, \cdot) = 0. \tag{3.2.14}$$

Here

$$\mu = 2\mu_f\theta + \gamma_0\chi\int_\Omega \chi\ d\mathbf{x}, \ \ \theta = 1 - \varphi - \chi, \ \ \gamma_0 = \frac{\gamma}{|\partial S_0|}.$$

Hence we have shown the following result.

Proposition 3.2 *Under the assumptions of Theorem 2.1 there exists the solution* $\{\rho, \varphi, \chi, \mathbf{u}\}$ *of system* (3.1.1), (3.2.12), (3.2.13), *satisfying the regularity properties (3.1.5), the energy inequality*

$$\begin{aligned} \frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r)\ d\mathbf{x} \quad &+ \quad \int_0^r\int_\Omega \mu|\mathbb{D}\mathbf{u}|^2\ dtd\mathbf{x} \\ &\leqslant \quad \frac{1}{2}\int_\Omega \rho_{0,\delta}|\mathbf{u}_0|^2\ d\mathbf{x} + \int_0^r\int_\Omega \rho\mathbf{g}\mathbf{u}\ dtd\mathbf{x} \end{aligned} \tag{3.2.15}$$

for a.a. $r \in (0, T)$ *and* $\rho, \varphi, \chi \in C(0, T;\ L^p_{loc}(\mathbb{R}^N)), \quad \forall p \in [1, \infty)$, *such that*

$$\begin{aligned} ||\rho(r, \cdot)||_{L^p(\mathbb{R}^N)} \quad &= \quad ||\rho_{0,\delta}||_{L^p(\mathbb{R}^N)}, \quad ||\varphi(r, \cdot)||_{L^p(\mathbb{R}^N)} = ||\varphi_{0,\delta}||_{L^p(\mathbb{R}^N)}, \\ ||\chi(r, \cdot)||_{L^p(\mathbb{R}^N)} \quad &= \quad ||\chi_{0,\delta}||_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{3.2.16}$$

Moreover there exists a preserving orientation isometry $A(t, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ *and a time dependent set* $S(t)$, *defined by (3.2.9)-(3.2.10) and related with* $\mathbf{u}$ *on* $S(t)$ *through formulas (3.2.5), (3.2.7). The function* $\varphi(t, \cdot)$ *and* $S(t)$ *for all* $t \in [0, T]$ *are related by (3.2.10).*

3.3 Limit transition on $\delta \to 0$

In this section the parameter τ continues to be fixed and the solution of problem (3.1.1), (3.2.12), (3.2.13) is denoted by

$$\rho_\delta,\ \varphi_\delta,\ \chi_\delta,\ \mathbf{u}_\delta,\ S_\delta(t) = A_\delta(t, [S_0]_\delta) \quad \text{for any } \delta \in (0, \tau). \tag{3.3.1}$$

The boundaries of $S_\delta(t)$ and $\widetilde{S}_\delta(t) = \boldsymbol{\eta}_\delta(t, S_0)$ are denoted by $\partial S_\delta(t)$ and $\partial \widetilde{S}_\delta(t)$, which are the interior and exterior boundaries of $S(\chi_\delta(t)) \equiv \boldsymbol{\eta}_\delta(t, U_\delta(\partial S_0))$, respectively. During all considerations in the section we will take into account that

$$\rho_\delta(t,\cdot) = \begin{cases} \rho_s(\boldsymbol{\eta}_\delta^{-1}(t,\mathbf{x})) \geqslant const > 0, & \mathbf{x} \in S_\delta(t); \\ 0, & \mathbf{x} \in S(\chi_\delta(t)); \\ \rho_f = const, & \mathbf{x} \in \mathbb{R}^N \backslash \widetilde{S}_\delta(t) \end{cases} \tag{3.3.2}$$

for a.e. $t \in (0,T)$, where $\boldsymbol{\eta}_\delta(t,\cdot)$ solves the system

$$\frac{\partial}{\partial t}\boldsymbol{\eta}_\delta(t,\mathbf{y}) = \overline{\mathbf{u}_\delta}(t, \boldsymbol{\eta}_\delta(t,\mathbf{y})), \qquad \boldsymbol{\eta}_\delta(0,\mathbf{y}) = \mathbf{y} \in \mathbb{R}^N. \tag{3.3.3}$$

By (3.2.5) and (3.2.7)-(3.2.10) we have

$$\int\limits_{S_\delta(t)} |\mathbf{u}_\delta|^2 \ d\mathbf{x} = |\frac{d\mathbf{q}_\delta}{dt}(t)|^2 |[S_0]_\delta| + \int\limits_{[S_0]_\delta} |\frac{d\mathbb{Q}_\delta}{dt}(t)(\mathbf{y} - \mathbf{q}_\delta(0))|^2 \ d\mathbf{y}$$

for a.a. $t \in (0,T)$. Moreover

$$\int\limits_{B_R(\mathbf{q}_\delta(0))} |\mathbb{Q}_\delta'(t)(\mathbf{y} - \mathbf{q}_\delta(0))|^2 \ d\mathbf{x} \geqslant C|\mathbb{Q}_\delta'(t)|^2,$$

where $R > 0$ is chosen, such that $B_R(\mathbf{q}_\delta(0)) \subset [S_0]_\delta$. Here $B_R(\mathbf{q}_\delta(0))$ is an open ball of radius $R > 0$ with the center $\mathbf{q}_\delta(0)$ and the constant C depends only on R and N. Due to (3.2.15) and (3.3.2), we have the uniform boundedness of $\{\mathbf{u}_\delta\}_{\delta>0}$ in $L^\infty(0,T;L^2(S_\delta(t)))$. From above considerations and (3.2.7) we conclude

$$|\mathbf{q}_\delta'| \leqslant C, \quad |\mathbb{Q}_\delta'| \leqslant C, \quad \sqrt{2}|\boldsymbol{\omega}_\delta| = |\mathbb{Q}_\delta'\mathbb{Q}_\delta^T| \leqslant C \tag{3.3.4}$$

for the constants C independent of δ. Therefore, there exists a subsequence of $\{\mathbf{q}_\delta, \mathbb{Q}_\delta, \boldsymbol{\omega}_\delta\}_{\delta>0}$, such that

$$\begin{aligned} \boldsymbol{\omega}_\delta &\rightharpoonup \boldsymbol{\omega} && \text{*-weakly in } L^\infty(0,T), \\ \mathbf{q}_\delta, \mathbb{Q}_\delta &\rightharpoonup \mathbf{q}, \mathbb{Q} && \text{*-weakly in } W^{1,\infty}(0,T) \quad \text{and} \quad \mathbb{Q}^T\mathbb{Q} = \mathbb{I}, \\ \mathbf{q}_\delta, \mathbb{Q}_\delta &\to \mathbf{q}, \mathbb{Q} && \text{in } C^\alpha(0,T) \qquad \text{for any } \alpha \in [0,1), \\ A_\delta &\to A = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) && \text{in } C^\alpha(0,T; C^2_{loc}(\mathbb{R}^N)). \end{aligned} \tag{3.3.5}$$

Since $\int_\Omega \chi_\delta |\mathbb{D}\mathbf{u}_\delta| \, d\mathbf{x} \leqslant \left(\int_\Omega \chi_\delta |\mathbb{D}\mathbf{u}_\delta|^2 \, d\mathbf{x}\right)^{1/2} \left(\int_\Omega \chi_\delta \, d\mathbf{x}\right)^{1/2}$, we derive

$$\int_0^T \left(\int_\Omega |\mathbf{u}_\delta| \, ^q d\mathbf{x}\right)^2 dt \leqslant \int_0^T \left(\int_\Omega |\mathbb{D}\mathbf{u}_\delta| \, d\mathbf{x}\right)^2 dt \leqslant C \qquad (3.3.6)$$

for any $q \leqslant N/(N-1)$ by the energy inequality (3.2.15) and the embedding results, obtained in Theorem 2.2, p. 152-153, of [27] (see also Proposition 1.2 [28]). Therefore there exists a subsequence of $\{\mathbf{u}_\delta\}_{\delta>0}$, such that

$$\begin{array}{rcll} \mathbf{u}_\delta & \rightharpoonup & \mathbf{u} & \text{weakly in } L^2(0,T;L^q(\Omega)), \\ \overline{\mathbf{u}_\delta} & \rightharpoonup & \overline{\mathbf{u}} & \text{weakly in } L^2(0,T;W^{k,q}_{loc}(\mathbb{R}^N)), \ \forall k \geqslant 0, \\ \mathbb{D}\mathbf{u}_\delta & \rightharpoonup & \mathbb{D}\mathbf{u} & \text{weakly in } L^2(0,T;\mathcal{M}(\Omega)). \end{array} \qquad (3.3.7)$$

Moreover, applying (3.2.15) and (3.2.16), we have that for any $p \in [1,\infty)$

$$\begin{array}{rcll} \rho_\delta, \ \varphi_\delta, \ \chi_\delta, \ \theta_\delta & \to & \rho, \ \varphi, \ 0, \ 1-\varphi & \text{in } C(0,T;\, L^p_{loc}(\mathbb{R}^N)), \\ \sqrt{\rho_\delta}\mathbf{u}_\delta & \rightharpoonup & \sqrt{\rho}\mathbf{u} & *\text{-weakly in } L^\infty(0,T;L^2(\Omega)), \end{array} \qquad (3.3.8)$$

which give

$$\begin{array}{rcl} S(t) & \equiv & S(\varphi(t)) = A(t,S_0) \quad \text{for } t \in [0,T] \\ \varphi(t,\mathbf{x}) & = & \varphi_0(A^{-1}(t,\mathbf{x})) \quad \text{for a.e. } (t,\mathbf{x}) \in \Omega_T. \\ \rho(t,\mathbf{x}) & = & \left\{ \begin{array}{ll} \rho_s(A^{-1}(t,\mathbf{x})), & \mathbf{x} \in [S(t)]_\tau; \\ \rho_f = const, & \mathbf{x} \in \mathbb{R}^N \setminus S(t), \end{array} \right. \quad \text{a.e. } t \in (0,T) \end{array} \qquad (3.3.9)$$

with the help of (3.2.10), (3.3.2), (3.3.5) and Lemma 4.6.

By (3.2.15), (3.3.6) and the semi-continuity property of integral there exist two matrix functions $\mathbb{M}_i = \mathbb{M}_i(t,\mathbf{x}) \in L^2(\Omega_T)$, such that

$$\begin{array}{rcl} \sqrt{\theta_\delta}\mathbb{D}\mathbf{u}_\delta & \rightharpoonup & \mathbb{M}_1, \quad \sqrt{\varphi_\delta}\mathbb{D}\mathbf{u}_\delta \rightharpoonup \mathbb{M}_2 \quad \text{weakly in } L^2(\Omega_T), \\ \displaystyle\int_{\Omega_T} (1-\varphi)|\mathbb{M}_1|^2 \, dtd\mathbf{x} & \leqslant & \displaystyle\liminf_{\delta\to 0} \int_{\Omega_T} \theta_\delta |\mathbb{D}\mathbf{u}_\delta|^2 \, dtd\mathbf{x} \leqslant C, \\ \displaystyle\int_{\Omega_T} \varphi|\mathbb{M}_2|^2 \, dtd\mathbf{x} & \leqslant & \displaystyle\liminf_{\delta\to 0} \int_{\Omega_T} \varphi_\delta |\mathbb{D}\mathbf{u}_\delta|^2 \, dtd\mathbf{x} = 0, \\ \displaystyle\int_0^T \left(\int_\Omega |\mathbb{D}\mathbf{u}|\right)^2 dt & \leqslant & \displaystyle\int_0^T \left(\int_\Omega |\mathbb{D}\mathbf{u}_\delta| \, d\mathbf{x}\right)^2 dt \leqslant C, \end{array} \qquad (3.3.10)$$

where the equalities $\theta_\delta = \theta_\delta^2$, $\varphi_\delta = \varphi_\delta^2$ and (3.3.8) have been used. From (3.3.9) we conclude

$$\mathbb{M}_1 = \mathbb{D}\mathbf{u} \quad \text{in } F(t) = \Omega \setminus \overline{S(t)} \quad \text{and} \quad \mathbb{M}_2 = \mathbb{D}\mathbf{u} = 0 \quad \text{in } S(t). \qquad (3.3.11)$$

Due to Theorems 1.4, 1.5 of [2] (see also theorem 2.1 , p. 148-150, of [28] and Theorem 1.1 of [27]) the Radon measure $\mathbb{D}\mathbf{u}$ has the following the Lebesgue-Radon-Nikodym decomposition

$$\mathbb{D}\mathbf{u} = \mathbb{D}\mathbf{u}\mathcal{L}^N + \mathbb{T}(\mathbf{u}_s - \mathbf{u}_f)\,\mathcal{H}^{N-1}\lfloor \partial S(t) \cap \Omega \qquad \text{for a.a. } t \in (0,T), \quad (3.3.12)$$

where $\mathbb{D}\mathbf{u}\mathcal{L}^N$ is the absolutely continuous part of $\mathbb{D}\mathbf{u}$ with respect to the Lebesgue measure $\mathcal{L}^N$. The density of $\mathbb{D}\mathbf{u}\mathcal{L}^N$ coincides with the function $\mathbb{D}\mathbf{u}$ in $F(t)$ and $\mathbb{D}\mathbf{u} = 0$ in $S(t)$; $\mathcal{H}^{N-1}\lfloor\partial S(t)\cap\Omega$ is the $(N-1)$-dimensional Hausdorff measure, restricted to the surface $\partial S(t)\cap\Omega$; the matrix $\mathbb{T}(\mathbf{p})$ has the components

$$\mathbb{T}_{ij}(\mathbf{p}) = (p_i n_j + p_j n_i)/2, \qquad i,j = 1,..,N \tag{3.3.13}$$

with $\mathbf{n}$ being the unit interior normal to $\partial S(t)$; $\mathbf{u}_s(t,\cdot)$ and $\mathbf{u}_f(t,\cdot)$ are the trace values of $\mathbf{u}$ on $\partial S(t)$ from the domain $S(t)$ and the domain $F(t)$, respectively. The functions $\mathbf{u}_s(t,\cdot)\in L^1(\partial S(t))$ and $\mathbf{u}_f(t,\cdot)\in L^1(\partial S(t))$ satisfy

$$||\mathbf{u}_s(t,\cdot)||_{L^1(\partial S(t))},\ ||\mathbf{u}_f(t,\cdot)||_{L^1(\partial S(t))} \leqslant C||\mathbf{u}(t,\cdot)||_{BD(\Omega)}$$

with the constant C depending only on the curvature of $\partial S(t)$, i.e. on the curvature of ∂S_0. Therefore (3.3.10) implies

$$||\mathbf{u}_s||_{L^2(0,T;L^1(\partial S(t)))},\ ||\mathbf{u}_f||_{L^2(0,T;L^1(\partial S(t)))} \leqslant C. \tag{3.3.14}$$

Moreover (3.3.5) and (3.3.11) give

$$\begin{aligned}\mathbf{u} &\equiv \mathbf{u}_s \quad \text{in } \overline{S(t)}, \quad \text{a.e. } t\in[0,T] \quad \text{with}\\ \mathbf{u}_s &= \mathbf{q}'(t) + \boldsymbol{\omega}(t)\times(\mathbf{x}-\mathbf{q}(t)) \quad \text{for all } \mathbf{x}\in S(t),\\ &\quad ||\mathbf{q}||_{W^{1,\infty}(0,T)}, ||\mathbb{Q}||_{W^{1,\infty}(0,T)}, ||\boldsymbol{\omega}||_{L^\infty(0,T)} \leqslant C.\end{aligned} \tag{3.3.15}$$

For simplicity of the notations, since the function $\mathbf{u}_s$ is continuous in $\overline{S(t)}$, we use here and below the same notation for the trace value of the velocity $\mathbf{u}$ from the "solid" side $S(t)$.

Since information (3.3.7)-(3.3.8) is not enough to pass to the limit on $\delta\to 0$ in (3.2.13), we consider some embedding results, which will be valid in the "fluid" domains $S(\theta_\delta(t)) \equiv \Omega\backslash\overline{\widetilde{S}_\delta(t)}$, $\forall\delta > 0$. The following embedding inequalities

$$\begin{aligned}||\mathbf{z}||_{L^r(B)} &\leqslant C||\mathbf{z}||_{W^{1,2}(B)} \leqslant C||\mathbf{z}||_{LD^2(B)},\\ ||\mathbf{z}||_{L^p(\partial B)} &\leqslant C||\mathbf{z}||_{W^{1,2}(B)}, \qquad \forall\mathbf{z}\in LD^2(B)\end{aligned} \tag{3.3.16}$$

are true for any finite numbers $r\leqslant 2N/(N-2)$ and $p\leqslant 2(N-1)/(N-2)$ in a bounded domain $B\subset\mathbb{R}^N$ with the boundary $\partial B\in C^1$. Moreover we have the interpolation inequality

$$||\mathbf{z}||_{L^q(B)} \leqslant C\left\{||\mathbf{z}||^{1-a}_{L^2(B)}||\nabla\mathbf{z}||^a_{L^2(B)} + ||\mathbf{z}||_{L^2(B)}\right\}, \tag{3.3.17}$$

where C denotes various positive constants, depending only on the diameter $d(B) = \sup_{\mathbf{x},\mathbf{y}\in B}\{|\mathbf{x}-\mathbf{y}|\}$ of B and $\partial B\in C^1$. Here $q\in[2, 2N/(N-2)]$ is a finite real and $a = N(1/2-1/q)$.

Since Ω is bounded, there exists $R>0$, such that $]\Omega[_{2\tau_0}\subset B_R(0)$. Obviously, (3.3.16)-(3.3.17) are valid in the following domains

1) $B = B_R(0)\backslash\overline{S_0}$; 2) $B = B_R(0)\backslash\overline{S(t)}$; 3) $B = B_R(0)\backslash\overline{S_\delta(t)}$.

Let us derive the following behavior of the set $\widetilde{S}_\delta(t)$.

Lemma 3.1 *The boundary $\partial\widetilde{S}_\delta(t)$ of the set $\widetilde{S}_\delta(t)$ is C^2, such that*

$$\widetilde{S}_\delta(t) \xrightarrow{b} S(t) \equiv \boldsymbol{\eta}(t, S_0) \text{ uniformly on } t \in [0,T]. \tag{3.3.18}$$

Moreover $\partial\widetilde{S}_\delta(t)$ converges to $\partial S(t)$, uniformly on $t \in [0,T]$ in C^1-norm.

Proof. Since $\{\mathbf{u}_\delta\}_{\delta>0}$ is uniformly bounded in $L^2(0,T;L^q(\Omega))$, $q \leqslant N/(N-1)$ on $\delta > 0$, then, using (3.3.3), we have that for the space derivatives

$$|D^k\boldsymbol{\eta}_\delta(t,\mathbf{x})| \leqslant \frac{C}{\tau^k} \quad \text{in } \Omega_T, \quad k = 1,2 \tag{3.3.19}$$

with the constants C independent of δ (and $\tau - fixed$). Property (3.3.18) is a direct consequence of Lemma 4.6.

For any $\mathbf{y}, \mathbf{y}_0 \in \partial S_0$ and the projection points $\mathbf{x} = \mathbf{p}(\mathbf{y})$, $\mathbf{x}_0 = \mathbf{p}(\mathbf{y}_0) \in \partial_\delta S_0$ we have the formulas of Taylor

$$\begin{aligned}
\boldsymbol{\eta}_\delta(t,\mathbf{y}) &= \boldsymbol{\eta}_\delta(t,\mathbf{x}) + D\boldsymbol{\eta}_\delta(t,\mathbf{x})\cdot(\mathbf{y}-\mathbf{x}) + \frac{D^2\boldsymbol{\eta}_\delta(t,\zeta)}{2!}\cdot(\mathbf{y}-\mathbf{x})^2,\\
\boldsymbol{\eta}_\delta(t,\mathbf{y}_0) &= \boldsymbol{\eta}_\delta(t,\mathbf{x}_0) + D\boldsymbol{\eta}_\delta(t,\mathbf{x}_0)\cdot(\mathbf{y}_0-\mathbf{x}_0) + \frac{D^2\boldsymbol{\eta}_\delta(t,\zeta_0)}{2!}\cdot(\mathbf{y}_0-\mathbf{x}_0)^2.
\end{aligned}$$

By Lemma (4.2) we have $\mathbf{x} = \mathbf{y} + \bigtriangledown\mathbf{d}_{S_0}(\mathbf{y})\delta$, $\mathbf{x}_0 = \mathbf{y_0} + \bigtriangledown\mathbf{d}_{S_0}(\mathbf{y_0})\delta$. Therefore if we devide the difference of the formulas of Taylor over $|\mathbf{y}-\mathbf{y}_0|$ and pass to the limit on $|\mathbf{y}-\mathbf{y}_0| \to 0$, we get

$$|\boldsymbol{\tau}(\boldsymbol{\eta}_\delta(t,\mathbf{y}_0)) - \boldsymbol{\tau}(\boldsymbol{\eta}_\delta(t,\mathbf{p}(\mathbf{y}_0)))| \leqslant C\frac{\delta}{\tau^2} \tag{3.3.20}$$

with the help of (3.3.19) and $\partial S_0 \in C^2$. Here $\boldsymbol{\tau}$ are tangents vectors coincide with arbitrary given direction $\mathbf{s} = (\mathbf{y}-\mathbf{y}_0)$. Accounting that $\boldsymbol{\eta}_\delta(t,\overline{[S_0]_\delta}) = A_\delta(t,\overline{[S_0]_\delta})$, we obtain the convergence of $\partial\widetilde{S}_\delta(t)$ to $\partial S(t)$, uniformly on $t \in [0,T]$ in C^1-norm (we refer to the description of C^1- surfaces in the book [22], p. 304-307). ■

Lemma 3.1 implies that the boundary $\partial\widetilde{S}_\delta(t)$ has the shape close to the shape of $\partial S(t)$ in C^1-norm. Therefore inequalities (3.3.16)-(3.3.17) are valid for the domains $B = B_R(0)\backslash\overline{\widetilde{S}_\delta(t)}$. In particular, taking into account (3.2.15), (3.3.2) and $\mathbf{u}_\delta(t,\cdot) = 0$ outside of Ω, we have the inequalities

$$\begin{aligned}
||\mathbf{u}_\delta||_{L^\infty(0,T;L^2(B))} &\leqslant C,\\
||\mathbf{u}_\delta||_{L^2(0,T;L^r(B))} &\leqslant C||\mathbf{u}_\delta||_{L^2(0,T;W^{1,2}(B))}\\
&\leqslant C||\mathbf{u}_\delta||_{L^2(0,T;LD^2(B))} \leqslant C,\\
||\mathbf{u}_\delta||_{L^2(0,T;L^p(\partial B))} &\leqslant C,\\
||\mathbf{u}_\delta||_{L^{8/3}((0,T)\times B)} &\leqslant C \quad \text{for } B = S(\theta_\delta(t)) \equiv \Omega\backslash\overline{\widetilde{S}_\delta(t)},
\end{aligned} \tag{3.3.21}$$

where r, p are defined in (3.3.16). As a consequence of (3.3.21), we have

$$||\overline{\mathbf{u}_\delta} - \mathbf{u}_\delta||_{L^2((0,T)\times B)} \leqslant C\tau^\alpha ||\mathbf{u}_\delta||_{L^2(0,T;W^{1,2}(B))} \text{ for some } \alpha > 0, \tag{3.3.22}$$

that can be shown as in Theorem 1, 4., page 272 of [9]. All constants C in (3.3.21), (3.3.22) depend only on $d(\Omega)$, $\partial S_0 \in C^1$, being independent of δ and also τ, it is enough to consider small δ, such that $C\frac{\delta}{\tau^2} \leqslant 10^{-3}||\partial S_0||_{C^1}$ in (3.3.20).

3.3.1 Approximation of test function

Let us consider an arbitrary function $\boldsymbol{\psi}$, such that

$$\begin{aligned} \boldsymbol{\psi} &\in L^2(0,T;KB(S(t))), \\ \boldsymbol{\psi}_t &\in L^2(0,T;L^2(\Omega\backslash\partial S(t))), \quad \boldsymbol{\psi}(T,\cdot) = 0. \end{aligned} \tag{3.3.23}$$

Lemma 3.2 *For a given* $\boldsymbol{\psi}$**,** *satisfying* (3.3.23)*, there exists a sequence of functions*

$$\boldsymbol{\psi}_\delta \in L^2(0,T;K(S_\delta(t))), \quad \partial_t\boldsymbol{\psi}_\delta \in L^2(\Omega_T), \quad \boldsymbol{\psi}_\delta(T,\cdot) = 0,$$

such that

$$\begin{aligned} \boldsymbol{\psi}_\delta &\to \boldsymbol{\psi} \quad \textit{in } L^2(\Omega_T), \\ (1-\chi_\delta)\nabla_{t,\mathbf{x}}\boldsymbol{\psi}_\delta &\to \nabla_{t,\mathbf{x}}\boldsymbol{\psi} \quad \textit{in } L^2(0,T;L^2(\Omega\backslash\partial S(t))). \end{aligned} \tag{3.3.24}$$

The functions $\boldsymbol{\psi}_\delta$ *are defined by* (3.3.28) *if* $N = 2$ *and* (3.3.29) *if* $N = 3$.

Proof. To construct $\{\boldsymbol{\psi}_\delta\}_{\delta>0}$, we use the stream function approach as in Proposition 4.3 of [25].

1st step) Let us define the approximations for the characteristic function of the set of non-positive reals $\mathbb{R}^+$ by

$$\widehat{\varrho}_\delta(s) = \begin{cases} 1, & s < 0; \\ -\frac{s^2}{2\delta^3(1-\delta)} + 1, & s \in [0,\delta^2]; \\ (-\frac{s}{\delta} + 1 - \delta/2)/(1-\delta), & s \in (\delta^2, \delta-\delta^2); \\ \frac{(s-\delta)^2}{2\delta^3(1-\delta)}, & s \in [\delta-\delta^2,\delta]; \\ 0, & s > \delta \end{cases}$$

and the approximation for the characteristic function of the set $S_\delta(t)$ by

$$\varrho_\delta(t,\mathbf{x}) = \widehat{\varrho}_\delta(d_{S_0}(\boldsymbol{\eta}_\delta^{-1}(t,\mathbf{x}))). \tag{3.3.25}$$

2^{nd} *step)* We have that $\mathbb{D}\boldsymbol{\psi}(t,\cdot) = 0$ on $S(t)$ and $\text{div}\boldsymbol{\psi}(t,\cdot) = 0$ in $\mathcal{D}'(\Omega)$, that permit to use Theorem 3.8, p 36 of [14]. Therefore in the case:

a) when $N = 2$, there exist a scalar stream function $\xi(t,\cdot) \in H^1(\Omega)$, a rigid velocity function $\boldsymbol{\psi}_S(t,\cdot)$ and a scalar function $\xi_S(t,\cdot)$, satisfying the conditions

$$\begin{cases} \boldsymbol{\psi} = \mathbf{curl}\xi = \begin{pmatrix} \partial_{x_2}\xi \\ -\partial_{x_1}\xi \end{pmatrix} \in KB(S(t)), \quad \boldsymbol{\psi}_S(t,\mathbf{x}) = \mathbf{curl}\xi_S(t,\mathbf{x}) \\ \xi_S = \frac{1}{2}\omega|\mathbf{x}|^2 + (\begin{pmatrix} -\boldsymbol{\psi}_2^0 \\ \boldsymbol{\psi}_1^0 \end{pmatrix}, \mathbf{x}) + \xi^0, \\ \boldsymbol{\psi} \equiv \boldsymbol{\psi}_S \quad \text{and} \quad \xi \equiv \xi_S \quad \text{in } S(t). \end{cases}$$

The scalar functions $\omega(t)$, $\psi_1^0(t), \psi_2^0(t)$, $\xi^0(t) \in W^{1,\infty}(0,T)$ are defined uniquely by the above relations. We can consider that the functions $\boldsymbol{\psi}_S$, ξ_S are defined in Ω_T.

Let us extend the function $\boldsymbol{\psi}$, restricted to the "fluid" part $F(t)$, inside of the "solid part" $S(t)$ for $t \in (0,T)$ by the following way: let $\widetilde{\boldsymbol{\psi}}$ be the solution of the problem

$$\begin{cases} div\widetilde{\boldsymbol{\psi}} = 0 \quad \text{in } S(t), \\ \\ \widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}_f(t,\cdot) \quad \text{on } \partial S(t), \end{cases} , \tag{3.3.26}$$

where $\boldsymbol{\psi}_f(t,\cdot)$ is the trace of $\boldsymbol{\psi}(t,\cdot)$ on $\partial S(t)$ from the "fluid" side $F(t)$. Let us consider the following extension

$$\boldsymbol{\psi}_F = \boldsymbol{\psi} \quad \text{in } F(t) \quad \text{and} \quad \boldsymbol{\psi}_F = \widetilde{\boldsymbol{\psi}} \quad \text{in } S(t), \quad \forall t \in (0,T). \tag{3.3.27}$$

Let $\xi_F \in L^2(0,T;W^{2,2}(\Omega))$ be a stream function of $\boldsymbol{\psi}_F$ in Ω_T. Since $\boldsymbol{\psi}$ satisfies (3.3.23) and the boundary $\cup_{t\in(0,T)}\partial S(t)$ is Lipschitz continuous on the time variable and is C^2 on the space variables $\mathbf{x}$, we can find ξ_F, such that $\xi = \xi_F$ in $\cup_{t\in(0,T)}F(t)$ and $\boldsymbol{\psi}_F = \mathbf{curl}\xi_F \in W^{1,2}(0,T;L^2(\Omega))$. Let us define the approximate functions

$$\xi_\delta = \xi_S + \varrho_\delta(\xi_F - \xi_S), \quad \boldsymbol{\psi}_\delta = \mathbf{curl}\xi_\delta \quad \text{in } \Omega_T \tag{3.3.28}$$

to ξ and to $\boldsymbol{\psi}$, respectively. Obviously that $\boldsymbol{\psi}_\delta$ satisfies (3.3.24).

b) when $N = 3$, there exist a vector function $\boldsymbol{\xi}(t,\cdot) = (\xi_1,\xi_2,\xi_3)^T \in H^1(\Omega)$, a rigid velocity function $\boldsymbol{\psi}_S(t,\cdot)$ and a vector function $\boldsymbol{\xi}_S(t,\cdot)$, satisfying the conditions

$$\begin{cases} \boldsymbol{\psi} = \mathbf{curl}\boldsymbol{\xi} = \begin{pmatrix} \partial_{x_2}\xi_3 - \partial_{x_3}\xi_2 \\ \partial_{x_3}\xi_1 - \partial_{x_1}\xi_3 \\ \partial_{x_1}\xi_2 - \partial_{x_2}\xi_1 \end{pmatrix} \in KB(S(t)), \quad \boldsymbol{\psi}_S(t,\mathbf{x}) = \mathbf{curl}\boldsymbol{\xi}_S(t,\mathbf{x}), \\ \boldsymbol{\xi}_S = \frac{1}{2} \begin{pmatrix} \omega_1(x_2^2 + x_3^2) \\ \omega_2(x_1^2 + x_3^2) \\ \omega_3(x_1^2 + x_2^2) \end{pmatrix} + A\mathbf{x} + \frac{1}{2}\boldsymbol{\psi}^0 \times \mathbf{x} + \boldsymbol{\xi}^0, \\ \boldsymbol{\psi} \equiv \boldsymbol{\psi}_S \quad \text{and} \quad \boldsymbol{\xi} \equiv \boldsymbol{\xi}_S \quad \text{in } S(t). \end{cases}$$

The vector-functions $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t), \omega_3(t))^T$, $\boldsymbol{\psi}^0(t)$, $\boldsymbol{\xi}^0(t) \in W^{1,\infty}(0,T)$ and the (3×3)-symmetric matrix $A(t) \in W^{1,\infty}(0,T)$ are uniquely defined by the above relations. We can consider that the functions $\boldsymbol{\psi}_S$, $\boldsymbol{\xi}_S$ are defined in Ω_T.

As in the case a), by the same way (3.3.26)-(3.3.27), we construct the extension $\boldsymbol{\psi}_F$ on the whole domain Ω_T for the function $\boldsymbol{\psi}$, which is restricted to the "fluid" part $\cup_{t\in(0,T)} F(t)$. Let $\boldsymbol{\xi}_F \in L^2(0,T; W^{2,2}(\Omega))$ be a stream function of $\boldsymbol{\psi}_F$ in Ω_T. We can find $\boldsymbol{\xi}_F$, such that $\boldsymbol{\xi} = \boldsymbol{\xi}_F$ in $\cup_{t\in(0,T)} F(t)$ and $\boldsymbol{\psi}_F = \mathbf{curl}\boldsymbol{\xi}_F \in W^{1,2}(0,T; L^2(\Omega))$. Let us define the approximate functions

$$\boldsymbol{\xi}_\delta = \boldsymbol{\xi}_S + \varrho_\delta(\boldsymbol{\xi}_F - \boldsymbol{\xi}_S), \quad \boldsymbol{\psi}_\delta = \mathbf{curl}\boldsymbol{\xi}_\delta \quad \text{in } \Omega_T \tag{3.3.29}$$

to $\boldsymbol{\xi}$ and to $\boldsymbol{\psi}$, respectively. Obviously, that $\boldsymbol{\psi}_\delta$ satisfies (3.3.24). ■

3.3.2 Convergence inside the mixture zone $S(\chi_\delta)$

In this subsection we derive the following convergence result, related with the "jump" term on $\partial S(t)$ in (2.3) of the definition of weak solution.

Lemma 3.3 *Let $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_\delta$ be the functions as in Lemma 3.2. Then there exists a suitable subsequence of $\{\mathbf{u}_\delta\}_{\delta>0}$, such that*

$$\int_{\Omega_T} \mu_\delta \mathbb{D}\mathbf{u}_\delta : \mathbb{D}\boldsymbol{\psi}_\delta \, dt d\mathbf{x} \to \int_0^T \Big\{ \int_{\Omega\setminus\partial S(t)} 2\mu_f \, \mathbb{D}\,\mathbf{u} : \mathbb{D}\boldsymbol{\psi} \; d\mathbf{x} + \int_{\partial S(t)} \gamma(\mathbf{u}_f - \mathbf{u}_s)(\boldsymbol{\psi}_f - \boldsymbol{\psi}_s) \; d\mathbf{x} \Big\} dt \quad \textit{for } \delta \to 0,$$

where $\mathbf{u}_s$, $\boldsymbol{\psi}_s$ and $\mathbf{u}_f$, $\boldsymbol{\psi}_f$ be trace values of $\mathbf{u}$, $\boldsymbol{\psi}$ on $\partial S(t)$ from the "solid" side $S(t)$ and the "fluid" side $F(t)$, respectively.

Proof. i) By (3.3.7), (3.3.8) and (3.3.12) we easily derive

$$\int_{\Omega_T} \theta_\delta \mathbb{D}\mathbf{u}_\delta : \mathbb{D}\boldsymbol{\psi}_\delta dt d\mathbf{x} \to \int_{\Omega_T} \theta \, \mathbb{D}\,\mathbf{u} : \mathbb{D}\boldsymbol{\psi} \; dt d\mathbf{x}; \tag{3.3.30}$$

ii) In the following considerations we use the notations of Lemma 3.2 and consider the case, when $N = 3$. Let us point that the case $N = 2$ can be transformed in 3-dimensional one, if all scalar stream functions, introduced in 2^{nd} *step),* a) of Lemma 3.2, are considered as vector functions. For instance: if ξ_δ is given by (3.3.28), then we define the vector function $\boldsymbol{\xi}_\delta = (0, 0, \xi_\delta)$.

By (3.3.29) and Lemma 4.4, we have

$$\mathbb{D}\boldsymbol{\psi}_\delta = D_1(\nabla\varrho_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S) + D_2(\varrho_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S) \tag{3.3.31}$$

in accordance with notations (4.2.4).

Since

$$\int_\Omega \chi_\delta(t)\ d\mathbf{x} = \int_\Omega \chi_\delta(0)\ d\mathbf{x} = \delta|\partial S_0| + O(\delta^2), \tag{3.3.32}$$

then the energy inequality (3.2.15), $\boldsymbol{\xi}_F - \boldsymbol{\xi}_S \in L^2(0,T;W^{2,2}(\Omega))$, $\partial S_0 \in C^2$ imply

$$\begin{aligned}\delta \int_0^T \int_{S(\chi_\delta(t))} \mathbb{D}\mathbf{u}_\delta \quad &: \quad D_1(\nabla \varrho_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)\, d\mathbf{x}dt \\ = \int_0^T \int_{S(\chi_\delta(t))} \mathbb{D}\mathbf{u}_\delta \quad &: \quad D_1(\mathbf{m}_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)\, d\mathbf{x}dt + O(\sqrt{\delta}),\end{aligned} \tag{3.3.33}$$

where $\mathbf{m}_\delta = -\nabla d_{S_0}(\boldsymbol{\eta}_\delta^{-1}(t,\mathbf{x}))$. Moreover, using that $\boldsymbol{\xi}_F = \boldsymbol{\xi}_S$ on $\partial S(t)$ (in view of $\boldsymbol{\xi}(t,\cdot) \in H^1(\Omega)$) and Theorem 1 of [15], we obtain

$$\delta \int_0^T \int_{S(\chi_\delta(t))} \mathbb{D}\mathbf{u}_\delta : D_2(\varrho_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)\, d\mathbf{x}dt = O(\sqrt{\delta}). \tag{3.3.34}$$

Since the matrix $D_1 = D_1(\mathbf{m}_\delta, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)$ is symmetric, applying the Gauss-Green formula, we have

$$\begin{aligned}\int_{S(\chi_\delta(t))} \mathbb{D}\mathbf{u}_\delta \quad &: \quad D_1\, d\mathbf{x} = - \int_{S(\chi_\delta(t))} \operatorname{div}(D_1) \cdot \mathbf{u}_\delta\, d\mathbf{x} \\ &+ \quad \int_{\partial S(\chi_\delta(t))} (D_1 \mathbf{n}_\delta) \cdot \mathbf{u}_\delta\, d\mathbf{x} = I_\delta^1 + I_\delta^2,\end{aligned} \tag{3.3.35}$$

where $\mathbf{n}_\delta$ is the exterior normal to the boundary of $S(\chi_\delta(t))$. Due to (3.2.15) and $\boldsymbol{\xi}_F - \boldsymbol{\xi}_S \in L^2(0,T;W^{2,2}(\Omega))$, $\partial S_0 \in C^2$, we easily obtain

$$|\int_0^T I_\delta^1(t)\, dt| \leqslant C||\boldsymbol{\xi}_F - \boldsymbol{\xi}_S||_{L^2(0,T;W^{2,2}(S(\chi_\delta(t))))} \to 0 \quad \text{for } \delta \to 0. \tag{3.3.36}$$

Moreover we have

$$I_\delta^2(t) = \int_{\partial S_\delta(t)} (D_1 \mathbf{n}_\delta) \cdot \mathbf{u}_{\delta,s}\, d\mathbf{x} + \int_{\partial \widetilde{S}_\delta(t)} (D_1 \mathbf{n}_\delta) \cdot \mathbf{u}_{\delta,f}\, d\mathbf{x} = J_\delta^1 + J_\delta^2, \tag{3.3.37}$$

where $\mathbf{u}_{\delta,s}(t,\mathbf{x}) \equiv \mathbf{q}_\delta'(t) + \boldsymbol{\omega}_\delta(t) \times (\mathbf{x} - \mathbf{q}_\delta(t))$ is the trace value of $\mathbf{u}_\delta$ on $\partial S_\delta(t)$ with $\mathbf{q}_\delta'$, $\boldsymbol{\omega}_\delta$ satisfying (3.3.4), and $\mathbf{u}_{\delta,f}$ is the trace value of $\mathbf{u}_\delta$ on $\partial \widetilde{S}_\delta(t)$ (see (3.3.21)), such that

$$||\mathbf{u}_{\delta,f}||_{L^2(0,T;L^4(\partial S(t)))} \leqslant C. \tag{3.3.38}$$

The integral J_δ^1 can be written as an integral on $\partial S(t)$, using the change of variables $T_t^{1,\delta} : \partial S_\delta(t) \to \partial S(t)$, defined as

$$T_t^{1,\delta}(\mathbf{x}) = A(t, \mathbf{p}_{\partial S_0}(A_\delta^{-1}(t,\mathbf{x}))) \quad \text{for } \mathbf{x} \in \partial S_\delta(t),$$

where $\mathbf{p}_{\partial S_0}$ is defined in Lemma 4.2. Due to (3.3.4) and (3.3.5) the operators $T_t^{1,\delta}$, $\left(T_t^{1,\delta}\right)^{-1}$ are Lipschitz one with the Lipschitz constants, independent of

$\delta > 0$ and $t \in [0,T]$. These operators converge point-wisely to the identity operators, when $\delta \to 0$. Hence Lemma 4.2 and (3.3.4) imply

$$\begin{aligned} \mathbf{m}_\delta \circ \left(T_t^{1,\delta}\right)^{-1}, \ \mathbf{n}_\delta \circ \left(T_t^{1,\delta}\right)^{-1} \quad &\to \quad -\mathbf{n}, \ \mathbf{n} \quad \text{ in } C([0,T] \times \partial S(t)), \\ \mathbf{u}_{\delta,s} \circ \left(T_t^{1,\delta}\right)^{-1} \quad &\rightharpoonup \quad \mathbf{u}_s \quad \text{ weakly-* in } L^\infty(0,T; L^\infty(\partial S(t))), \end{aligned}$$

where $\mathbf{n}$ is the interior normal to $\partial S(t)$ ($\mathbf{n}$ is directed inside of $S(t)$) and $\mathbf{u}_s$ is defined in (3.3.15). Therefore

$$\int_0^T J_\delta^1(t)\ dt \to -\int_0^T \int_{\partial S(t)} (D_1(\mathbf{n}, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)\mathbf{n}) \cdot \mathbf{u}_s \, d\mathbf{x}dt. \tag{3.3.39}$$

By the similar way as above the integral J_δ^2 can be rewritten as an integral on $\partial S(t)$, using the change of variables $T_t^{2,\delta} : \partial \widetilde{S}_\delta(t) \to \partial S(t)$, defined as

$$T_t^{2,\delta}(\mathbf{x}) = A(t, \boldsymbol{\eta}_\delta^{-1}(t,\mathbf{x})) \quad \text{for } \mathbf{x} \in \partial \widetilde{S}_\delta(t).$$

The operators $T_t^{2,\delta}$, $\left(T_t^{2,\delta}\right)^{-1}$ are Lipschitz one with the Lipschitz constants, independent of $\delta > 0$ and $t \in [0,T]$. These operators converge pointwisely to the identity operators for $\delta \to 0$. We have

$$\mathbf{m}_\delta \circ \left(T_t^{2,\delta}\right)^{-1}, \ \mathbf{n}_\delta \circ \left(T_t^{2,\delta}\right)^{-1} \to -\mathbf{n}, \ -\mathbf{n} \qquad \text{in } C([0,T] \times \partial S(t)).$$

Applying the same approach as in Theorem 6.1, p. 438 of [6] and using (3.3.38), we obtain

$$\mathbf{u}_{\delta,f} \circ \left(T_t^{2,\delta}\right)^{-1} \rightharpoonup \mathbf{u}_f \quad \text{ weakly in } L^2(0,T; L^4(\partial S(t))),$$

where $\mathbf{u}_f$ is defined in (3.3.12). Therefore

$$\int_0^T J_\delta^2(t)\ dt \to \int_0^T \int_{\partial S(t)} (D_1(\mathbf{n}, \boldsymbol{\xi}_F - \boldsymbol{\xi}_S)\mathbf{n}) \cdot \mathbf{u}_f \, d\mathbf{x}dt. \tag{3.3.40}$$

Combining (3.3.39) and (3.3.40) with the help of Lemma 4.5, we obtain

$$\int_0^T \int_{S(\chi_\delta(t))} \mathbb{D}\mathbf{u}_\delta : D_1 \, d\mathbf{x}dt \to \int_0^T \int_{\partial S(t)} (\mathbf{u}_f - \mathbf{u}_s) \cdot (\boldsymbol{\psi}_f - \boldsymbol{\psi}_s) \, d\mathbf{x}dt,$$

that, jointly with (3.3.30), (3.3.32)-(3.3.34), implies the result of this lemma. ■

Corollary 3.1 *There exists a suitable subsequence of* $\{\mathbf{u}_\delta\}_{\delta>0}$, *such that*

$$\liminf_{\delta\to 0} \int_{\Omega_T} \mu_\delta \mathbb{D}\mathbf{u}_\delta : \mathbb{D}\mathbf{u}_\delta \, dtd\mathbf{x} \geqslant \int_0^T \left\{ \int_{\Omega\setminus\partial S(t)} 2\mu_f \left|\mathbb{D}\,\mathbf{u}\right|^2 \, d\mathbf{x} \right.$$
$$\left. + \int_{\partial S(t)} \gamma |\mathbf{u}_f - \mathbf{u}_s|^2 \, d\mathbf{x} \right\} dt. \tag{3.3.41}$$

Proof. Since $\mathbf{u}$ satisfies (3.3.23), then due to Lemma 3.2 there exists an approximate sequence $\{\boldsymbol{\psi}_\delta = \widetilde{\mathbf{u}}_\delta\}_{\delta>0}$ to $\boldsymbol{\psi} = \mathbf{u}.$ Applying Lemma 3.3 to $\mathbf{u}_\delta$, $\widetilde{\mathbf{u}}_\delta$ and $\boldsymbol{\psi}_\delta = \widetilde{\mathbf{u}}_\delta$ with the help of $\int_{\Omega_T} \mu_\delta |\mathbb{D}(\mathbf{u}_\delta - \widetilde{\mathbf{u}}_\delta)|^2 \, dtd\mathbf{x} \geqslant 0$, we obtain (3.3.41). ■

As a consequence of (3.3.41) and the energy inequality (3.2.15), we obtain that the limit pair $\{\rho, \mathbf{u}\}$ satisfies

$$\frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2(r) \, d\mathbf{x} + \int_0^r \left\{ \int_{\Omega\setminus\partial S(t)} 2\mu_f \left|\mathbb{D}\,\mathbf{u}\right|^2 \, d\mathbf{x} + \int_{\partial S(t)} \gamma |\mathbf{u}_f - \mathbf{u}_s|^2 \, d\mathbf{x} \right\} dt$$
$$\leqslant \frac{1}{2}\int_\Omega \rho_0 |\mathbf{u}_0|^2 \, d\mathbf{x} + \int_0^r \int_\Omega \rho \mathbf{g}\mathbf{u} \, dtd\mathbf{x} \tag{3.3.42}$$

for a.a. $r \in (0, T)$.

3.3.3 Convergence of the convective term

In this subsection we show the convergence result for the convective term of (3.2.13).

Lemma 3.4 *Let the function* $\boldsymbol{\psi}$ *and the sequence of approximate functions* $\{\boldsymbol{\psi}_\delta\}_{\delta>0}$ *be as in Lemma 3.2. Let us assume additionally that*

$$\boldsymbol{\psi} \in L^{2(N-1)}(0, T; KB(S(t))). \tag{3.3.43}$$

Then for $\delta \to 0$ *we have*

$$\int_{\Omega_T} \rho_\delta \mathbf{u}_\delta (\overline{\mathbf{u}_\delta} \cdot \nabla)\boldsymbol{\psi}_\delta \, dtd\mathbf{x} \to \int_0^T \left\{ \int_{\Omega\setminus\partial S(t)} \rho\mathbf{u}(\mathbf{u}\cdot\nabla)\boldsymbol{\psi} \, d\mathbf{x} \right\} dt + o(1), \tag{3.3.44}$$

where $o(1) \to 0$ *as* $\tau \to 0$.

Proof. It is easy to check that if $\boldsymbol{\psi} \in L^{2(N-1)}(0, T; KB(S(t)))$, then $\boldsymbol{\psi}_\delta \in L^{2(N-1)}(0, T; K(S_\delta(t)))$ by the construction.

Due to (3.3.7), since $\tau > 0$ is fixed, applying Lemma 4.6, there exist a positive $\delta_0 = \delta_0(\tau)$ and an integer $M = M(\tau)$, such that

$$S(t) \subset]\widetilde{S}_\delta(t)[_{\tau/4}, \quad \widetilde{S}_\delta(t) \subset]S(t)[_{\tau/4} \quad \forall t \in [0,T], \quad \forall \delta < \delta_0,$$

and the partition $0 < s < 2s < ... < Ms = T$ satisfies

$$S(t) \subset]S(ms)[_{\tau/4}, \quad S(ms) \subset]S(t)[_{\tau/4}, \qquad \forall t \in I_m = [ms, (m+1)s]$$

for any $m = 0, ..., M-1$. As a consequence we have

$$\widetilde{S}_\delta(t) \subset]S(t)[_{\tau/4} \subset]S(ms)[_{\tau/2} \subset]\widetilde{S}_\delta(t)[_\tau, \quad \forall t \in I_m, \quad \forall m. \tag{3.3.45}$$

Let $F_{m,\tau} = \Omega \backslash]S(ms)[_{\tau/2}$. We have $\Omega \subset F_{m,\tau} \cup]S(ms)[_{3\tau/4}$, then there exist two functions $\phi_m^f(\mathbf{x}) \in C_0^\infty(F_{m,\tau})$, $\phi_m^s \in C_0^\infty(]S(ms)[_{3\tau/4})$ for each $m = 0, \dots, M-1$, which form a partition of the unity in Ω :

$i)$ $0 \leqslant \phi_m^f(\mathbf{x}),\ \phi_m^s(\mathbf{x}) \leqslant 1, \quad \forall\, \mathbf{x} \in \mathbb{R}^N; \qquad ii)$ $\phi_m^f(\mathbf{x}) + \phi_m^s(\mathbf{x}) = 1, \quad \forall \mathbf{x} \in \Omega.$

The set $\cup_{m=0}^{M-1}]I_m[_{\tau/8}$ is an open cover of the time interval $[0,T]$, therefore there exist functions $\varphi_m \in C_0^\infty(]I_m[_{\tau/8})$, $m = 0, \dots, M-1$, which form a partition of the unity in $[0,T]$:

$$i)\ \ 0 \leqslant \varphi_m(t) \leqslant 1, \quad \forall\, t \in \mathbb{R}; \qquad ii)\ \ \sum_{m=0}^{M-1} \varphi_m(t) = 1, \quad \forall\, t \in [0,T].$$

Let ξ (or $\boldsymbol{\xi}$) be the stream function of $\boldsymbol{\psi}$, if $N = 2$ (or if $N = 3$, respectively), which is introduced in Lemma 3.2. We denote by $\boldsymbol{\psi}_m^f = \varphi_m \mathbf{curl}(\phi_m^f \xi)$ and $\boldsymbol{\psi}_m^s = \varphi_m \mathbf{curl}(\phi_m^s \xi)$ (or $\boldsymbol{\psi}_m^f = \varphi_m \mathbf{curl}(\phi_m^f \boldsymbol{\xi})$ and $\boldsymbol{\psi}_m^s = \varphi_m \mathbf{curl}(\phi_m^s \boldsymbol{\xi})$, respectively).

$$\boldsymbol{\psi} = \boldsymbol{\psi}^f + \boldsymbol{\psi}^s \quad \text{with} \quad \boldsymbol{\psi}^f = \sum_{m=0}^{M-1} \boldsymbol{\psi}_m^f, \quad \boldsymbol{\psi}^s = \sum_{m=0}^{M-1} \boldsymbol{\psi}_m^s \quad \text{in } \Omega_T.$$

Using the stream function (3.3.28) if $N = 2$ (or (3.3.29), if $N = 3$) and the same partitions ϕ_m^f, ϕ_m^s, φ_m, defined above, we construct the sequence of functions $\boldsymbol{\psi}_{\delta,m}^f$, $\boldsymbol{\psi}_{\delta,m}^s$ and

$$\boldsymbol{\psi}_\delta = \boldsymbol{\psi}_\delta^f + \boldsymbol{\psi}_\delta^s \quad \text{with} \quad \boldsymbol{\psi}_\delta^f = \sum_{m=0}^{M-1} \boldsymbol{\psi}_{\delta,m}^f, \quad \boldsymbol{\psi}_\delta^s = \sum_{m=0}^{M-1} \boldsymbol{\psi}_{\delta,m}^s \quad \text{in } \Omega_T.$$

By (3.3.21) we see that the quantities

$$\mathbf{U}_\delta = \rho_\delta \mathbf{u}_\delta, \quad \mathbb{T}_\delta = 2\mu_f \mathbb{D}\mathbf{u}_\delta - \rho_\delta \overline{\mathbf{u}_\delta} \mathbf{u}_\delta^T$$

in the sub-domain $I \times B =]I_m[_{\tau/8} \times F_{m,\tau}$ satisfy the hypotheses of Lemma 4.7 with $q = \frac{4}{3}$. Hence we can apply Lemma 4.8, pass to a suitable subsequence if necessary, and derive the following convergence

$$\int\limits_{\Omega_T} (\mathbf{U}_\delta \otimes \mathbf{U}_\delta) : \mathbb{D}\boldsymbol{\psi}_{\delta,m}^f \, dtd\mathbf{x} \to \int\limits_{\Omega_T} (\mathbf{U} \otimes \mathbf{U}) : \mathbb{D}\boldsymbol{\psi}_m^f \, dtd\mathbf{x} \quad \text{as } \delta \to 0, \quad \forall m, \tag{3.3.46}$$

with $\mathbf{U} = \rho\mathbf{u}$. Here we use $\boldsymbol{\psi}_\delta \equiv \boldsymbol{\psi}$ in $\Omega \backslash S(\chi_\delta(t))$ by the construction of $\boldsymbol{\psi}_\delta$ in Lemma 3.2.

Let us define the cut-off function $\phi_L(t, \mathbf{x}) = \min \ \{|\phi(t, \mathbf{x})|, L\}$ for $L > 0$. If we denote $g_\delta = \rho_\delta \mathbf{u}_\delta((\overline{\mathbf{u}_\delta} - \mathbf{u}_\delta) \cdot \nabla)\, \boldsymbol{\psi}_\delta^f$, then

$$|\int\limits_{\Omega_T} g_\delta \, dtd\mathbf{x}| \ \ \leqslant \ \ L \int_0^T ||\mathbf{u}_\delta||_{L^2(B)} ||\overline{\mathbf{u}_\delta} - \mathbf{u}_\delta||_{L^2(B)} dt$$

$$+C \int_0^T ||\mathbf{u}_\delta||^2_{L^4(B)} ||\phi - \phi_L||_{L^2(B)} dt \quad \text{with } \phi = |\nabla \boldsymbol{\psi}|.$$

From (3.3.21), (3.3.22), (3.3.43) and (3.3.17) with $q = 4$, $\mathbf{z} = \mathbf{u}_\delta$ for $B = S(\theta_\delta(t))$, we have

$$|\int\limits_{\Omega_T} g_\delta \, dtd\mathbf{x}| \leqslant CL\tau^\alpha + C||\phi - \phi_L||_{L^{2(N-1)}(0,T;L^2(B))}.$$

Choosing $L = \tau^{-\alpha/2}$, we derive

$$|\int\limits_{\Omega_T} g_\delta \, dtd\mathbf{x}| \leqslant o(1) \to 0 \quad \text{as } \tau \to 0 \quad (\text{independently of } \delta). \tag{3.3.47}$$

Hence (3.3.46) and (3.3.47) imply

$$\int\limits_{\Omega_T} \rho_\delta \mathbf{u}_\delta \, (\overline{\mathbf{u}_\delta} \cdot \nabla)\, \boldsymbol{\psi}_\delta^f \, dtd\mathbf{x} \to \int\limits_{\Omega_T} \rho(\mathbf{u} \otimes \mathbf{u}) : \mathbb{D}\boldsymbol{\psi}^f \, dtd\mathbf{x} + o(1). \tag{3.3.48}$$

Since $\mathbb{D}\boldsymbol{\psi}_\delta = 0$ in $S_\delta(t)$, $\mathbb{D}\boldsymbol{\psi} = 0$ in $S(t)$ and $\overline{\mathbf{u}_\delta} = \mathbf{u}_\delta = \mathbf{q}_\delta'(t) + \boldsymbol{\omega}_\delta(t) \times (\mathbf{x} - \mathbf{q}_\delta(t))$ in $[S_\delta(t)]_\tau$, we have

$$\int\limits_0^T \int\limits_{[S_\delta(t)]_\tau} \mathbf{u}_\delta \, (\overline{\mathbf{u}_\delta} \cdot \nabla)\, \boldsymbol{\psi}_\delta \, d\mathbf{x}dt = 0 = \int\limits_0^T \int\limits_{S(t)} \mathbf{u}(\mathbf{u} \cdot \nabla)\boldsymbol{\psi} \, d\mathbf{x}dt.$$

Using (3.3.2), (3.3.45) and the boundedness of $\{\mathbf{u}_\delta\}_{\delta > 0}$ in $L^\infty(0,T; L^\infty(S_\delta(t))) \cap L^2(0,T; L^r(S(\theta_\delta(t))))$ (see (3.2.5), (3.3.4), (3.3.21)), we obtain

$$\lim_{\delta \to 0} \int\limits_{\Omega_T} \rho_\delta \mathbf{u}_\delta \, (\overline{\mathbf{u}_\delta} \cdot \nabla)\, \boldsymbol{\psi}_\delta^s \, dtd\mathbf{x} = \int\limits_{\Omega_T} \rho\mathbf{u} \, (\mathbf{u} \cdot \nabla)\boldsymbol{\psi}^s \, dtd\mathbf{x} + O(\tau^\alpha). \tag{3.3.49}$$

From (3.3.48) and (3.3.49), we derive (3.3.44). ■

Finally, taking the limit transition $\delta \to 0$ in system (3.1.1), (3.2.12), (3.2.13) and combining the convergence results (3.3.5), (3.3.7), (3.3.8) and Lemmas 3.3, 3.4, we show the following result.

Proposition 3.3 *Under the assumptions of Theorem 2.1 for the fixed* $\tau > 0$ *there exists the solution*

$$\begin{aligned} \rho &\in L^{\infty}((0,T)\times\mathbb{R}^N), \qquad \varphi \in L^{\infty}(0,T;Char(\mathbb{R}^N)),\\ \mathbf{u} &\in C_{\rm weak}(0,T;V^{0,2}(\Omega))\cap L^2(0,T;KB(S(t))), \end{aligned}$$

of the system

$$\begin{aligned} \partial_t\rho + {\rm div}(\rho\overline{\mathbf{u}}) &= 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^N), \quad \rho(0,\cdot)=\rho_0 \quad \text{in } \mathbb{R}^N,\\ \partial_t\varphi + {\rm div}(\varphi\overline{\mathbf{u}}) &= 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^N), \quad \varphi(0,\cdot)=\varphi_0 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

$$\begin{aligned} &\int_0^T\int_{\Omega\setminus\partial S(t)}\{\rho\mathbf{u}\partial_t\boldsymbol{\psi} + \rho\mathbf{u}\,(\mathbf{u}\cdot\nabla)\,\boldsymbol{\psi} - \mu_f\mathbb{D}\mathbf{u}:\mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}\}\,d\mathbf{x}dt + o(1)\\ &= -\int_\Omega \rho_0\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\;d\mathbf{x} + \int_0^T\int_{\partial S(t)}\gamma(\mathbf{u}_f-\mathbf{u}_s)(\boldsymbol{\psi}_f-\boldsymbol{\psi}_s)\,d\mathbf{x}dt \qquad (3.3.50) \end{aligned}$$

which holds for any test function $\boldsymbol{\psi}$, *such that*

$$\begin{aligned} \boldsymbol{\psi} &\in L^{2(N-1)}(0,T;KB(S(t))),\\ \boldsymbol{\psi}_t &\in L^2(0,T;L^2(\Omega\setminus\partial S(t))), \quad \boldsymbol{\psi}(T,\cdot)=0. \qquad (3.3.51) \end{aligned}$$

Here $o(1)\to 0$ *as* $\tau\to 0$. *The solution of this system satisfies (3.3.42) and* $\rho,\varphi\in C(0,T;\,L^p_{loc}(\mathbb{R}^N))$, $\forall p\in[1,\infty)$, *such that for any* $r\in[0,T]$

$$||\rho(r,\cdot)||_{L^p(\mathbb{R}^N)} = ||\rho_0||_{L^p(\mathbb{R}^N)}, \quad ||\varphi(r,\cdot)||_{L^p(\mathbb{R}^N)} = ||\varphi_0||_{L^p(\mathbb{R}^N)}.$$

Here φ_0 *is the characteristic functions of* S_0, *defined in the whole space* R^N.

Moreover there exist a preserving orientation isometry $A(t,\cdot):\mathbb{R}^N\to\mathbb{R}^N$, *which is defined by (3.3.5) and related with* $\mathbf{u}$ *on the set* $S(t)=A(t,S_0)$ *through formulas (3.3.15). The matrix* $\mathbb{Q}=\mathbb{Q}(t)$, *the vector* $\boldsymbol{\omega}=\boldsymbol{\omega}(t)$ *and a skew-symmetric matrix* $\mathbb{P}=\mathbb{P}(t)$ *satisfy relationships (1.4), (1.5). The functions* $\rho(r,\cdot)$, $\varphi(t,\cdot)$ *and the set* $S(t)$ *are related by (3.3.9) for all* $t\in[0,T]$.

3.4 Limit transition on $\tau\to 0$

In this subsection the solution of problem (3.3.50) is denoted by ρ_τ, φ_τ, $\mathbf{u}_\tau$ and $S_\tau(t)\equiv S(\varphi_\tau(t)) = A_\tau(t,S_0)$ for any $\tau>0$. From Proposition 3.3, following the same stroke as in the subsection 3.3, we derive for $\{\mathbf{q}_\tau,\mathbb{Q}_\tau,\boldsymbol{\omega}_\tau\}_{\tau>0}$ a similar estimate as (3.3.4), independent of τ. Therefore there exists a suitable subsequence of $\{\mathbf{q}_\tau,\mathbb{Q}_\tau,\boldsymbol{\omega}_\tau\}_{\tau>0}$, such that

$$\begin{aligned} \boldsymbol{\omega}_\tau &\rightharpoonup \boldsymbol{\omega} \qquad \text{*-weakly in } L^{\infty}(0,T),\\ \mathbf{q}_\tau,\mathbb{Q}_\tau &\rightharpoonup \mathbf{q},\mathbb{Q} \qquad \text{*-weakly in } W^{1,\infty}(0,T) \quad \text{and} \quad \mathbb{Q}^T\mathbb{Q}=\mathbb{I},\\ \mathbf{q}_\tau,\mathbb{Q}_\tau &\to \mathbf{q},\mathbb{Q} \qquad \text{in } C^\alpha(0,T), \qquad \forall\alpha\in[0,1),\\ A_\tau &\to A=\mathbf{q}(t)+\mathbb{Q}(t)(\mathbf{y}-\mathbf{q}(0)) \quad \text{in } C^\alpha(0,T;C^2_{loc}(\mathbb{R}^N)).(3.4.1) \end{aligned}$$

that, using (3.3.9), implies

$$\begin{aligned} \rho_\tau,\ \varphi_\tau &\rightarrow \rho,\ \varphi \quad \text{in } C(0,T;\ L^p_{loc}(\mathbb{R}^N)), \quad \forall p \in [1,\infty), \\ S(t) &\equiv S(\varphi(t)) = A(t,S_0) \quad \text{for all } \ t \in [0,T] \end{aligned}$$

and (2.6). Moreover, applying (3.3.7) and (3.3.42), we have that

$$\begin{aligned} \mathbf{u}_\tau, \overline{\mathbf{u}_\tau}^\tau &\rightharpoonup \mathbf{u} \qquad \text{weakly in } \ L^2(0,T;L^q(\Omega)), \\ (1-\varphi_\tau)\mathbb{D}\mathbf{u}_\tau, \varphi_\tau\mathbb{D}\mathbf{u}_\tau &\rightharpoonup (1-\varphi)\mathbb{D}\mathbf{u}, \varphi\mathbb{D}\mathbf{u} \quad \text{weakly in } \ L^2(\Omega_T), \\ \sqrt{\rho_\tau}\mathbf{u}_\tau &\rightharpoonup \sqrt{\rho}\mathbf{u} \qquad \text{*-weakly in } \ L^\infty(0,T;L^2(\Omega)) \end{aligned}$$

with q defined in (3.3.16).

Lemma 3.5 *For any given* $\boldsymbol{\psi}$*, satisfying* (2.4)*, there exist a sequence of functions*

$$\begin{aligned} \boldsymbol{\psi}_\tau &\in L^{2(N-1)}(0,T;KB(S_\tau(t))), \\ \partial_t\boldsymbol{\psi}_\tau &\in L^2(0,T;L^2(\Omega\backslash\partial S_\tau(t))), \quad \boldsymbol{\psi}_\tau(T,\cdot) = 0, \end{aligned} \tag{3.4.2}$$

such that

$$\begin{aligned} \boldsymbol{\psi}_\tau,\ \ \varphi_\tau\nabla_{\mathbf{x}}\boldsymbol{\psi}_\tau &\rightarrow \boldsymbol{\psi},\ \ \varphi\nabla_{\mathbf{x}}\boldsymbol{\psi} \\ (1-\varphi_\tau)\nabla_{\mathbf{x}}\boldsymbol{\psi}_\tau &\rightarrow (1-\varphi)\nabla_{\mathbf{x}}\boldsymbol{\psi} \quad \textit{in } \ L^{2(N-1)}(0,T;L^2(\Omega)), \\ (1-\varphi_\tau)\partial_t\boldsymbol{\psi}_\tau,\ \ \varphi_\tau\partial_t\boldsymbol{\psi}_\tau &\rightarrow (1-\varphi)\partial_t\boldsymbol{\psi},\ \ \varphi\partial_t\boldsymbol{\psi} \quad \textit{in } L^2(\Omega_T). \end{aligned} \tag{3.4.3}$$

Proof. We can consider that the function $\boldsymbol{\psi}(t,)$, $t \in (0,T)$, is extended by zero outside Ω. Let $\boldsymbol{\xi}$ be a stream function of $\boldsymbol{\psi}$ in $(0,T)\times\mathbb{R}^N$. We introduce $\boldsymbol{\psi}_{1,\tau} = \mathbf{curl}(\boldsymbol{\xi}\circ A_\tau \circ A^{-1})$.

Note that any function $\mathbf{z} \in LD^2(\Omega\backslash S_\tau(t))$ with zero values on $\partial S_\tau(t)$ can be extended by zero inside $S_\tau(t)$, i.e. the embedding inequalities (3.3.16) are true for this extended $\mathbf{z}$ in $B = \Omega$. Therefore the existence of the solution $\boldsymbol{\psi}_{2,\tau}$ of the problem

$$\begin{cases} -div(\mathbb{D}\boldsymbol{\psi}_{2,\tau}) + \nabla p_\tau = 0, \quad div\boldsymbol{\psi}_{2,\tau} = 0 \quad \text{in } \Omega\backslash S_\tau(t), \\ \boldsymbol{\psi}_{2,\tau} = 0 \quad \text{on } \partial S_\tau(t), \quad \boldsymbol{\psi}_{2,\tau} = -\boldsymbol{\psi}_{1,\tau}(t,\cdot) \quad \text{on } \partial\Omega, \end{cases} \qquad t \in (0,T)$$

follows from the Lax-Milgram theorem. Due to the regularity of A, A_τ and $\boldsymbol{\psi}$, we show that $\boldsymbol{\psi}_{2,\tau} \in L^{2(N-1)}(0,T;LD^2(\Omega\backslash S_\tau(t))) \cap W^{1,2}(0,T;L^2(\Omega\backslash S_\tau(t)))$, such that

$$\begin{aligned} ||\boldsymbol{\psi}_{2,\tau}||_{L^{2(N-1)}(0,T;LD^2(\Omega\backslash S_\tau(t)))} &\leqslant C||\boldsymbol{\psi}_{1,\tau}||_{L^{2(N-1)}(0,T;W^{1/2,2}(\partial B))} \rightarrow 0, \\ ||\boldsymbol{\psi}_{2,\tau}||_{W^{1,2}(0,T;L^2(\Omega))} &\leqslant C||\boldsymbol{\psi}_{1,\tau}||_{W^{1,2}(0,T;W^{-1/2,2}(\partial\Omega))} \rightarrow 0 \end{aligned}$$

as $\tau \rightarrow 0$. The constants C are independent of τ.

Let us put $\boldsymbol{\psi}_{2,\tau}(t,\cdot) = 0$ in $S_\tau(t)$, a.e. $t \in (0,T)$. Finally, we define $\boldsymbol{\psi}_\tau = \boldsymbol{\psi}_{1,\tau} + \boldsymbol{\psi}_{2,\tau}$ in Ω_T, which satisfies (3.4.2)-(3.4.3). ■

Now, using the constructed sequence of test functions $\{\boldsymbol{\psi}_\tau\}_{\tau<\tau_0}$ in Lemma 3.5, the rest of the convergence proof as $\tau \rightarrow 0$ can be done repeating step by step the arguments of the preceding subsection 3.3.3, using (3.3.42) for ρ_τ, φ_τ, $\mathbf{u}_\tau$ and we show Theorem 2.1.

4 Appendix. Technical results

4.1 Justification of Definition 2.1

In the following result we present the justification of Definition 2.1 for the weak solution of (1.6)-(1.9).

Lemma 4.1 *The triple* $\{S, \rho, \mathbf{u}\}$ *is the solution of system* (1.6)-(1.9)*, if and only if, the triple* $\{A, \rho, \mathbf{u}\}$ *is the weak solution of* (1.6)-(1.9) *in the sense of Definition 2.1.*

Proof. To show this lemma we use the following well-known result from the fluid mechanics theory: *Let* $V(t)$ *be a time dependent volume moved by a smooth velocity* $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$*. Then*

$$\frac{d}{dt}\int_{V(t)} f(t,\mathbf{x})\, d\mathbf{x} = \int_{V(t)} \frac{df}{dt}\, d\mathbf{x} \tag{4.1.1}$$

for any smooth function $f = f(t, \mathbf{x})$*. Here* $\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla)f$ *is the total time derivative.*

Here we consider that $\mathbf{u}$ is "smooth" in the sense that $\mathbf{u}$ is a C^1-differentiable function over Ω_T, being discontinuous across the smooth surface $\partial S(t)$.

$\Longrightarrow$ Let $\{S, \rho, \mathbf{u}\}$ be the solution of system (1.6)-(1.9). The isometry A is defined by (1.1).

Let ξ, $\boldsymbol{\psi}$ be test functions, defined in the definition 2.1. Identity (2.2) is a direct consequence of formula (4.1.1), applied to the function $f = \rho\xi$. Now if we apply formulae (4.1.1) to the function $\rho\mathbf{u}$ in the volumes $S(t)$ and $F(t)$, respectively, we derive

$$\begin{aligned} -\int_{S_0} \rho_s \mathbf{u}_0 \boldsymbol{\psi}(0,\cdot)\, d\mathbf{x} = \int_0^T \int_{S(t)} & \rho_s\{\boldsymbol{\psi}(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) \\ & + (\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi})\mathbf{u}\}\, d\mathbf{x}dt, \end{aligned} \tag{4.1.2}$$

$$\begin{aligned} -\int_{F_0} \rho_f \mathbf{u}_0 \boldsymbol{\psi}(0,\cdot)\, d\mathbf{x} = \int_0^T \int_{F(t)} & \rho_f\{\boldsymbol{\psi}(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) \\ & + (\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi})\mathbf{u}\}\, d\mathbf{x}dt. \end{aligned} \tag{4.1.3}$$

For the stress tensor $P = P(\mathbf{u}, p) = -pI + 2\mu\,\mathbb{D}\mathbf{u}$ we have the identity

$$-\int_{\partial S(t)} (P\mathbf{n}\cdot\boldsymbol{\psi})_f\, d\mathbf{x} = -\int_{F(t)} \{2\mu\,\mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \mathrm{div}P\cdot\boldsymbol{\psi}\}\, d\mathbf{x} \tag{4.1.4}$$

The sum of (4.1.2), (4.1.3) and (4.1.4) gives

$$
\begin{aligned}
&- \int_0^T \int_{\partial S(t)} (P\mathbf{n}\cdot\boldsymbol{\psi})_f \, d\mathbf{x} - \int_\Omega \rho_0 \mathbf{u}_0 \boldsymbol{\psi}(0,\cdot)\, d\mathbf{x} \\
&= \int_0^T \int_{S(t)} \rho\boldsymbol{\psi}(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \rho\mathbf{u}(\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi})\, d\mathbf{x}dt \\
&\quad + \int_0^T \int_{F(t)} \boldsymbol{\psi}\{\rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mathrm{div}P\}\, d\mathbf{x}dt \\
&\quad + \int_0^T \int_{F(t)} \rho\mathbf{u}(\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi}) - 2\mu\, \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi}\, d\mathbf{x}dt. \qquad (4.1.5)
\end{aligned}
$$

Observing that

$$
\mathbf{u} = \mathbf{a}(t) + \boldsymbol{\omega}(t)\times(\mathbf{x}-\mathbf{q}(t)), \qquad \boldsymbol{\psi} = \mathbf{b}(t) + \boldsymbol{\varpi}(t)\times(\mathbf{x}-\mathbf{q}(t)) \quad \text{on } S(t) \quad (4.1.6)
$$

with $\mathbf{b}, \boldsymbol{\varpi}$ being arbitrary function of t, it is easy to show that (1.6) can be written as the integral identity

$$
\begin{aligned}
&\int_0^T \int_{S(t)} \rho_s \boldsymbol{\psi}(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u})\, d\mathbf{x}dt \\
&= - \int_0^T \int_{\partial S(t)} P_f \mathbf{n}\cdot\boldsymbol{\psi}_s\, d\mathbf{x}dt + \int_0^T \int_{S(t)} \rho_s \mathbf{g}\boldsymbol{\psi}\, d\mathbf{x}dt. \qquad (4.1.7)
\end{aligned}
$$

Therefore from (1.7), (4.1.5) and (4.1.7) we deduce the identity

$$
\begin{aligned}
&- \int_0^T \int_{\partial S(t)} P_f \mathbf{n}\cdot\boldsymbol{\psi}_f\, d\mathbf{x}dt - \int_\Omega \rho_0\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\, d\mathbf{x} = - \int_0^T \int_{\partial S(t)} P_f \mathbf{n}\cdot\boldsymbol{\psi}_s\, d\mathbf{x}dt \\
&\quad + \int_0^T \int_{\Omega\setminus\partial S(t)} \rho\mathbf{u}(\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi}) - 2\mu\, \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \rho\mathbf{g}\boldsymbol{\psi}\, d\mathbf{x}dt.
\end{aligned}
$$

By (1.9) we derive that A, ρ, $\mathbf{u}$ satisfy (2.3).

$\Longleftarrow$ Now we prove the inverse result. Let us choose in (2.3) the test function $\boldsymbol{\psi}$, such that $\boldsymbol{\psi} = 0$ on $S(t)$. Since

$$
\mathrm{div}\boldsymbol{\psi} = 0, \qquad \mathrm{div}\boldsymbol{u} = 0 \qquad \text{in } \mathcal{D}'(\Omega),
$$

then

$$
\boldsymbol{\psi}_f\cdot\mathbf{n} = \boldsymbol{\psi}_s\cdot\mathbf{n} = 0, \qquad \boldsymbol{u}_f\cdot\mathbf{n} = \boldsymbol{u}_s\cdot\mathbf{n} \qquad \text{on } \partial S(t).
$$

Therefore identity (2.3) can be written as

$$
\begin{aligned}
&\int_0^T \int_{F(t)} \rho_f \mathbf{u}(\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\psi}) - 2\mu\, \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} + \rho_f\mathbf{g}\boldsymbol{\psi}\, d\mathbf{x}dt \\
&= \int_0^T \int_{\partial S(t)} \gamma(\mathbf{u}_f - \mathbf{u}_s)\cdot\boldsymbol{\tau}\ \boldsymbol{\psi}_f\cdot\boldsymbol{\tau}\, d\mathbf{x}dt - \int_{F_0} \rho_f\mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\, d\mathbf{x}dt.
\end{aligned}
$$

Due to (4.1.3), (4.1.4) we derive equations (1.7) and the boundary conditions (1.9).

Returning back to (2.3) and using (1.7), (1.9), (4.1.5), we derive the integral identity (4.1.7). Finally choosing $\boldsymbol{\psi}$ in (4.1.6)-(4.1.7) at first with $\boldsymbol{\varpi}(t) = 0$ and then with $\mathbf{b} = 0$, we obtain equations (1.6). ■

4.2 Properties of smooth surfaces

Let us recall useful result, related with the properties of C^2-smooth surfaces.

Lemma 4.2 *(p. 354-357 of [13]) There exists a small $\tau_0 > 0$, depending only on the curvature of ∂S, such that*

$$d_S \in C^2(U_{\tau_0}(\partial S)), \qquad \textit{where } \; U_{\tau_0}(\partial S) = d_S^{-1}((-\tau_0, \tau_0)). \tag{4.2.1}$$

For any $\mathbf{x} \in U_{\tau_0}(\partial S)$ there exists a unique nearest point $\mathbf{p}_{\partial S}(\mathbf{x}) \in \partial S$, such that

$$|\mathbf{p}_{\partial S}(\mathbf{x}) - \mathbf{x}| = \mathrm{dist}[\mathbf{x}, S], \qquad \mathbf{p}_{\partial S}(\mathbf{x}) \in C^1(U_{\tau_0}(\partial S)).$$

Further the mapping

$$\mathbf{x} \longmapsto \begin{pmatrix} \mathbf{p}_{\partial S}(\mathbf{x}) \\ d_S(\mathbf{x}) \end{pmatrix} : \quad U_{\tau_0}(\partial S) \to \partial S \times (-\tau_0, \tau_0)$$

is C^1-diffeomorphism with the inverse

$$\begin{pmatrix} \mathbf{y} \\ \xi \end{pmatrix} \longmapsto \mathbf{y} + \xi \mathbf{n}(\mathbf{y}) : \quad \partial S \times (-\tau_0, \tau_0) \to U_{\tau_0}(\partial S),$$

where the function $\mathbf{n}(\mathbf{y}) = \bigtriangledown d_S(\mathbf{y}) \in C^1(\partial S)$ is the unit interior normal to ∂S.

The following Lemmas 4.3, 4.4 and 4.3 are related with the approximation of the "jump" term on $S(t)$ in (2.3) of Definition 2.1 by the third term in the viscosity μ_ε, introduced in (3.1.4).

Lemma 4.3 *Let $S \subset \mathbb{R}^N$ be an open simply-connected set, having C^2-smooth boundary ∂S. Let $\mathbf{n}$ be the unit interior normal to ∂S. For any two functions $\mathbf{p}, \mathbf{g} \in C(\partial S)$ with $\mathbf{p} \cdot \mathbf{n} = 0$ or $\mathbf{g} \cdot \mathbf{n} = 0$ on the boundary ∂S, we have*

$$\mathbb{T}(\mathbf{p}) : \mathbb{T}(\mathbf{g}) = \frac{1}{2} \mathbf{p} \cdot \mathbf{g}, \qquad \mathbb{T}(\mathbf{p}) \mathbf{n} \cdot \mathbf{g} = \frac{1}{2} \mathbf{p} \cdot \mathbf{g} \quad \textit{at any point of } \; \partial S. \tag{4.2.2}$$

The matrix $\mathbb{T}(\mathbf{p})$ is defined by (3.3.13).

Proof. By the definition we have

$$\begin{aligned} \mathbb{T}(\mathbf{p}) \quad &: \quad \mathbb{T}(\mathbf{g}) = \frac{1}{4} \sum_{i,j=1}^{N} (p_i n_j + p_j \, n_i) \, (g_i \, n_j + g_j \, n_i) = \\ &= \quad \frac{1}{4} \sum_{i,j=1}^{N} \left\{ (p_i g_i) \; n_j^2 + (g_i \, n_i) \, (p_j \, n_j) + (p_i \, n_i) \, (g_j \, n_j) + (p_j g_j) \; n_i^2 \right\} \\ &= \quad \frac{1}{2} \left\{ (\mathbf{p} \cdot \mathbf{g}) + (\mathbf{p} \cdot \mathbf{n}) \, (\mathbf{g} \cdot \mathbf{n}) \right\}, \end{aligned}$$

that implies the first formula of (4.2.2). By the same way we obtain the second one

$$\begin{aligned}\mathbb{T}(\mathbf{p})\mathbf{n}\cdot\mathbf{g} &= \sum_{i=1}^{N}\left\{\sum_{j=1}^{N}\frac{1}{2}\left(p_i n_j + p_j\, n_i\right) n_j\right\} g_i \\ &= \frac{1}{2}\left\{(\mathbf{p}\cdot\mathbf{g}) + (\mathbf{p}\cdot\mathbf{n})\,(\mathbf{g}\cdot\mathbf{n})\right\}. \qquad \blacksquare\end{aligned}$$

In Lemmas 4.4, 4.5 we consider that $N=3$, taking into account that the case $N=2$ is a particular one of the case $N=3$ (see the note, given in Lemma 3.3, ii)).

Lemma 4.4 *For any smooth function χ and vector-function $\boldsymbol{\xi}$, we have*

$$\mathbb{D}(\mathbf{curl}(\chi\boldsymbol{\xi})) = D_1(\nabla\chi,\boldsymbol{\xi}) + D_2(\chi,\boldsymbol{\xi}), \tag{4.2.3}$$

where the symmetric matrices D_1, D_2 are defined as

$$\begin{aligned} D_1(\nabla\chi,\boldsymbol{\xi}) &= \frac{1}{2}\left\{\boldsymbol{\psi}\otimes\nabla\chi + \nabla\chi\otimes\boldsymbol{\psi} + \nabla\chi\times S(\boldsymbol{\xi})\right\} + \nabla\chi\times\nabla\boldsymbol{\xi}, \\ D_2(\chi,\boldsymbol{\xi}) &= \chi\mathbb{D}\boldsymbol{\psi} + \frac{1}{2}\left\{\nabla(\nabla\chi)\times\boldsymbol{\xi} + [\nabla(\nabla\chi)\times\boldsymbol{\xi}]^T\right\} \\ \textit{with } \boldsymbol{\psi} &= \mathbf{curl}(\boldsymbol{\xi}) \textit{ and } S(\boldsymbol{\xi}) = [\nabla\boldsymbol{\xi}]^T - \nabla\boldsymbol{\xi}. \end{aligned} \tag{4.2.4}$$

The symbols $\otimes$ and $\times$ denote the dyadic and the cross product of vectors, respectively. In particular, the components $(\boldsymbol{u}\otimes\boldsymbol{v})_{i,j} = u_i v_j, \quad i,j=1,2,3.$

Proof. The following three identities are valid

$$\begin{cases}\mathbf{curl}(\chi\boldsymbol{\xi}) = \chi\mathbf{curl}\boldsymbol{\xi} + \nabla\chi\times\boldsymbol{\xi}, \quad \nabla(\chi\boldsymbol{\psi}) = \chi\nabla\boldsymbol{\psi} + \nabla\chi\otimes\boldsymbol{\psi} \\ \nabla(\nabla\chi\times\boldsymbol{\xi}) = \nabla(\nabla\chi)\times\boldsymbol{\xi} + \nabla\chi\times\nabla\boldsymbol{\xi},\end{cases}$$

that implies

$$\nabla(\mathbf{curl}(\chi\boldsymbol{\xi})) = \{\nabla\chi\otimes\boldsymbol{\psi} + \nabla\chi\times\nabla\boldsymbol{\xi}\} + \{\chi\nabla\boldsymbol{\psi} + \nabla(\nabla\chi)\times\boldsymbol{\xi}\},$$

$$[\nabla(\mathbf{curl}(\chi\boldsymbol{\xi}))]^T = \{\boldsymbol{\psi}\otimes\nabla\chi + [\nabla\chi\times\nabla\boldsymbol{\xi}]^T\} + \{\chi\,[\nabla\boldsymbol{\psi}]^T + [\nabla(\nabla\chi)\times\boldsymbol{\xi}]^T\}. \tag{4.2.5}$$

In these formulas we consider that $\nabla\chi$ is a column vector and

$$\boldsymbol{u}\times\nabla\boldsymbol{v} = -\nabla\boldsymbol{v}\times\boldsymbol{u}, \quad \boldsymbol{u}\times\nabla\boldsymbol{v} = \begin{pmatrix}[\boldsymbol{u}\times\partial_{x_1}\boldsymbol{v}]^T \\ [\boldsymbol{u}\times\partial_{x_2}\boldsymbol{v}]^T \\ [\boldsymbol{u}\times\partial_{x_3}\boldsymbol{v}]^T\end{pmatrix} \tag{4.2.6}$$

for smooth vectors $\boldsymbol{u}$, $\boldsymbol{v}$. Hence

$$[\nabla\chi\times\nabla\boldsymbol{\xi}]^T = \nabla\chi\times[\nabla\boldsymbol{\xi}]^T = \nabla\chi\times S(\boldsymbol{\xi}) + \nabla\chi\times\nabla\boldsymbol{\xi}. \tag{4.2.7}$$

Combining (4.2.5) and (4.2.7), we derive (4.2.3)-(4.2.4). $\blacksquare$

Lemma 4.5 *Let $S \subset \mathbb{R}^3$ be an open simply-connected set, having C^2-smooth boundary ∂S. Let $\mathbf{n}$ be the unit interior normal to ∂S. For any two functions $\boldsymbol{\xi} \in C^1(\partial S)$, $\mathbf{g} \in C(\partial S)$, such that $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ or $\mathbf{g} \cdot \mathbf{n} = 0$ on the boundary ∂S with $\boldsymbol{\psi} = \mathbf{curl}(\boldsymbol{\xi})$, we have*

$$(D_1(\mathbf{n}, \boldsymbol{\xi})\mathbf{n}) \cdot \mathbf{g} = \boldsymbol{\psi} \cdot \mathbf{g} \qquad \text{at any point of } \partial S, \tag{4.2.8}$$

where $D_1(\mathbf{n}, \boldsymbol{\xi})\mathbf{n} = \frac{1}{2}\{\boldsymbol{\psi} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\psi} + \mathbf{n} \times S(\boldsymbol{\xi})\} + \mathbf{n} \times \nabla \boldsymbol{\xi}$.

Proof. Since $\boldsymbol{\psi} \cdot \mathbf{n} = 0$, then we can check that $\mathbf{n} \times S(\boldsymbol{\xi}) = \boldsymbol{\psi} \otimes \mathbf{n}$. By the same way as in Lemma 4.3, we obtain

$$\begin{aligned}
\frac{1}{2}\left[\boldsymbol{\psi} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\psi} + \mathbf{n} \times S(\boldsymbol{\xi})\right] \mathbf{n} \cdot \mathbf{g} \quad &= \quad \sum_{i=1}^{3} \left\{ \sum_{j=1}^{3} \left[\psi_i n_j + \frac{1}{2} \psi_j \, n_i \right] n_j \right\} g_i \; = \\
&= \quad (\boldsymbol{\psi} \cdot \mathbf{g}) + \frac{1}{2} (\boldsymbol{\psi} \cdot \mathbf{n}) (\mathbf{g} \cdot \mathbf{n}) \, .
\end{aligned} \tag{4.2.9}$$

By the definition of $\boldsymbol{u} \times \nabla \boldsymbol{v}$ (see (4.2.6)), we have

$$(\mathbf{n} \times \nabla \boldsymbol{\xi}) \cdot \mathbf{n} = \begin{pmatrix} [\mathbf{n} \times \partial_{x_1} \boldsymbol{\xi}] \cdot \mathbf{n} \\ [\mathbf{n} \times \partial_{x_2} \boldsymbol{\xi}] \cdot \mathbf{n} \\ [\mathbf{n} \times \partial_{x_3} \boldsymbol{\xi}] \cdot \mathbf{n} \end{pmatrix} = \mathbf{0}, \tag{4.2.10}$$

since $[\mathbf{n} \times \partial_{x_i} \boldsymbol{\xi}] \cdot \mathbf{n} = [\mathbf{n} \times \mathbf{n}] \cdot \partial_{x_i} \boldsymbol{\xi} = 0$. Combining (4.2.9)-(4.2.10), we derive (4.2.8). ■

In fact Lemmas 4.4, 4.5 can be shown directly for the case $N = 2$ with significantly less calculations, than we present in the proof of these lemmas.

4.3 Transport equation

In this subsection let us consider open simply-connected sets S, S_n being subsets of $\mathbb{R}^N$ with the boundaries ∂S, $\partial S_n \in C^2$ for any integer index $n \geqslant 1$. In the sequel we say that a sequence of sets S_n converges to S, $S_n \xrightarrow{b} S$, in the sense of boundaries if

$$d_{S_n} \to d_S \quad \text{ in } \; C_{\mathrm{loc}}(\mathbb{R}^N).$$

Let us recall classical results of characteristic curves (see, for instance, [10, Proposition 5.1], [25, Lemma 5.2]).

Lemma 4.6 *Let $\{\mathbf{v}_n(t, \mathbf{x})\}_{n=1}^{\infty}$ be a family of divergence free vector fields, uniformly bounded in $L^2(0, T; C^2_{loc}(\mathbb{R}^N))$. Let $\boldsymbol{\eta}_n(t, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ be the solutions of the Cauchy problem*

$$\frac{\partial}{\partial t} \boldsymbol{\eta}_n(t, \mathbf{y}) = \mathbf{v}_n(t, \boldsymbol{\eta}_n(t, \mathbf{y})), \qquad \boldsymbol{\eta}_n(0, \mathbf{y}) = \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^N.$$

Then: 1) the mappings $\boldsymbol{\eta}_n(t,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ *are* C^2*-diffeomorphism, such that*

$$det\left(\frac{\partial \boldsymbol{\eta}_n(t,\mathbf{y})}{\partial \mathbf{y}}\right) = 1 \quad \forall (t,\mathbf{y}) \in [0,T] \times \mathbb{R}^N; \tag{4.3.1}$$

2) for a suitable subsequence,

$$\begin{aligned} \mathbf{v}_n &\rightharpoonup \mathbf{v} \quad \textit{weakly in } L^2(0,T; W^{2,p}_{loc}(\mathbb{R}^N)), \quad \forall p \in [1,\infty), \\ \boldsymbol{\eta}_n,\ \boldsymbol{\eta}_n^{-1} &\to \boldsymbol{\eta},\ \boldsymbol{\eta}^{-1} \quad \textit{in } C(0,T; C^1_{loc}(\mathbb{R}^N)), \end{aligned} \tag{4.3.2}$$

where $\boldsymbol{\eta}(t,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ *is* C^2*-diffeomorphism, being the unique solution of*

$$\frac{\partial}{\partial t}\boldsymbol{\eta}(t,\mathbf{y}) = \mathbf{v}(t,\boldsymbol{\eta}(t,\mathbf{y})), \qquad \boldsymbol{\eta}(0,\mathbf{y}) = \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^N$$

and satisfying (4.3.1) *too;*

3) if, in addition, $S_n \xrightarrow{b} S$*, then* $\boldsymbol{\eta}_n(t,S_n) \equiv S_n(t) \xrightarrow{b} S(t) \equiv \boldsymbol{\eta}(t,S)$*, uniformly in* $t \in [0,T]$*. In particular, for arbitrary* $\sigma > 0$ *there exists* $n_0(\sigma) > 0$*, such that for any* $n > n_0$ *we have*

$$S(t) \subset]S_n(t)[_\sigma, \qquad S_n(t) \subset]S(t)[_\sigma \quad \textit{for all } t \in [0,T];$$

4) let $\{\phi_{0,n}\}_{n=1}^\infty$ *be a sequence, uniformly bounded in* $L^\infty(\mathbb{R}^N)$*. Let* $\{\phi_n\}_{n=1}^\infty$ *be unique solutions of*

$$\partial_t \phi_n + \operatorname{div}(\phi_n \mathbf{v}_n) = 0 \quad \textit{in } \mathcal{D}'((0,T)\times\mathbb{R}^N), \quad \phi_n(0,\mathbf{x}) = \phi_{0,n}(\mathbf{x}) \quad \textit{in } \mathbb{R}^N.$$

If

$$\phi_{0,n} \to \phi_0 \quad \textit{in } L^1_{loc}(\mathbb{R}^N) \quad \textit{for some } \phi_0 \in L^\infty_{loc}(\mathbb{R}^N),$$

then for any $p \in [1,\infty)$

$$\phi_n(t,\mathbf{x}) = \phi_{0,n}(\boldsymbol{\eta}_n^{-1}(t,\mathbf{x})) \to \phi(t,\mathbf{x}) = \phi_0(\boldsymbol{\eta}^{-1}(t,\mathbf{x})) \quad \textit{in } C(0,T; L^p_{loc}(\mathbb{R}^N)),$$

where $\phi \in L^\infty(\mathbb{R}^N)$ *is the unique solution of*

$$\partial_t \phi + \operatorname{div}(\phi \mathbf{v}) = 0 \quad \textit{in } \mathcal{D}'((0,T)\times\mathbb{R}^N), \quad \phi(0,\mathbf{x}) = \phi_0(\mathbf{x}) \quad \textit{in } \mathbb{R}^N.$$

Moreover, if $\phi_0 \in Char(\mathbb{R}^N)$*, then* $\phi \in L^\infty(0,T; Char(\mathbb{R}^N))$*.*

4.4 The compactness of the convective term

Let us start this section by the following very important result, in which we introduce a "local" pressure p for the Navier-Stokes equations, written in a sub-domain of Ω_T. The pressure p will be decomposed on the components $p = p_{\rm reg} + \partial_t p_{\rm harm}$, where $p_{\rm reg}$ enjoys the same regularity properties as the convective-viscous terms, while $p_{\rm harm}$ is a harmonic function. The basic idea of the concept of local pressure for the incompressible Navier-Stokes equations was developed by Koch, Solonnikov [21] and Wolf [29].

Lemma 4.7 *Let us consider a time interval $I = (t_1, t_2)$ and a domain $B \subset \mathbb{R}^N$ with a regular C^2 boundary. Assume that $\mathbf{U} \in L^\infty(I; L^2(B))$, $\mathrm{div}\boldsymbol{U} = 0$, $\mathbb{T} \in L^q(I \times B)$, $1 < q < 2$ and $\mathbf{g} \in L^2(I \times B)$ satisfy the integral identity*

$$\int_{I\times B} \Big(\mathbf{U}\partial_t\boldsymbol{\psi} + \mathbb{T} : \nabla\boldsymbol{\psi} + \mathbf{g}\boldsymbol{\psi}\Big)\, dtd\mathbf{x} = 0, \tag{4.4.1}$$

for all $\boldsymbol{\psi} \in \mathcal{D}(I \times B)$, $\mathrm{div}\boldsymbol{\psi} = 0$. Then there exist two functions $p_{\mathrm{reg}} \in L^q(I \times B)$ and $p_{\mathrm{harm}} \in L^\infty(I; L^q(B))$, satisfying

$$\Delta p_{\mathrm{harm}} = 0 \quad \text{in } \mathcal{D}'(I \times B), \qquad \int_B p_{\mathrm{harm}}\; d\mathbf{x} = 0, \tag{4.4.2}$$

$$\int_{I\times B} \Big(\mathbf{U}\partial_t\boldsymbol{\psi} + \mathbb{T} : \nabla\boldsymbol{\psi} + \mathbf{g}\boldsymbol{\psi}\Big)\, dtd\mathbf{x} = \int_{I\times B} \Big(p_{\mathrm{reg}}\mathrm{div}\boldsymbol{\psi} + p_{\mathrm{harm}}\partial_t\mathrm{div}\boldsymbol{\psi}\Big)\, dtd\mathbf{x}$$

for any $\boldsymbol{\psi} \in \mathcal{D}(I \times B)$. In addition,

$$\begin{aligned}
&\|p_{\mathrm{reg}}\|_{L^q(I\times B)} \leqslant C(\|\mathbb{T}\|_{L^q(I\times B)} + \|\mathbf{g}\|_{L^2(I\times B)}) =: M,\\
&\|p_{\mathrm{harm}}\|_{L^\infty(I;L^q(B))} \leqslant C\Big(\|\mathbf{U}\|_{L^\infty(I;L^2(B))} + M\Big) =: L,\\
&\|p_{\mathrm{harm}}\|_{L^\infty(I;C^2(G))} \leqslant CL \qquad \text{for any open} \quad G \subset \overline{G} \subset B
\end{aligned} \tag{4.4.3}$$

with constants C depending only on q, I, B and G.

We are in a position to state the local stability property of solutions to Navier-Stokes equations. We show the "weak" compactness of the convective term stated in what follows. In the following lemma we follow [5], where it was given a correction of the proof, given in [29].

Lemma 4.8 *Let us assume that $\{\mathbf{U}_n, \mathbb{T}_n\}_{n=1}^\infty$ on some cylinder $I \times B \subset \Omega_T$ satisfies the integral identity*

$$\int_{I\times B} [\mathbf{U}_n\partial_t\boldsymbol{\psi} + \mathbb{T}_n : \nabla\boldsymbol{\psi} + \mathbf{g}\boldsymbol{\psi}]\; dtd\mathbf{x} = 0,$$

for any $\boldsymbol{\psi} \in \mathcal{D}(I \times B)$, such that $\mathrm{div}\boldsymbol{\psi} = 0$. Furthermore, assume that

$$||\mathbf{U}_n||_{L^\infty(I;L^2(B))} + ||\nabla\mathbf{U}_n||_{L^2(I\times B)} \leqslant C, \quad \mathrm{div}\mathbf{U}_n = 0, \tag{4.4.4}$$

$$||\mathbb{T}_n||_{L^q(I\times B)} \leqslant C, \quad 1 < q < 2, \tag{4.4.5}$$

where the constants C are independent of n. Then, passing to a suitable subsequence, we have

$$\begin{aligned}
&\mathbf{U}_n \rightharpoonup \mathbf{U} \quad \text{weakly-(*) in} \quad L^\infty(I; L^2(B)) \quad \text{and weakly in} \quad L^2(I; W^{1,2}(B)),\\
&\mathbf{U}_n \otimes \mathbf{U}_n \rightharpoonup \overline{\mathbf{U} \otimes \mathbf{U}} \quad \text{weakly in } L^{3/2}(I \times B),
\end{aligned} \tag{4.4.6}$$

where

$$\int_{I\times B} \overline{\mathbf{U} \otimes \mathbf{U}} : \nabla\boldsymbol{\psi}\; dtd\mathbf{x} = \int_{I\times B} \mathbf{U} \otimes \mathbf{U} : \nabla\boldsymbol{\psi}\; dtd\mathbf{x} \tag{4.4.7}$$

for any $\boldsymbol{\psi} \in \mathcal{D}(I \times B)$, $\mathrm{div}\boldsymbol{\psi} = 0$.

Proof. By Lemma 4.7 there exist the functions $p_{\mathrm{reg},n}$, $p_{\mathrm{harm},n}$ satisfying the integral identity

$$\int_{I\times B}\Big(\mathbf{U}_n\partial_t\boldsymbol{\psi}+\mathbb{T}_n:\nabla\boldsymbol{\psi}+\mathbf{g}\boldsymbol{\psi}\Big)\ dtd\mathbf{x}=$$
$$\int_{I\times B}\Big(p_{\mathrm{reg},n}\mathrm{div}\boldsymbol{\psi}+p_{\mathrm{harm},n}\partial_t\mathrm{div}\boldsymbol{\psi}\Big)\ dtd\mathbf{x} \tag{4.4.8}$$

for any $\boldsymbol{\psi}\in\mathcal{D}(I\times B)$. Moreover, in accordance with (4.4.3), (4.4.4), we have that

$$\|p_{\mathrm{reg},n}\|_{L^q(I\times B)}\leqslant C,\quad \|p_{\mathrm{harm},n}\|_{L^\infty(I;L^q(B))}\leqslant C,$$
$$\|p_{\mathrm{harm},n}\|_{L^\infty(0,T;C^2(G))}\leqslant C(G)\quad\text{for any open}\ \ G\subset\overline{G}\subset B. \tag{4.4.9}$$

By (4.4.4) and (4.4.9), passing to a suitable subsequence, we have

$$\begin{aligned}
&\mathbf{U}_n\rightharpoonup\mathbf{U}\ \ \text{weakly-(*) in}\ \ L^\infty(I;L^2(B))\quad\text{and weakly in}\ \ L^2(I;W^{1,2}(B)),\\
&\mathbf{U}_n\otimes\mathbf{U}_n\rightharpoonup\overline{\mathbf{U}\otimes\mathbf{U}}\quad\text{weakly in}\ L^{3/2}(I\times B),\\
&p_{\mathrm{harm},n}\rightharpoonup p_{\mathrm{harm}}\quad\text{weakly-(*) in}\ \ L^\infty(I;L^q(B))\ \text{and}\ L^\infty(0,T;C^2(G)),\\
&p_{\mathrm{reg},n}\rightharpoonup p_{\mathrm{reg}}\quad\text{weakly in}\ L^q(I\times B).
\end{aligned} \tag{4.4.10}$$

Hence, using (4.4.8)-(4.4.10), we can apply the Lions-Aubin argument [4] and obtain

$$\int_{I\times B}\varphi|\mathbf{U}_n+\nabla p_{\mathrm{harm},n}|^2\ dtd\mathbf{x}\to\int_{I\times B}\varphi|\mathbf{U}+\nabla p_{\mathrm{harm}}|^2\ dtd\mathbf{x},$$

for any $\varphi\in\mathcal{D}(I\times B)$ with $supp(\varphi)\subseteq\overline{G}$. In other words, we have

$$\mathbf{U}_n+\nabla p_{\mathrm{harm},n}\to\mathbf{U}+\nabla p_{\mathrm{harm}}\quad\text{strongly in}\ L^2(I\times G). \tag{4.4.11}$$

Thus

$$\begin{aligned}
&\int_{I\times B}\overline{\mathbf{U}\otimes\mathbf{U}}:\nabla\boldsymbol{\psi}\ dtd\mathbf{x}=\lim_{n\to\infty}\int_{I\times B}(\mathbf{U}_n\otimes\mathbf{U}_n):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}=\\
&\lim_{n\to\infty}\int_{I\times B}\Big((\mathbf{U}_n+\nabla p_{\mathrm{harm},n})\otimes\mathbf{U}_n\Big):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}-\\
&\lim_{n\to\infty}\int_{I\times B}\Big(\nabla p_{\mathrm{harm},n}\otimes(\mathbf{U}_n+\nabla p_{\mathrm{harm},n})\Big):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}-\\
&\lim_{n\to\infty}\int_{I\times B}(\nabla p_{\mathrm{harm},n}\otimes\nabla p_{\mathrm{harm},n}):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}=\int_{I\times B}(\mathbf{U}\otimes\mathbf{U}):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}
\end{aligned}$$

for any $\boldsymbol{\psi}\in\mathcal{D}(I\times G)$, $\mathrm{div}\boldsymbol{\psi}=0$. Indeed

$$\int_{I\times B}(\nabla p\otimes\nabla p):\nabla\boldsymbol{\psi}\ dtd\mathbf{x}=-\int_{I\times B}\Big(\frac{1}{2}\nabla|\nabla p|^2\cdot\boldsymbol{\psi}+\Delta p(\nabla p\cdot\boldsymbol{\psi})\Big)\ dtd\mathbf{x}=0$$

for $p=p_{\mathrm{harm},n},\ p_{\mathrm{harm}}$, respectively. Using that G is arbitrary, we finally deduce (4.4.6)-(4.4.7). ■

5 Conclusion

In the article we have shown the global solvability of the motion of *one* rigid body in the fluid, which includes collisions of the body with the boundary of the domain. Our proof is based on the embedding results (3.3.16), (3.3.17), playing a crucial role. We may generalize the result on the problem of the motion of *several* rigid bodies, but one of main obstacles is absence of embedding results for the space of bounded deformations $LD^2(\Omega)$ in domains with cusps. We can refer to a particular result [3], obtained for two-dimensional domains.

6 Acknowledgment

The work of N.V. Chemetov was supported through the project POCTI/ISFL/209 of CMAF/UL and the project PTDC/MAT/110613/2009 of FFC/FC/UL.
The work of Š. Nečasová was supported by Grant P201/11/1304 of GAČR.

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