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Abstract: In this paper we focus on an existence of a weak solution to a system describing a self-propelled motion of a single deformable body in a viscous compressible fluid which occupies a bounded domain in the 3 dimensional Euclidean space. The considered governing system for the fluid is the isentropic compressible Navier-Stokes equation. We prove an existence of a weak solution up to a collision.

Keywords: Self-propelled motion, compressible fluid, deformable structure

1 Introduction

This paper is devoted to a self-propelled motion of a body S in a viscous compressible fluid which is contained in a bounded domain $\Omega \subset \mathbb{R}^3$.

The problem of self-propelled motion or self-propulsion is a common means of locomotion of macroscopic objects. Typical examples are motions performed by birds, fishes, airplanes, rockets and submarines. In the microscopic world, many minute organism, like flagellates and ciliates move by self-propulsion, were studied by many authors. Even though the hydrodynamical mechanism of self-propulsion may be different for macroscopic and microscopic bodies see [26], the self-propelled motion of a body into a viscous liquid is essentially due to the interaction between the boundary of the body and a liquid. Hence, the boundary of body serves as the driver of body and the distribution V_* of the velocity on the boundary of the body, as its thrust. The thrust can be generated by muscular action, as in animal locomotion or by mechanical device, as in an airplane. In a famous experiment by Taylor [27], a mechanical fish can happily swim in water but makes no progress in a very viscous liquid like corn syrup. The fish consists of a cylindrical body with a plane tail which flaps to and fro. Due to the reversibility of flow in a liquid with no inertia or, mathematically, due to the linearity of the equations, whatever the fish achieves by one flap or tail, he/she will immediately lose with the next flap. In a commonly accepted model of Ciliata, the layer model, the motion of the cilia produces a distribution of velocity on a surface enclosing the layer of cilia, which serves to propel the animal [1, 19]. A principal characteristic of "flight' is that a significant part of the aerodynamic force is needed to cancel the weight of the organism. Thus, certain features of flying apply to buoyant fish. In forward flight such a force can be obtained by creating horizontal vorticity, this being the main purpose of the lifting surface of the body. The soaring and gliding of birds provides a familiar example where the classical aerodynamics of fixed-wing aircraft can be applied at once. The observations of birds led to Lanchester (notion of circulation and induced drag of wings). For more detail see [3].

The system composed by a swimming or flying creature can be considered as a fluid-structure system. In the recent years, many mathematical works have been published in the field of fluid–structure interaction problems, many of them tackling the well-posedness of the corresponding equations of motion. The main difficulties to obtain well-posedness of such systems are the non-linearity coming from the fluid equations (the Navier-Stokes or the Euler equations), the coupling between the equations of the fluid and the equations of the structure and the fact that the spatial domain of the fluid is moving and unknown. The last problem is simpler in the case of a rigid

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body for the structure since in that case, the motion of the structure is completely described by the rotation and the translation of the structure. In the case where the structure is deformable, for instance for an elastic structure, the existence of weak solutions could be very difficult to obtain: if the displacement of the structure is not regular, neither is the domain of the fluid. In [4] and [2], some approximated models are considered for the motion of an elastic structure in a viscous incompressible fluid. More precisely, the equations of the elasticity are modified in order to gain some regularity for the elastic deformation. Note that in the case of plate equations, it is possible to obtain the existence of weak solution without these approximations (see [16]). Concerning the mathematical theory of compressible fluids the fundamental results on Newtonian case were obtained in the last two decades by P. L. Lions [21] (barotropic case with $p(\rho) = \rho^\gamma$) and E. Feireisl et al. [12] (generalization to a larger class of exponents γ), E. Feireisl [8] and E. Feireisl, A. Novotný [10] (heat conductive fluids, singular limits). Based on the entropy inequality, the concept was further generalized to the notion of dissipative solutions and of the weak–strong uniqueness, see [9, 11].

The case of 2 dimensions was studied, for example, in [17]. Except for an existence result, authors prove a uniqueness of solution and they also provide some numerical simulations. In 3 space dimensions, Starovoitov in [23] studies a motion of several rigid bodies whereas in [18] authors provide an existence result of equation describing self-propelled motion of a body in an incompressible fluid. The problem of existence of the strong solution of self-propelled motion was study by Galdi, Silvestre. In [24] the Stokes approximation of the self-propelled motion of a rigid body in a viscous liquid that fills all the three-dimensional space exterior to the body was studied. Precisely, the existence and uniqueness of strong solution to the coupled systems of equations describing the motion of the system body–liquid, for any time and any regular distribution of velocity on the boundary of the body was proved. In [25] the motion of a self-propelled rigid body through a Navier- Stokes fluid that fills all the three-dimensional space exterior domain was investigated. The existence of a weak solution that is defined globally in time, provided that the net flux across the boundary, of the prescribed boundary values for the velocity, is zero. In the work of Galdi [14, 15] they were devoted to the self-propulsion of a rigid body at vanishing Reynolds number. They considered that the shape of the body is constant during the motion, the thrust is produced either because the body generates a nonzero momentum flux through its boundary, or/and because it moves portions of its boundary. As it was already mentioned, in the limit of zero Reynolds number, the importance of inertia in determining the motion of the fluid, and consequently, the motion of the body, becomes negligible. The motion of the body is therefore completely determined by its geometry and by the distribution of velocity on its boundary. In fact, it has been shown in [14] that, in the steady case, the motion of the body can be completely decoupled from that of the liquid, and the method used in [15] can also be extended to unsteady self-propelled motion to separate the motions of the body and the liquid.

The main aim of this paper is to provide a similar result as presented in [18] for the case of a compressible fluid surrounding a body. In order to prove a main theorem, we use a method presented in [7]. This method is based on an approximative system and on a high viscosity limit which simulates rigid body. Many parts of the proof are done similarly as in [7] and thus these parts are only sketched without any rigorous details. However, there are some problems which appear due to the self-propelled motion and coupling with the compressible fluid. In this paper we focus ourselves on this differences coming from the non-rigid motion rather than on problems which was solved in [7].

This paper is organized as follows. In Section 2 we introduce a setting and a governing system. The main theorem is presented in Section 3. Further, we introduce an approximative system in Section 4. The deformable body in an approximative system is treated as a part of a fluid which has tremendous viscosity. In Section 5 we deal with limiting processes in order to obtain the main result.

2 Setting

We consider a flying body with the deformable structure which occupies a bounded open connected set $\mathcal{S}_t$ in an instant $t \in [0,T]$. The body is surrounded by a viscous compressible fluid in a bounded domain $\Omega \subset \mathbb{R}^3$, i.e. the fluid fills a domain $\mathcal{F}_t := \Omega \backslash \mathcal{S}_t$ in an instant t. A function $\rho_{\mathcal{S}_t} : \mathcal{S}_t \mapsto (0,\infty)$ stands for a density of the body. We consider that S_t and Ω are locally Lipschitzian domains in $\mathbb{R}^3$.

The motion of the body consists of three elements: a translation described by $a \in R^3$, a rotation represented by $Q \in SO(3)$ and a smooth deformation $\mathcal{A} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ i.e. $\mathcal{A}$ is a smooth orientation-preserving diffeomorphism which is prescribed and stands for the self-propelled motion. Thus, the domain $\mathcal{S}_t$ can be described using a function $\eta[t] : \mathbb{R}^3 \mapsto \mathbb{R}^3$ as follows

$$\mathcal{S}_t = \eta[t]\mathcal{S}_0$$

i.e. every point $\mathbf{x} \in \mathcal{S}_t$ can be expressed as

$$\mathbf{x} = \eta[t](\mathbf{y}) = \mathbf{a}(t) + Q(t)\mathcal{A}_t(\mathbf{y}),$$

where $\mathbf{y} \in \mathcal{S}_0$ (S_0 is an initial position of the body). The velocity of the point $\mathbf{x}$ is

$$\mathbf{x}'(t) = \eta'[t](\eta^{-1}[t](\mathbf{x})) = \mathbf{a}'(t) + Q'(t)\mathcal{A}_t(\mathbf{y}) + Q(t)\partial_t\mathcal{A}_t(\mathbf{y}) = \mathbf{a}'(t) + \omega(t) \times (\mathbf{x} - \mathbf{a}(t)) + \mathbf{w}(t,\mathbf{x}),$$

where $\mathbf{w}(t,\mathbf{x}) = Q(t)\partial_t\mathcal{A}_t\left(\mathcal{A}_t^{-1}\left(Q^*(t)(\mathbf{x} - \mathbf{a}(t))\right)\right)$ and

$$\mathbb{S}(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \mathbb{S}(\omega(\mathbf{t})) = Q'(t)Q^T(t).$$

In what follows, we use overlined letters for quantities related to the body, which is considered without any rotation and translation. i.e. in a deformed configuration. Namely:

$$\begin{aligned} \overline{\mathcal{S}}_t &= \mathcal{A}_t(\mathcal{S}_0) \\ \overline{\mathbf{w}}(t,\overline{\mathbf{x}}) &= \frac{\partial}{\partial t}\mathcal{A}_t\left(\mathcal{A}_t^{-1}(\overline{\mathbf{x}})\right) \quad \forall \overline{\mathbf{x}} \in \mathcal{S}_t. \end{aligned}$$

We assume $\operatorname{div}\overline{\mathbf{w}} = 0$. Moreover, there exists a smooth divergence-less function $\overline{\mathbf{\Lambda}}$ which coincides with $\overline{\mathbf{w}}$ on a set $\overline{\mathcal{S}}_t$ and is supported on a neighborhood of $\overline{\mathcal{S}}_t$, i.e.:

$$\overline{\mathbf{\Lambda}}(t,\overline{\mathbf{x}}) = \begin{cases} \overline{\mathbf{w}}(t,\overline{\mathbf{x}}) & \text{for } \overline{\mathbf{x}} \in \overline{\mathcal{S}}_t \\ 0 & \text{if } \operatorname{dist}(\overline{\mathbf{x}},\overline{\mathcal{S}}_t) \geq \sigma, \end{cases}$$

where σ is sufficiently small. The existence of this smooth function is discussed in [18].
We define

$$\mathbf{\Lambda}(t,x) = Q(t)\overline{\mathbf{\Lambda}}(t,(Q^*(t)(\mathbf{x} - \mathbf{a}(t)))\,.$$

We denote the density of a fish in an instant $t \in [0,T]$ by $\rho_S := \rho_S(.,t) : S_t \mapsto (0,\infty)$. This density is given by

$$\rho_{\mathcal{S}}(t,\mathbf{x}) = \frac{\rho_{\mathcal{S}_0}\left(\mathcal{A}_t^{-1}(Q(t)^T[x - a(t)])\right)}{\det(\nabla\mathcal{A}_t(\mathcal{A}_t^{-1}(Q(t)^T[x - a(t)])))}.$$

Consequently, the density in a deformed configuration could be expressed as

$$\overline{\rho}_{\mathcal{S}}(t,\mathbf{x}) = \frac{\rho_{\mathcal{S}_0}(\mathcal{A}_t^{-1}(\overline{\mathbf{x}}))}{\det(\nabla\mathcal{A}_t(\mathcal{A}_t^{-1}(\overline{\mathbf{x}})))}.$$

In what follows, we assume that $\mathcal{A}_t$ is prescribed and we try to establish equations for $a(t)$ and $Q(t)$. Moreover, we assume that $\mathcal{A}_t$ satisfies hypothesis presented in [18], namely

(H1) For every $t \geq 0$, the mapping $\mathbf{y} \mapsto \mathcal{A}(t, \mathbf{y})$ is a smooth diffeomorphism from $\mathbb{R}^3$ onto $\mathbb{R}^3$. Moreover, for every $\mathbf{y} \in \mathbb{R}^3$, the mapping $t \mapsto \mathcal{A}(t, \mathbf{y})$ is smooth.

(H2) The total volume of the body remains constant, i.e.:

$$\left|\overline{S}_t\right| = |S_0| .$$

(H3) The center of gravity and the angular momenta of the body can not be changed by interior forces:

$$\begin{aligned}
\int_{\overline{S}_t} \overline{\rho}_S(t, \overline{\mathbf{x}}) \overline{\mathbf{w}}(t, \overline{\mathbf{x}}) d\overline{\mathbf{x}} &= 0, \\
\int_{\overline{S}_t} \overline{\rho}_S(t, \overline{\mathbf{x}}) \left[\overline{\mathbf{x}} \times \overline{\mathbf{w}}(t, \overline{\mathbf{x}})\right] d\overline{\mathbf{x}} &= 0.
\end{aligned}$$

For $\mathbf{x} \in \Omega$ and $t \in [0, T]$, we set[1]

$$\begin{aligned}
\mathbf{u}(t, \mathbf{x}) &= \chi_{\mathcal{F}_t} \mathbf{u}_{\mathcal{F}}(t, \mathbf{x}) + \chi_{S_t} \left(\partial_t \eta[t]\right) \left(\left(\eta[t]\right)^{-1}(\mathbf{x}) \right), \\
\rho(t, \mathbf{x}) &= \chi_{\mathcal{F}_t} \rho_{\mathcal{F}}(t, \mathbf{x}) + \chi_{S_t} \rho_{\mathcal{S}}(t, \mathbf{x}),
\end{aligned}$$

where $\mathbf{u}_{\mathcal{F}}$, $\rho_{\mathcal{F}}$ is velocity resp. density of the surrounding fluid. We assume that the following equations hold: Balance of mass:

$$\partial_t \rho_{\mathcal{F}} + \operatorname{div}(\rho_{\mathcal{F}} \mathbf{u}_{\mathcal{F}}) = 0 \text{ on } \mathcal{F}_t. \tag{2.1}$$

Balance of linear momentum:

$$\partial_t(\rho_{\mathcal{F}} \mathbf{u}_{\mathcal{F}}) + \operatorname{div}(\rho_{\mathcal{F}} \mathbf{u}_{\mathcal{F}} \otimes \mathbf{u}_{\mathcal{F}}) + \nabla p = \operatorname{div} T(\mathbf{u}) + \rho_{\mathcal{F}} \mathbf{g}_{\mathcal{F}} \text{ on } \mathcal{F}_t. \tag{2.2}$$

The tensor T is given via

$$T(\mathbf{u}) := 2\mu \mathbf{D}\mathbf{u} + \lambda I \operatorname{div} \mathbf{u}, \tag{2.3}$$

where $2\mathbf{D} = \nabla + \nabla^T$, $\mu \in (0, \infty)$, $\lambda \in \mathbb{R}$ and $\mu + \lambda \geq 0$. A pressure p is given by

$$p = \alpha \rho_{\mathcal{F}}^{\gamma}, \quad a > 0, \tag{2.4}$$

with $\gamma \in \mathbb{R}$ restricted below. We consider the following boundary conditions

$$\begin{aligned}
\mathbf{u}_{\mathcal{F}}(t, \mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \\
\mathbf{u}_{\mathcal{F}}(t, \mathbf{x}) &= \mathbf{a}'(t) + \omega(t) \times (\mathbf{x} - \mathbf{a}(t)) + \mathbf{w}(t, \mathbf{x}) = \mathbf{u}_{\mathcal{S}}, \mathbf{x} \in \partial\mathcal{S}_t.
\end{aligned} \tag{2.5}$$

Since the motion $\overline{\mathcal{A}}_t$ is prescribed, we have to introduce equations for unknowns $a(t)$ and $\omega(t)$ which describe the movement of the body. Before we write down the equations, we set

$$\begin{aligned}
M &:= \int_{S_t} \rho_S, \\
J(t) &:= \int_{S_t} \rho_S(t, \mathbf{x}) \left(|\mathbf{x} - \mathbf{a}(t)|^2 - (\mathbf{x} - \mathbf{a}(t)) \otimes (\mathbf{x} - \mathbf{a}(t)) \right) d\mathbf{x}.
\end{aligned}$$

Finally, the functions $\mathbf{a}(t)$, $\omega(t)$ should satisfy

$$\begin{aligned}
M\mathbf{a}''(t) &= -\int_{\partial S_t} (T - pI)\, \mathbf{n} + \int_{S_t} \rho_S \mathbf{g}, \\
(J\omega)'(t) &= -\int_{\partial S_t} (\mathbf{x} - \mathbf{a}(t)) \times (T - pI) \mathbf{n} d\Gamma + \int_{S_t} \rho_S (\mathbf{x} - \mathbf{a}(t)) \times \mathbf{g} d\mathbf{x}.
\end{aligned} \tag{2.6}$$

[1] By χ_M we denote the characteristic function of a set M.

The initial state is described through

$$\begin{array}{cccc} \mathbf{a}(0)=0, & Q(0)=I, & \mathcal{A}_0=I, & \rho_S(0)=\rho_{S0}, \\ \mathbf{a}'(0)=\mathbf{a}_0, & \omega(0)=\omega_0, & \rho_{\mathcal{F}}(0)=\rho_{\mathcal{F}0}, & \rho(0)\mathbf{u}(0)=\mathbf{m}_0. \end{array} \tag{2.7}$$

For abbreviation, $\rho_0=\chi_{\mathcal{F}_0}\rho_{\mathcal{F}0}+\chi_{\mathcal{S}_0}\rho_{S0}$. We also assume that

$$\mathbf{m}_0=0 \text{ a.e. on the set } \{\mathbf{x}\in\Omega,\rho_0(\mathbf{x})=0\}, \quad \frac{|\mathbf{m}|^2}{\rho_0}\in L^1(\Omega)$$

and that there exist $c_1,c_2\in(0,\infty)$ such that

$$\rho_0\chi_{\mathcal{S}_0}\in[c_1,c_2].$$

We define

$$\begin{gathered} \mathcal{H}_\sigma(\psi)=\{v\in L^2(\Omega)\ ;\ \nabla\cdot v=0 \quad \text{in } \Omega, \\ v\cdot n=0 \quad \text{on } \partial\Omega, \quad D(v)=0 \quad \text{in } \mathcal{S}(\psi)\}, \\ \mathcal{K}_\sigma(\psi)=\mathcal{H}_\sigma(\psi)\cap H_0^1(\Omega), \end{gathered}$$

where L^η, H_0^η, H^η are the classical Lebesgue and Sobolev spaces. Further,

$$L^2_\sigma(\Omega)=\mathcal{H}_\sigma(\psi), \quad H^1_\sigma(\Omega)=\mathcal{K}_\sigma(\psi).$$

We set

$$\rho(t,x)=\begin{cases}\rho_{\mathcal{F}}(t,x) & \text{if } x\in\mathcal{F}(t),\\ \rho_{\mathcal{S}}(t,x) & \text{if } x\in\mathcal{S}(t),\end{cases} \qquad u(t,x)=\begin{cases} u(t,x) & \text{if } x\in\mathcal{F}(t),\\ a'(t)+\omega(t)\times(x-a(t))+w(t,x) & \text{if } x\in\mathcal{S}(t).\end{cases}$$

2.1 Definition – weak solution. *We say that a pair* $(\rho,\mathbf{u})\in L^\infty(L^\gamma)\times(L^2(0,T_*;\mathcal{K}_\sigma(\psi))\cap L^\infty(0,T_*;\mathcal{H}_\sigma(\psi)))$ *is a weak solution of ((2.1) - (2.7)) if*

- $\rho\geq 0$;
- *A renormalized equation of continuity equation holds in a weak sense, i.e.*

$$\partial_t b(\rho)+\operatorname{div}(b(\rho)\mathbf{u})+(b'(\rho)\rho-b(\rho))\operatorname{div}\mathbf{u}=0 \ \textit{in } \mathcal{D}'((0,T)\times\Omega), \tag{2.8}$$

  where $b\in C^1(\mathbb{R})$;
- *Balance of linear momentum holds in a weak sense, i.e.*

$$\int_0^T\int_\Omega(\rho\mathbf{u})\partial_t\varphi+[\rho\mathbf{u}\otimes\mathbf{u}]:\mathbf{D}\varphi+p\operatorname{div}\varphi d\mathbf{x}dt = \int_0^T\int_\Omega T(\mathbf{u}):\mathbf{D}\varphi-\rho\mathbf{g}\varphi d\mathbf{x}dt+\int_\Omega\mathbf{m_0}\varphi(0,.)d\mathbf{x}, \quad \forall\varphi\in\mathcal{R}(S_t), \tag{2.9}$$

  where

$$\mathcal{R}(S_t)=\{\varphi\in C_0^\infty([0,T)\times\Omega),\mathbf{D}\varphi(x)=0 \textit{ on an open neighborhood of } S_t\}\,; \tag{2.10}$$

- *The energy inequality*

$$\frac{1}{2}\int_\Omega\rho(\tau)|\mathbf{u}(\tau)|^2+\frac{a}{\gamma-1}\rho^\gamma(\tau)d\mathbf{x}+\int_0^\tau\int_\Omega 2\mu|\mathbf{D}(\mathbf{u})|^2+\lambda(\operatorname{div}\mathbf{u})^2d\mathbf{x}dt\leq C(\varepsilon,\rho(0),\mathbf{u}(0),\mathbf{g})$$

  holds for a.e. $\tau\in[0,T]$;

- *The movement of the body $\mathcal{S}$ is compatible with* $\mathbf{u}$ *in following sense*

$$\mathbf{u}_{\mathcal{F}}(t,.) - \mathbf{u}_S(t,.) \text{ belongs locally to the space } W_0^{1,2}(\Omega \setminus S_t). \tag{2.11}$$

Remarks:

- There is no a priori reason to assume that the momentum $(\rho\mathbf{u})$ is continuous in time. We can only have that a function
$$t \longmapsto \int_{\mathbb{R}^3} (\rho\mathbf{u}) \cdot \psi$$
is continuous in a certain neighborhood of a point t_0 provided $\psi = \psi(x) \in \mathcal{D}(\Omega)$ and $\psi = 0$ on a neighbourhood of $\overline{\mathbf{S}}(t_0)$.

- An alternative condition to the concept of compatibility of the velocity $\mathbf{u}$ with the rigid objects was used in [5], [22], namely
$$\mathbf{u} \in L^2((0,T); W_0^{1,2} \cap V^s(\Omega)),$$
where the sets $V^s = V^s(t)$ are defined as
$$V^s = \{\mathbf{u} \in W^{1,2}(\Omega) | \mathcal{D}(\mathbf{u})\rho_{\mathbf{S}}(t) = 0\}.$$

3 Main result

3.1 Theorem – Main result. *Let Ω be a $\mathcal{C}^{2+\nu}$ domain, $\nu > 0$, $\gamma > 3/2$ and $\mathcal{S}_0 \subset\subset \Omega$ be a compact connected set. Let there exist $c_1, c_2 > 0$ and initial data ρ_0, $\mathbf{m}_0$ be such that*

$$\rho_0 \geq 0, \quad \rho_0\chi_{\mathcal{S}_0} \in [c_1, c_2], \quad \rho_0 \in L^\gamma(\Omega), \tag{3.1}$$

$$\mathbf{m}_0 = 0 \quad \text{a.e. on the set } \{x \in \Omega \mid \rho_0 = 0\}, \quad \frac{\mathbf{m}_0^2}{\rho_0} \in L^1(\Omega). \tag{3.2}$$

Then there exists $T_* \in (0, \infty)$ such that there exists a weak solution $(\rho, \mathbf{u})$ of (2.1) – (2.7) on an interval $(0, T_*)$.

Remarks: The approximation of the problem (2.1) – (2.7) is constructed by the following way:

- d - approximation

  We approximate the continuity equation by adding the term $d\Delta\rho$ and we also add the term $d\nabla\rho\nabla\mathbf{u}$ to the momentum equation.

- β -approximation

  We introduce the artificial pressure to adding a term $b\rho^\beta$, with $\beta > 2$ to the constitutive equation.

- n approximation

  We use the penalization method introduced by Starovoitov et all. [22] to consider the viscosity coefficients dependent on the distance to the boundary.

Letting $n \to \infty$, $d \to 0$ and $\beta \to 0$ we get the existence of the weak solution of the problem.

4 Approximative problem

4.1 Approximation (d, β, n)

We use an approximative scheme which is proposed in [18] in Remark 11, i.e. we suppose that the viscosity of a compressible fluid rapidly increases on the body $\mathcal{S}_t$ together with known approximation scheme [20]. The part of the velocity which is zero on a "fluid domain" and grows rapidly on a "body domain" is denoted by μ_χ, resp. λ_χ. These viscosities are defined precisely later. Now it is enough to assume that functions $\mu_\chi : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ and $\lambda_\chi : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ obey

$$\mu_\chi \geq 0, \qquad \lambda_\chi + \mu_\chi + \mu + \lambda \geq 0, \tag{4.1}$$

where the variable χ depends on $\mathbf{u}$ and will be specified later.

The approximative problem consists of following equations:

- A continuity equation

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= d\Delta\rho, \quad d > 0, \\ \nabla\rho \cdot \mathbf{n}|_{\partial\Omega} &= 0. \end{aligned} \tag{4.2}$$

- A momentum equation (we define $\mathbf{v} = \mathbf{u} - \mathbf{\Lambda}$)

$$\begin{aligned} \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) + d\nabla\rho\nabla\mathbf{u} &= \operatorname{div}\left(\mu\mathbf{D}(\mathbf{u}) + \mu_\chi(\mathbf{D}\mathbf{v})\right) \\ &\quad + \operatorname{div}\left(\lambda I \operatorname{div}\mathbf{u} + \lambda_\chi I \operatorname{div}(\mathbf{v})\right) \\ &\quad + \operatorname{div} F, \\ \mathbf{u}|_{\partial\Omega} &= 0, \end{aligned} \tag{4.3}$$

  where $\mathbf{\Lambda} := \mathbf{\Lambda}_\mathbf{u} : [0, T] \times \Omega \mapsto \mathbb{R}^n$ is a given function depending on $\mathbf{u}$.

- A constitutive relation for a pressure

$$p = p(\rho) = a\rho^\gamma + b\rho^\beta, \quad a, b > 0, \beta > \max\{4, \gamma\}. \tag{4.4}$$

- We complement this system by the following initial data

$$\rho(0) = \rho_0, \quad (\rho\mathbf{u})(0) = \mathbf{m}_0, \tag{4.5}$$

  where $\rho_0 \in C^{2+\nu}(\overline{\Omega})$, $0 < c_2 \leq \rho_0 \leq c_3$, $\nabla\rho_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\mathbf{m}_0 \in C^2(\overline{\Omega})$.

4.1 Lemma - existence of solution to approximative system. *Let $\Omega \subset \mathbb{R}^3$ and S_t be bounded $C^{2+\nu}$, $\nu > 0$ domains, $g \in \mathcal{D}((0,T) \times \Omega)$ be given. Let* (4.1) *hold and let $\beta > \max\{4, \gamma\}$, $\gamma > \frac{3}{2}$. Moreover, let $\mathbf{\Lambda}$ satisfy*

$$\begin{aligned} \|\partial_t\mathbf{\Lambda}\|_{L^2(0,T,L^\infty(\Omega))} &\leq C(1 + \|\mathbf{u}\|_{L^2((0,T,L^2(\Omega)))}), \\ \|\mathbf{\Lambda}\|_{L^\infty(0,T0,L^\infty(\Omega)} + \|\nabla\mathbf{\Lambda}\|_{L^\infty(0,T,L^\infty(\omega))} + \|\Delta\mathbf{\Lambda}\|_{L^\infty(0,T,(L^\infty)(\Omega)} &\leq C, \\ \mathbf{\Lambda}|_{\partial\Omega} &= 0. \end{aligned}$$

Then there exists a weak solution $(\rho, \mathbf{u}) \in L^\infty(0, T, L^\beta(\Omega)) \times L^2(0, T, W_0^{1,2}(\Omega))$ to a problem (4.2)–(4.5).

Proof. The proof can be done in a similar way as in [20]. The presence of two unknowns $\mathbf{\Lambda}$ and χ does not bring any crucial problem. □

5 Proof of the main theorem

5.1 Average rigid motion

Let $\mathcal{S}_t$ be a set defined for all times t. Hereinafter, we write simply χ instead of $\chi_{\mathcal{S}_t}$:

$$
\begin{aligned}
M_{[\chi,\rho]} &= \int_\Omega \rho\chi, \quad \mathbf{a}_{[\chi,\rho]} = \frac{1}{M_{[\chi,\rho]}}\int_\Omega \rho(\mathbf{x})\chi(\mathbf{x})\mathbf{x}d\mathbf{x},\\
I_{[\chi,\rho]} &= \int_\Omega \rho(\mathbf{x})\chi(\mathbf{x})\left(|\mathbf{x}-\mathbf{a}_{[\chi,\rho]}|^2 - (\mathbf{x}-\mathbf{a}_{[\chi,\rho]})\otimes(\mathbf{x}-\mathbf{a}_{[\chi,\rho]})\right)d\mathbf{x},\\
\mathbf{l}_{[\chi,\rho,\mathbf{u}]} &= \frac{1}{M_{[\chi,\rho]}}\int_\Omega \rho\chi\mathbf{u},\\
\omega_{[\chi,\rho,\mathbf{u}]} &= \left(I_{[\chi,\rho]}\right)^{-1}\int_\Omega \rho(\mathbf{x})\chi(\mathbf{x})\left((\mathbf{x}-\mathbf{a}_{[\chi,\rho]})\times\mathbf{u}(\mathbf{x})\right)d\mathbf{x}.
\end{aligned}
$$

The quantities $\mathbf{l}_{[\chi,\rho]}$ and $\omega_{[\chi,\rho,\mathbf{u}]}$ express an average transition and rotation of $\mathcal{S}_t$. Thus, an average rigid motion of a body $\mathcal{S}$ can be described by a function $\mathbf{\Pi}_{[\chi,\rho,\mathbf{u}]}$ which is defined as follows:

$$\mathbf{\Pi}_{[\chi,\rho,\mathbf{u}]}(\mathbf{x}) = \mathbf{l}_{[\chi,\rho,\mathbf{u}]} + \omega_{[\chi,\rho,\mathbf{u}]}\times(\mathbf{x}-\mathbf{a}_{[\chi,\rho]}).$$

Further, we define a function $Q_{[\chi,\rho,\mathbf{u}]}:\mathbb{R}\mapsto\mathbb{R}^{n\times n}$ as a solution to the following ODE

$$Q'_{[\chi,\rho,\mathbf{u}]}(t) = \mathbb{S}(\omega_{[\chi,\rho,\mathbf{u}]}Q_{[\chi,\rho,\mathbf{u}]}(t), \quad Q_{[\chi,\rho,\mathbf{u}]}(0) = I,$$

and a function $\mathbf{c}_{[\chi,\rho,\mathbf{u}]}:\mathbb{R}\mapsto\mathbb{R}^3$ as a solution of

$$\mathbf{c}'_{[\chi,\rho,\mathbf{u}]}(t) = \omega_{[\chi,\rho,\mathbf{u}]}(t)\times\mathbf{c}_{[\chi,\rho,\mathbf{u}]}(t) + \mathbf{l}_{[\chi,\rho,\mathbf{u}]}(t), \quad \mathbf{c}(0) = 0.$$

We set $\mathbf{\Lambda}_{[\chi,\rho,\mathbf{u}]}(t,\mathbf{x}) = Q_{[\chi,\rho,\mathbf{u}]}(t)\overline{\mathbf{\Lambda}}(t, Q^*_{[\chi,\rho,\mathbf{u}]}(\mathbf{x}-\mathbf{c}_{[\chi,\rho,\mathbf{u}]}))$.

Let $\mathcal{S}_0\in\Omega$ and $\mathbf{u}$, ρ_0 be given with $\rho_0(\mathbf{x})\in[c_1,c_2]$ for all $\mathbf{x}\in\mathcal{S}_0$. We prescribe the movement of body $\mathcal{S}_t$ by the following system of equations

$$
\begin{aligned}
\partial_t\tilde\rho + \operatorname{div}(\tilde\rho(\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]}+\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]})) &= 0 \text{ on } \mathcal{S}_t\\
\partial_t\chi + \operatorname{div}(\chi(\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]}+\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]})) &= 0,
\end{aligned}
\tag{5.1}
$$

where $\mathcal{S}_t = \operatorname{supp}\chi(t)$. We complete (5.1) with the following initial conditions

$$\tilde\rho(0) = \rho_0 \text{ in } \mathcal{S}_0 \text{ and } \chi(0) = \chi_{\mathcal{S}_0}. \tag{5.2}$$

According to Lemma 6.4 a solution to (5.1), (5.2) exists. Moreover, since $\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]} + \mathbf{\Lambda}_{\chi,\tilde\rho,\mathbf{u}}$ is solenoidal and $\tilde\rho\in[C_1,C_2]$, we use Lemma 6.2 and Lemma 6.3 in order to obtain

$$
\begin{aligned}
\|\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]}\|_{L^\infty(\Omega)} &\le c\|\mathbf{u}\|_{L^2(\Omega)}, && (5.3)\\
\|\partial_t\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\|_{L^2(0,T,L^\infty(\Omega)} &\le c(1+\|\mathbf{u}\|_{L^2(\Omega)}), && (5.4)\\
\|\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\|_{L^\infty(0,T,L^\infty(\Omega))} + \|\nabla\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\|_{L^\infty(0,T,L^\infty(\Omega)} & &&\\
+\|\Delta\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\|_{L^\infty(0,T,L^\infty(\Omega))} &\le c. && (5.5)
\end{aligned}
$$

Moreover, since $\mathbf{\Pi}$ is a linear function, we get $\nabla_x\mathbf{\Pi} = \mathbb{S}(\omega)$ and one may derive that

$$\|\nabla_x\mathbf{\Pi}\|_{L^2(0,T,L^\infty(\Omega))} \le c\|u\|_{L^2(0,T,L^2(\Omega)}. \tag{5.6}$$

For details we refer reader to the proof of Lemma 4 in [18].

From (5.3) one may also derive that there exists $T>0$ such that $\mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}(t)|_{\partial\Omega} = 0$ for all $t<T$.

5.1 Proposition. *For every* $\mathbf{u} \in L^1(0,T,L^1)$ *it holds that*

$$\mathbf{\Pi}_{[\chi,\tilde{\rho},\mathbf{u}]} = \mathbf{\Pi}_{[\chi,\tilde{\rho},\mathbf{u}+\mathbf{\Lambda}_{[\chi,\tilde{\rho},\mathbf{u}]}]}.$$

Consequently,

$$\mathbf{\Lambda}_{[\chi,\tilde{\rho},\mathbf{u}]} = \mathbf{\Lambda}_{[\chi,\tilde{\rho},\mathbf{u}+\mathbf{\Lambda}_{[\chi,\tilde{\rho},\mathbf{u}]}]}.$$

Proof. The first identity follows from the definition of $\mathbf{\Pi}$ and from hypothesis (H3). The second identity is an easy consequence of the definition of $\mathbf{\Lambda}$. □

5.2 High viscosity limit - approximation n

Let $\{\mu_{\chi_n}\}_{n=1}^{\infty}$ and $\{\lambda_{\chi_n}\}_{n=1}^{\infty}$ be sequences of viscosities specified later. Let $\mathbf{u_n}$ and ρ_n be corresponding weak solution to (4.1) – (4.5), where $\mathbf{\Lambda}_n := \mathbf{\Lambda}_{[\chi_n,\tilde{\rho}_n,\mathbf{u}_n]}$ and $\mathbf{\Pi}_n := \mathbf{\Pi}_{[\chi_n,\tilde{\rho}_n,\mathbf{u}_n]}$ are constructed as in subsection 5.1. Further, we define a set $\mathcal{S}_{nt}$ as $\mathcal{S}_{nt} = \operatorname{supp}\chi_n(t,.)$. We set $\mathbf{u_n} = \mathbf{v_n} + \mathbf{\Lambda_n}$. In what follows, we assume

$$\mathbf{u}_n|_{\partial\Omega} = \mathbf{v}_n|_{\partial\Omega} = \mathbf{\Lambda}_n|_{\partial\Omega} = 0,$$

at least on some time interval $(0,T_n)$. In order to proceed to the limit (letting $n \mapsto \infty$), we have to estimate norms of solutions independently on n. From (4.1) we get (testing by 1)

$$\|\rho_n\|_{L^\infty(0,T,L^1(\Omega))} \le C.$$

We multiply (4.3) by $\mathbf{v}$ in order to get

$$\begin{aligned}\int_\Omega \frac{1-\varepsilon}{2}\rho_n(T)|\mathbf{u}_n(T)|^2 + \frac{a}{\gamma-1}(\rho_n^\gamma)(T) + \frac{b}{\beta-1}\rho_n^\beta(T)d\mathbf{x}\\ + \int_0^T\int_\Omega (2(\mu+\mu_{\Psi_n})-\varepsilon)|D\mathbf{v}_n|^2 + (\lambda+\lambda_{\Psi_n}-\varepsilon)|\operatorname{div}\mathbf{v}_n|d\mathbf{x}dt \le\\ C(\varepsilon,\rho(0),\mathbf{u}(0),\mathbf{m}_0,\mathbf{\Lambda}) + C(\varepsilon)\int_0^T\int_\Omega \rho_n|\mathbf{u}_n|^2 d\mathbf{x}dt + C(\varepsilon,a,b)\int_0^T\int_\Omega \rho_n^\gamma + \rho_n^\beta d\mathbf{x}dt.\end{aligned} \tag{5.7}$$

Using Gronwall's inequality we obtain

$$\|\mathbf{v}_n\|_{L^2(0,T,W^{1,2}(\Omega))} + \|\rho_n|\mathbf{u}_n|^2\|_{L^\infty(0,T,L^1(\Omega))} + \|\rho_n^\beta\|_{L^\infty(0,T,L^1(\Omega))} \le C(T_0), \tag{5.8}$$

where the constant on the right hand side is independent on n and d. Further, using this and (4.1), we get

$$\|\nabla\rho_n\|_2^2 \le C, \tag{5.9}$$

where the right hand side again does not depend on n and d. According to (5.3) – (5.5) the quantities $\mathbf{\Lambda}_n$ and $\mathbf{\Pi}_n$ are estimated uniformly and thus there exists $T_* > 0$ such that

$$\mathbf{\Lambda}_n(t)|_{\partial\Omega} = 0$$

for all $t \in (0,T_*)$ and $\forall\ n \in \mathbb{N}$. From now we work on this time interval unless stated otherwise.

We define viscosities $\mu_n := \mu_{\chi_n}$ and $\lambda_n := \lambda_{\chi_n}$ by the following formula

$$\begin{aligned}\lambda_n &= n\chi_n,\\ \mu_n &= n\chi_n.\end{aligned}$$

We also define the following distance

$$db_{\mathcal{S}}(\mathbf{x}) = d_{\overline{R^N\setminus\mathcal{S}}}(\mathbf{x}) - d_{\overline{\mathcal{S}}}(\mathbf{x}),$$

where $d_K(\mathbf{x}) = \min_{\mathbf{y}\in K}|\mathbf{x}-\mathbf{y}|$, provided $K \subset \mathbb{R}^n$ is a closed set. We use $db_{\mathcal{S}}$ to define convergence of sets. We write $\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t$ if and only if $db_{\mathcal{S}_{nt}} \to db_{\mathcal{S}_t}$ in $C_{\mathrm{loc}}(\mathbb{R}^3)$.

We proceed to a limit as n tends to ∞ (passing to a subsequence if needed). Since the limiting process is the same as in [7], we present here only conclusions without detailed proof. From (5.7), (5.8) and (5.9) it follows, passing to a subsequences if necessary,

$$\begin{gathered}\rho_n \to \rho \text{ in } L^\beta((0,T)\times\Omega),\\ \nabla\rho_n \to \nabla\rho \text{ weakly in } L^2((0,T)\times\Omega),\\ \mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^2(0,T,W_0^{1,2}(\Omega)),\\ \rho_n\mathbf{u}_n \to \rho\mathbf{u} \text{ weakly in } L^2((0,T)\times\Omega),\\ \rho_n\mathbf{v}_n\otimes\mathbf{v}_n \to \mathbb{P} \text{ weakly in} L^{\frac{2N}{2N-1}}((0,T)\times\Omega).\end{gathered}$$

From (4.2) one may derive

$$\|\rho_n(\tau)\|^2_{L^2(\Omega)} + 2d\int_0^\tau\int_\Omega \|\nabla\rho_n\|^2 = -\int_0^\tau\int_\Omega \operatorname{div}\mathbf{u}_n|\rho_n|^2 + \|\rho_0\|^2_{L^2(\Omega)}$$

and also

$$\|\rho(\tau)\|^2_{L^2(\Omega)} + 2d\int_0^\tau\int_\Omega \|\nabla\rho\|^2 = -\int_0^\tau\int_\Omega \operatorname{div}\mathbf{u}|\rho|^2 + \|\rho_0\|^2_{L^2(\Omega)}.$$

Thus, $\|\nabla\rho_n\|_{L^2(0,T,L^2(\Omega))} \to \|\nabla\rho\|_{L^2(0,T,L^2(\Omega))}$ and since L^2 is a strictly convex space, we get $\nabla\rho_n \to \nabla\rho$ strongly in $L^2((0,T)\times\Omega)$. Consequently

$$\nabla\mathbf{v}_n\nabla\rho_n \to \nabla\mathbf{v}\nabla\rho \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

According to (5.3), (5.5) and (5.6), $\mathbf{\Pi}_n + \mathbf{\Lambda}_n$ is bounded in $L^2(0,T,W^{1,\infty}(\Omega))$ independently of n. Thus, the hypothesis of Lemma 6.1 are fulfilled and one may derive that

$$\mathbf{\Pi}_n + \mathbf{\Lambda}_n \to \mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]} + \mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]} \text{ in } C_{\mathrm{loc}}(\mathbb{R}^3) \text{ uniformly in } t,$$

and also

$$\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t \text{ uniformly in } t.$$

We define

$$\begin{aligned} P^s &= \{(t,\mathbf{x}), \mathbf{x}\in\mathcal{S}_t\},\\ P^f &= ([0,T_*]\times\Omega)\setminus P^s.\end{aligned}$$

It follows easily that P^s is closed and P^f is open. Thus, for a point $(t,\mathbf{x})\in P^f$ there exists an open interval $J\subset[0,T]$ and $U\subset P^f$ such that

$$(t,\mathbf{x})\in J\times U\subset\overline{J\times U}\subset P^f.$$

We have $\partial_t\rho_n$ bounded in $L^q(J,W^{-k,q}(U))$ for some $q>1$, $k\geq 1$ and, consequently

$$(\rho_n\mathbf{u}_n)\to(\rho\mathbf{u}) \text{ in } C(\overline{J},L^{\frac{2\beta}{\beta+1}}(U)).$$

Due to a compact embedding $L^{\frac{2\beta}{\beta+1}}\subset W^{-1,2}$ we get

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u} \text{ weakly in } L^{\frac{6}{5}}(J\times U).$$

Thus $\mathbb{P} = \rho\mathbf{u}\otimes\mathbf{u}$ on P^f. Moreover, since μ_{Ψ_n} and λ_{Ψ_n} tends to infinity on every compact $\mathcal{K}^s\subset \mathrm{int}P^s$, we derive from (5.7) that $D\mathbf{v}_n\to 0$ in $L^2(\mathcal{K}^s)$. Therefore,

$$\int_0^T\int_\Omega(\rho\mathbf{u})\partial_t\varphi + [\rho\mathbf{u}\otimes\mathbf{u}]:D\varphi + p\operatorname{div}\varphi + d\nabla\mathbf{u}\nabla\rho\varphi d\mathbf{x}dt$$
$$= \int_0^T\int_\Omega \mu D\mathbf{u}D\varphi + \lambda\operatorname{div}\mathbf{u}\operatorname{div}\varphi + \mathbf{g}\varphi d\mathbf{x}dt + \int_\Omega \mathbf{m_0}\varphi(0,.)d\mathbf{x},$$

whenever $\varphi\in\mathcal{R}(\mathcal{S}_t)$ (see (2.10) for a definition).

We have just proven the following lemma.

5.2 Lemma. *Let $\Omega \subset \mathbb{R}^3$ be a bounded $\mathcal{C}^{2+\nu}$ domain with $\nu > 0$. Let p be given by (4.4) with $\beta > \max\{4, \gamma\}$, $\gamma > 3/2$. Let (4.5) hold and let $\mathcal{S}_0 \subset \Omega$. Then there exists time T_* and such functions $\rho \in L^\infty(0, T_*, L^\beta)$, $\mathbf{u} \in L^2(0, T_*, W_0^{1,2}) \cap L^\infty(0, T_* L^2)$, $\tilde{\rho} \in L^\infty(0, T_*, L^\infty)$ and $\chi \in Char(0, T_*, \mathbb{R}^3)$ that*

- *$\rho, \mathbf{u}$ satisfy (4.2), (2.9) and initial condition (4.5) in a weak sense, $\rho \in C([0, T_*], L^1)$,*
- *$\tilde{\rho}$ and χ satisfy (5.1).*

5.3 Vanishing viscosity limit

In this subsection, we proceed to a limit with the parameter d. Let $d_n \to 0$ and let $\mathbf{u}_n$ and ρ_n be corresponding weak solution to (4.2)–(4.5) which are constructed as in Lemma 5.2. Further, let $\mathcal{S}_{nt}$ be bodies with corresponding motion described by $\mathbf{\Pi}_n = \mathbf{\Pi}_{[\chi_n, \tilde{\rho}_n, \mathbf{u}_n]}$ and $\mathbf{\Lambda}_n = \mathbf{\Lambda}_{[\chi_n, \tilde{\rho}_n, \mathbf{u}_n]}$. From estimates (5.8) and (5.9) we get following convergences

$$d_n \nabla \mathbf{u}_n \nabla \rho_n \to 0 \text{ in } L^1((0,T) \times \Omega)$$

$$d_n \Delta \rho_n \to 0 \text{ in } L^2((0,T), W^{-1,2}(\Omega))$$

$$\rho_n \to \rho \text{ in } C([0,T], L^\beta_{weak})$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2((0,T), W_0^{1,2})$$

and, consequently,

$$(\rho_n \mathbf{u}_n) \to (\rho \mathbf{u}) \text{ weakly star in } L^\infty((0,T), L^{\frac{2\beta}{\beta+1}}(\Omega)).$$

Thus, ρ and $\mathbf{u}$ satisfy the continuity equation in $\mathcal{D}'((0,T) \times \Omega)$, and using same regularization procedure as in [6], it can be derive that ρ and $\mathbf{u}$ satisfy also renormalized continuity equation.

According to Lemmata 6.2 and 6.3, it holds that

$$\left\| \mathbf{\Pi}_{[\chi_n, \rho_n, \mathbf{u}_n]} + \mathbf{\Lambda}_{[\chi_n, \rho_n, \mathbf{u}_n]} \right\|_{L^2(L^\infty)} + \left\| \nabla \left(\mathbf{\Pi}_{[\chi_n, \rho_n, \mathbf{u}_n]} + \mathbf{\Lambda}_{[\chi_n, \rho_n, \mathbf{u}_n]} \right) \right\|_{L^2(L^\infty)} \leq C.$$

It follows from Lemma 6.1 that

$$\mathcal{S}_{nt} \stackrel{b}{\to} \mathcal{S}_t$$

and

$$\mathbf{\Pi}_{[\chi_n, \rho_n, \mathbf{u}_n]} + \mathbf{\Lambda}_{[\chi_n, \rho_n, \mathbf{u}_n]} \to \mathbf{\Pi}_{[\chi, \tilde{\rho}, \mathbf{u}]} + \mathbf{\Lambda}_{[\chi, \tilde{\rho}, \mathbf{u}]} \text{ weakly star in } L^2(0, T, W^{1,\infty}(\Omega)).$$

Further, we define

$$P^s = \{(t, \mathbf{x}), x \in \mathcal{S}_t\},$$
$$P^f = ((0,T) \times \Omega) \setminus P^s.$$

Similarly as in previous subsection (see also section 8 in [7]) we have

$$\rho_n \mathbf{u}_n \otimes \mathbf{u}_n \to \mathbb{P} \text{ in } L^{\frac{6}{5}}((0,T) \times \Omega)$$

and

$$\mathbb{P} = \rho \mathbf{u} \otimes \mathbf{u} \text{ on } P^f.$$

Following step by step the procedure in section 8 in [7], we derive that $p(\rho_n) \to p(\rho)$ weakly in $L^{\frac{\beta+1}{\beta}}(K^f)$ for any compact $K^f \subset P^f$.

Precisely, we claim that the pressure $p(\rho_n)$ is locally bounded in $L^{\frac{\beta+1}{\beta}}_{loc}(P^f)$.

Similarly as in [7], one may derive the following lemma.

5.3 Lemma. *For any compact $K^f \subset P^f$, there exists a constant c independent of ϵ, such that*

$$\|\rho_n\|_{L^{\beta+1}(K^f)} + \|\rho_n\|_{L^{\gamma+1}(K^f)} \leq c(K^f, E_0).$$

This implies that

$$p(\rho_n) \to \overline{p(\rho)} \text{ weakly in } L^{\frac{\beta+1}{\beta}}(K_f) \text{ for any compact } K^f \subset P^f.$$

Then we can pass to the limit

$$\int_0^T \int_{\mathbb{R}^N} (\rho\mathbf{u})\partial_t\varphi + [\rho\mathbf{u}\otimes\mathbf{u}] : \mathbf{D}(\varphi) + \overline{p(\rho)} \operatorname{div}\varphi dxdt = \int_0^t \int_{\mathbb{R}^N} \mathbf{T}(\mathbf{u}) : \mathbf{D}(\varphi) - \rho F\varphi dxdt.$$

Our final task is the strong convergence of density. Similarly as in [7] we apply the following result

5.4 Lemma. *Let $\beta > 7$.*
Then

$$\lim_{n\to\infty} \int_0^T \int_{R^n} \phi(p(\rho_n) - (\lambda+2\mu)\ div\ \mathbf{u}_n)\rho_n dxdt = \int_0^T \int_{R^n} \phi((\overline{p(\rho)} - (\lambda+2\mu) div\ \mathbf{u})\rho dxdt$$

for any $\phi \in \mathcal{D}(Q^f)$.

This lemma implies that

$$\rho_n \to \rho \text{ in } L^1((0,T)\times\Omega)$$

and

$$\overline{p(\rho)} = p(\rho) \text{ on } \text{ on } P^f.$$

Thus the functions $\mathbf{u}$ and ρ satisfy (2.8) and (2.9).

We are in the best position to prove $\mathbf{u}\chi = \left(\mathbf{\Pi}_{[\chi,\rho,\mathbf{u}]} + \mathbf{\Lambda}_{[\chi,\rho,\mathbf{u}]}\right)\chi$. We point out that $D\mathbf{v}\chi = 0$ a.e. and thus $\mathbf{v}$ is a rigid velocity on a body $\mathcal{S}$. According to considerations in section 3.1 in [18], it holds that $\mathbf{v}\chi = \left(\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{v}]}\right)\chi$. Due to Proposition 5.1 we have

$$\mathbf{u}\chi = \left(\mathbf{v} + \mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\right)\chi = \left(\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{v}]} + \mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\right)\chi = \left(\mathbf{\Pi}_{[\chi,\tilde\rho,\mathbf{u}]} + \mathbf{\Lambda}_{[\chi,\tilde\rho,\mathbf{u}]}\right)\chi.$$

Moreover, from the uniqueness of a solution to a transport equation, we get $\tilde\rho\chi = \rho\chi$. In order to conclude this subsection, we formulate all results into the following lemma.

5.5 Lemma. *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded $\mathcal{C}^{2+\nu}$ domain with $\nu > 0$. Let p be given by* (4.4) *with $\beta > \max\{4,\gamma\}$, $\gamma > 3/2$. Then there exists time T_* and such functions $\rho \in L^\infty(0,T_*,L^\beta)$, $\mathbf{u} \in L^2(0,T_*,W_0^{1,2}) \cap L^\infty(0,T_*,L^2)$ and $\chi \in Char(0,T_*,\mathbb{R}^3)$ that ρ and $\mathbf{u}$ solve* (2.8), (2.9) *and the compatibility condition* (2.11) *is satisfied.*

5.4 Limit in pressure and domain

Our final task is to prove an existence of a solution for a pressure given by (2.4) and for a general domain Ω. We take a sequence of real numbers $b_n \to 0$, asequence of domains Ω_n, $\Omega_n \subset \Omega_{n+1}$, $\Omega_n \xrightarrow{b} \Omega$ and a sequence of weak solutions $\mathbf{u}_n$, ρ_n constructed in Lemma 5.5. This idea is summarized in the following lemma.

5.6 Lemma. *Let $\Omega_n, \Omega \subset \mathbb{R}^3$, be a bounded domains such that*

$$\Omega_n \subset \Omega_{n+1}, \quad \Omega_n \xrightarrow{b} \Omega \text{ as } n \to \infty.$$

Let the pressure $p = p_n$ is given by

$$p_n(\rho) = \alpha\rho^\gamma + b_n\rho^\beta$$

with

$$\gamma > 3/2,\ \ \beta > 1,\ \ b_n \to 0 \ \textit{as}\ n \to \infty.$$

Let ρ_n*,* $\mathbf{u}_n$ *be solutions to* (2.1)–(2.3)*,* (2.4)–(2.7)*.*

Then there is a sub-sequence such that

$$\rho_n \to \rho \ \textit{in}\ C([0,T_*],\mathbb{R}^3),$$

$$\mathbf{u}_n \to \mathbf{u} \ \textit{weakly in}\ L^2(0,T,W_0^{1,2}),$$

where ρ *and* $\mathbf{u}$ *is a weak solution to* (2.1)–(2.7)*.*

Proof. The proof can be done in a similar way as the proof of Theorem 9.1. in [7], since there is no difficulty arising from a self-deformation of the body. □

Proof of Theorem 3.1. We approximate a general bounded domain Ω by a sequence of smooth domains Ω_n, $\Omega_n \xrightarrow{b} \Omega$, $\Omega_n \subset \Omega_{n+1}$. This approximation exists according to Lemma 7.1 in [13]. According to Lemma 5.5 there exists a solution ρ_n, $\mathbf{u}_n$ on Ω_n which satisfy hypothesis of Lemma 5.6. In order to get a claim of the main result, it suffices to proceed to a limit with $n \to \infty$.

Precisely, similarly as in Section 3.6 we get

$$\rho_n \to \rho \text{ in } C([0,T];L^\gamma_{\text{weak}}(\Omega)) \tag{5.10}$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T,W_0^{1,2}(\Omega)) \tag{5.11}$$

$$\rho_n\mathbf{u}_n \to \rho\mathbf{u} \text{ weakly star in } L^\infty(0,T,L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \tag{5.12}$$

and $\rho,\mathbf{u}$ solve the continuity equation (2.8) in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$. Again we get

$$\mathbf{\Pi}_{[\chi_n,\rho_n,\mathbf{u}_n]} + \mathbf{\Lambda}_{[\chi_n,\rho_n,\mathbf{u}_n]} \to \mathbf{\Pi}_{[\chi,\rho,\mathbf{u}]} + \mathbf{\Lambda}_{[\chi,\rho,\mathbf{u}]} \text{ weakly star in } L^2(W^{1,2}) \tag{5.13}$$

$$\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t \text{ for any } t \in [0,T] \tag{5.14}$$

and $\mathbf{\Pi}_{[\chi,\rho,\mathbf{u}]} + \mathbf{\Lambda}_{[\chi,\rho,\mathbf{u}]} = u\chi_{\mathcal{S}_t}$ for all $t \in [0,T]$. Finally

$$\rho_n\mathbf{u}_n \otimes \mathbf{u}_n \to Q \text{ weakly in } L^2(0,T,L^q(\Omega)),\ q = \frac{6\gamma}{6\gamma - 2\gamma + 3}$$

and

$$Q = \rho\mathbf{u}\otimes\mathbf{u} \text{ a.e. on } P^f.$$

For the pressure we get that

$$\|\rho_n\|^{\gamma+\theta}_{L^{\gamma+\theta}(K^f)} + b_n\|\rho_n\|^{\beta+\theta}_{L^{\beta+\theta}(K^f)} \le C(K^f, E_{n,0}),$$

and

$$p(\rho_n) \to \overline{p(\rho)} = a\overline{\rho^\gamma} \text{ weakly in } L^{\frac{\gamma+\theta}{\gamma}}(\mathcal{K}) \text{ for any compact } K^f \subset P^f.$$

Moreover, the limit functions satisfy the integral identity

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3} (\rho\mathbf{u})\partial_t\varphi + (\rho\mathbf{u}\otimes\mathbf{u})\nabla\varphi + \overline{p(\rho)}\ \operatorname{div}\varphi\ dxdt \\ &= \int_0^t\int_{\mathbb{R}^3} \mathbf{T}(\mathbf{u}) : \mathbf{D}(\varphi) - \rho F\varphi\ dxdt,\end{aligned}$$

for all $\varphi \in \mathcal{R}(S_t)$. Similarly to [7], we can prove the strong convergence of density

$$\rho_n \to \rho \text{ in } C([0,T];L^1(\mathbb{R}^3)).$$

□

6 Appendix

6.1 Lemma. *(Proposition 5.1 [7]) Let* $\vec{u}_n(t, \mathbf{x})$ *be a family of functions such that* $t \to \vec{u}_n(t, \cdot)$ *is continuous from* $[0, T]$ *to* $\mathbb{R}^3$*,* $\mathbf{x} \to \vec{u}_n(\cdot, \mathbf{x})$ *is measurable from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$ *and*

$$t \to \|\vec{u}_n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \vec{u}_n(t, \cdot)\|_{L^\infty(\mathbb{R}^3)},$$

is bounded in $L^2(0, T)$*.*

Let $\vec{\eta}_n[t] : \mathbb{R}^3 \to \mathbb{R}^3$ *be the solution of the problem*

$$\frac{d}{dt}\, \vec{\eta}_n[t](x) = \vec{u}_n(t, \vec{\eta}_n[t](\mathbf{x})), \quad \eta_n[0](\mathbf{x}) = \mathbf{x} \;\; \mathbf{x} \in \mathbb{R}^3.$$

Let also $\mathbf{B}_n \subset \mathbb{R}^3$ *be a sequence such that* $\mathbf{B}_n \overset{b}{\to} \mathbf{B}$*, and denote by* $\mathbf{B}_n(t) = \eta_n[t](\mathbf{B}_n)$ *the image of* $\mathbf{B_n}$ *by the flow* $\vec{u}_n$*.*

Then passing to sub-sequences

$$\eta_n[t] \to \eta[t] \;\; \textit{in } C_{loc}(\mathbb{R}^3) \textit{ as } n \to \infty \;\; \textit{uniformly in } [0, T],$$

where $\eta[t]$ *solves*

$$\frac{d}{dt}\, \vec{\eta}[t](\mathbf{x}) = \vec{u}(t, \vec{\eta}[t](\mathbf{x})), \quad \eta[0](\mathbf{x}) = \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3,$$

and $\vec{u}_n \to \vec{u}$ *weakly-star in* $L^2(0, T; W^{1,\infty}(\mathbb{R}^3))$*.*

Moreover, $\mathbf{B}_n(t) \overset{b}{\to} \mathbf{B}(t)$ *uniformly in* $[0, T]$*, where* $\mathbf{B}(t) = \eta[t](\mathbf{B})$*.*

6.2 Lemma. *(Lemma 4, [18]) Assume* ψ_0 *is the characteristic function of* $\mathcal{S}_0$*. Then, there exists a positive constant* $C = C(\Omega, \mathcal{S}_0, C_1, C_2, \mathcal{A})$ *such that for all* $\rho \in L^\infty((0, T) \times \Omega)$*,* $\mathbf{v} \in L^\infty(0, T; L^2(\Omega))$ *and* $\rho(t, \cdot) \in [C_1, C_2]$ *for a.e.* $t \in [0, T]$*, we have*

$$\left\|\mathbf{\Pi}_{[\psi,\rho,\mathbf{v}]}\right\|_{L^\infty(\Omega)} \leq C \left\|\mathbf{v}\right\|_{L^2(\Omega)},$$

where χ *is the solution of (*5.1*).*

6.3 Lemma. *(Lemma 5 [18]) Assume* ψ_0 *is the characteristic function of* $\mathcal{S}_0$*. Then, there exists a positive constant* $C = C(\Omega, \mathcal{S}_0, C_1, C_2, \mathcal{A})$ *such that for all* $\rho \in L^\infty((0, T) \times \Omega)$*,* $\mathbf{v} \in L^\infty(0, T; L^2(\Omega))$ *such that* $\rho(t, \cdot) \in [C_1, C_2]$ *for a.e.* $t \in [0, T]$*, we have*

$$\left\|\frac{\partial \mathbf{\Lambda}_{[\psi,\rho,\mathbf{v}]}}{\partial t}\right\|_{L^2(0,T;L^\infty(\Omega))} \leq C \left(1 + \|\mathbf{v}\|_{L^2(0,T;L^2(\Omega))}\right),$$
$$\left\|\mathbf{\Lambda}_{[\psi,\rho,\mathbf{v}]}\right\|_{L^\infty(0,T;L^\infty(\Omega))} + \left\|\nabla\mathbf{\Lambda}_{[\psi,\rho,\mathbf{v}]}\right\|_{L^\infty(0,T;L^\infty(\Omega))} + \left\|\Delta\mathbf{\Lambda}_{[\psi,\rho,\mathbf{v}]}\right\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C.$$

6.4 Lemma. *(Lemma 8, [18]) Assume* $\mathbf{v} \in L^\infty(0, T; L^2_\sigma(\Omega))$*,* $\rho_{0\varepsilon} \in C^\infty(\mathbb{R}^3)$*,* $\rho_{0\varepsilon} \in [C_1, C_2] \subset (0, \infty)$ *for a.e.* $x \in \mathbb{R}^3$*,* $\psi_0 \in Char(\mathbb{R}^3)$*, and* $\mathcal{S}(\psi_0)$ *is bounded and of nonempty interior. Then the problem* 5.1 *admits a unique solution* $(\rho, \psi) \in L^\infty((0, T) \times \mathbb{R}^3)$*. Moreover, for a.e.* $t \in (0, T)$*,*

$$\rho(t) \in [C_1, C_2] \quad \textit{for a.e. } x \in \mathbb{R}^3, \quad \psi(t) \in Char(\mathbb{R}^3). \tag{6.1}$$

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