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Olsen's problem and essentially power bounded operators

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OLSEN'S PROBLEM AND ESSENTIALLY POWER BOUNDED OPERATORS

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Abstract. Let T be a non-Riesz operator on an infinite-dimensional Hilbert space. Then there exists a compact operator K such that $\|(T+K)^n\| = \|T^n\|_e$ for all n. In particular, every essentially power bounded operator is a compact perturbation of a power bounded operator.

1. Introduction

Let H be an infinite-dimensional Hilbert space. Denote by $B(H)$, $\mathcal{K}(H)$ and $\mathcal{F}(H)$ the set of all bounded, compact and finite-rank operators on H, respectively.

For $T \in B(H)$ denote by $\|T\|_e$ the essential norm of T, i.e., $\|T\|_e = \inf\{\|T+K\| : K \in \mathcal{K}(H)\}$. Let $\sigma_e(T)$ and $r_e(T)$ denote the essential spectrum and essential spectral radius, respectively, i.e., the spectrum and the spectral radius of the class $T + \mathcal{K}(H)$ in the Calkin algebra $B(H)/\mathcal{K}(H)$. Recall that $T \in B(H)$ is called Riesz if $r_e(T) = 0$.

For a subspace $M \subset H$ denote by P_M the orthogonal projection onto M.

Properties of an operator $T \in B(H)$ can be frequently improved by a suitable compact perturbation. By [W], any Riesz operator $T \in B(H)$ can be written as a sum $T = Q + K$ where K is a compact and Q quasinilpotent operator. More generally, by [S], for any $T \in B(H)$ there exists $K \in \mathcal{K}(H)$ such that $\sigma(T+K)$ is equal to the Weyl spectrum of T,

$$\sigma_W(T) = \bigcap\{\sigma(T+K) : K \in \mathcal{K}(H)\}.$$

By [CSSW], for any $T \in B(H)$ there exists a compact operator $K \in \mathcal{K}(H)$ such that the closure of the numerical range $\overline{W(T+K)}$ is equal to the essential numerical range $W_e(T)$.

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By [O1], if $T \in B(H)$ is polynomially compact, i.e., $p(T) \in \mathcal{K}(H)$ for some polynomial p, then there exists $K \in \mathcal{K}(H)$ such that $T+K$ is algebraic, $p(T+K)=0$ (with the same polynomial p).

The following problem was raised by C. Olsen [O2], cf. also [O1], [OP]:

Problem. Let $T \in B(H)$. Does there exist a compact operator $K \in \mathcal{K}(H)$ such that $\|p(T+K)\| = \|p(T)\|_e$ for all polynomials p? Less ambitiously, if T and p are both given, is there a compact operator K_p such that $\|p(T+K_p)\| = \|p(T)\|_e$?

Even the more modest formulation of the problem is surprisingly difficult. A positive answer was given in [O2] for the polynomials $p(z) = z, z^2, z^3$ and in [CLSW] for all linear polynomials. In [M], a positive answer was given for any power $p(z) = z^n$.

In the present paper we refine the estimates of [M] and show that for any non-Riesz operator $T \in B(H)$ there exists a compact operator $K \in \mathcal{K}(H)$ such that $\|(T+K)^n\| = \|T^n\|_e$ for all n simultaneously.

In particular, this implies that for any essentially power bounded operator $T \in B(H)$, (i.e., $\sup_n \|T^n\|_e < \infty$) there exists a compact operator K such that $T+K$ is power bounded, $\sup_n \|(T+K)^n\| < \infty$.

2. Main result

We need several lemmas that were proved in [CLSW] and [M].

Proposition 1. (see [M], Proposition 6) Let H be a separable infinite-dimensional Hilbert space, let $(e_1, e_2, \dots)$ be an orthonormal basis in H. Let $S \in B(H)$. Then

$$\|S\|_e = \lim_{k\to\infty} \|P_{H_k^\perp} S P_{H_k^\perp}\|,$$

where $H_k = \bigvee_{j=1}^k e_j \qquad (k \in \mathbb{N})$.

The next lemma is a slight modification of [M], Lemma 8, cf. also [CLSW], Lemma 5. For the sake of convenience we give a proof here.

Lemma 2. Let H be a separable infinite-dimensional Hilbert space, $S_1, \dots, S_n \in B(H)$. Let $F_0 \subset H$ be a finite-dimensional subspace. Then there exist mutually orthogonal finite-dimensional subspaces $F_k \subset H$ such that $H = \bigoplus_{k=0}^\infty F_k$ and $P_{F_r} S_j P_{F_s} = 0$ for all $r, s \in \mathbb{N}$, $|r-s| \ge 2$ and $j = 1, \dots, n$ (i.e., the operators $S_1 \dots, S_n$ are simultaneously block 3-diagonal).

Proof. Let $(e_1, e_2, \dots)$ be an orthonormal basis in H.

We construct the subspaces F_k inductively. If $k \geq 1$ and the subspaces $F_0, \dots, F_{k-1}$ have already been constructed, then set

$$G_k = \bigvee \{F_0, \dots, F_{k-1}, S_j F_{k-1}, S_j^* F_{k-1} \quad (j = 1, \dots, n), e_k\}$$

and $F_k = G_k \ominus (F_0 \oplus \cdots \oplus F_{k-1})$. Then $\dim F_k < \infty$, $S_j F_{k-1} \subset F_0 \oplus \cdots \oplus F_k$ and $S_j^* F_{k-1} \subset F_0 \oplus \cdots \oplus F_k$ for all $j = 1, \dots, n$.

If we continue this construction for all $k \in \mathbb{N}$ then we get the required decomposition. Note that $\bigoplus_{k=0}^{\infty} F_k = H$ since $e_k \in F_0 \oplus \cdots \oplus F_k$ for each k.

Let $r, s \geq 0$, $|r - s| \geq 2$. If $r > s$ then $P_{F_r} S_j P_{F_s} = 0$. If $r < s$ then $P_{F_s} S_j^* P_{F_r} = 0$, and so $P_{F_r} S_j P_{F_s} = 0$. □

Lemma 3. (see [M], Lemma 9, cf. also [CLSW], Lemma 6) Let $S \in B(H)$ be block 3-diagonal, i.e., there are mutually orthogonal finite-dimensional subspaces F_j such that $H = \bigoplus_{j=0}^{\infty} F_j$ and $P_{F_r} S P_{F_s} = 0$ whenever $|r - s| \geq 2$. Denote by Q_k the orthogonal projection onto $\bigoplus_{j=k+1}^{\infty} F_j$.

Let $l, d \in \mathbb{N}$, $k = l + 2d$ and let $V \in B(H)$ satisfy $V = Q_k V Q_k$. Then $\|S + V\| \leq \max\{\|S\|, \|Q_l(S + V)Q_l\|\} + \frac{\|S\|}{\sqrt{d}}$.

The next result is a modification of [M], Theorem 16.

Theorem 4. Let H be an infinite-dimensional separable Hilbert space, $S \in B(H)$, $m \in \mathbb{N}$, $\|S^m\|_e \neq 0$, $0 \leq r < r' < 1$. Let $T \in B(H)$ satisfy $\|T^j\| < \|S^j\|_e \quad (j = 1, \dots, m)$ and $T - rS \in \mathcal{F}(H)$. Then there exists $T' \in B(H)$ such that

$$\|T - T'\| \leq (r' - r)\|S\|,$$

$$T' - r'S \in \mathcal{F}(H)$$

and

$$\|T'^j\| < \|S^j\|_e \qquad (j = 1, \dots, m).$$

Proof. Without loss of generality we may assume that $\|S\| = 1$.

Let $F_0 = R(T - rS) \vee R(T^* - rS^*)$. By Lemma 2, there exist mutually orthogonal finite-dimensional subspaces $F_1, F_2, \dots$ such that $H = \bigoplus_{i=0}^{\infty} F_i$ and $S, S^2, \dots, S^m$ are simultanously block 3-diagonal with respect to this decomposition. Denote by Q_l the orthogonal projection onto $\bigoplus_{i=l+1}^{\infty} F_i$. We have $Q_0(T - rS)Q_0 = 0$.

Choose ε such that

$$0 < \varepsilon < \frac{1}{3} \min\{\|S^j\|_e - \|T^j\|, (1 - r')\|S^j\|_e \;\; (j = 1, \dots, m)\}.$$

Choose $n \in \mathbb{N}$ such that

$$\frac{(2^m-1)(r'-r)}{n} < \varepsilon$$

and $d \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{d}} < \frac{\varepsilon}{n}.$$

By Proposition 1, there exists k_0 such that $\|Q_l S^j Q_l\| < \|S^j\|_e + \varepsilon$ for all $l \geq k_0$ and $j = 1, \dots, m$.

For $t = 0, \dots, n$ let $s_t = r + \frac{t(r'-r)}{n}$. So $s_0 = r$ and $s_n = r'$. Choose numbers $k_1, \dots, k_n \in \mathbb{N}$ such that $k_{t+1} > k_t + 2d + 1$.

Define inductively operators $S_0, \dots, S_n$ by $S_0 = T$ and

$$S_{t+1} = S_t + (s_{t+1} - s_t) Q_{k_{t+1}} S Q_{k_{t+1}}.$$

Let $T' = S_n$.

Clearly $\|S_{t+1} - S_t\| \leq (s_{t+1} - s_t)\|S\| \leq \frac{r'-r}{n}$. So

$$\|T' - T\| \leq \|S_n - S_{n-1}\| + \dots + \|S_1 - S_0\| \leq r' - r.$$

For each t we have $Q_{k_t}(S_t - s_t S)Q_{k_t} = 0$. In particular, $Q_{k_n}(T' - r'S)Q_{k_n} = 0$, and so $T' - r'S \in \mathcal{F}(H)$.

We prove by induction on t that

$$\|S_t^j\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n} \qquad (j = 1, \dots, m). \tag{1}$$

For $t = 0$ this follows by the definition of ε. Suppose that (1) is true for some t, $0 \leq t \leq n-1$. Fix $j \in \{1, \dots, m\}$. By Lemma 3,

$$\|S_{t+1}^j\| \leq \max\{\|S_t^j\|, \|Q_{k_{t+1}-2d} S_{t+1}^j Q_{k_{t+1}-2d}\|\} + \frac{\|S\|}{\sqrt{d}},$$

where $\|S_t^j\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n}$ by the induction assumption and $\frac{\|S\|}{\sqrt{d}} < \frac{\varepsilon}{n}$. So it is sufficient to show that

$$\|Q_{k_{t+1}-2d} S_{t+1}^j Q_{k_{t+1}-2d}\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n}.$$

Write $V = S_{t+1} - S_t = (s_{t+1} - s_t)\|Q_{k_{t+1}} S^j Q_{k_{t+1}}\|$. Clearly $\|V\| \leq (s_{t+1} - s_t)\|S\| = \frac{r'-r}{n}$. Since $S_{t+1}^j - S_t^j = (S_t + V)^j - S_t^j$ can be expressed as a sum of $2^j - 1$ products, each of them containing V, we have

$$\|S_{t+1}^j - S_t^j\| \leq (2^j - 1)\frac{r'-r}{n} \leq \frac{(2^m-1)(r'-r)}{n} < \varepsilon.$$

For $x \in \bigoplus_{i=k_{t+1}-2d+1}^{\infty} F_i$ we have $Sx, S^2x, \dots, S^jx \in \bigoplus_{i=k_{t+1}-2d}^{\infty} F_i \subset \bigoplus_{i=k_t+2}^{\infty} F_i$. Moreover, S_t behaves on $\bigoplus_{i=k_t+2}^{\infty} F_i$ as $s_t S$. Thus $S_t^j x =$

$s_t^j S^j x$. Hence

$$\begin{aligned}\|Q_{k_{t+1}-2d}S_{t+1}^jQ_{k_{t+1}-2d}\| &\le \|Q_{k_{t+1}-2d}S_t^jQ_{k_{t+1}-2d}\|+\varepsilon\\ &= s_t^j\|Q_{k_{t+1}-2d}S^jQ_{k_{t+1}-2d}\|+\varepsilon \le s_t(\|S^j\|_e+\varepsilon)+\varepsilon\\ \le s_t\|S^j\|_e+2\varepsilon &\le \|S^j\|_e+2\varepsilon-(1-r')\|S^j\|_e \le \|S^j\|_e-\varepsilon.\end{aligned}$$

This proves (1).

In particular, for $t=n$ we get $\|T'^j\| < \|S^j\|_e \quad (j=1,\dots,m)$. □

Let (c_j) be a submultiplicative sequence of positive numbers, i.e., $c_{i+j} \le c_i c_j$ for all $i,j \in \mathbb{N}$. It is well known that the limit $\lim_{j\to\infty} c_j^{1/j}$ exists and it is equal to $\inf_j c_j^{1/j}$.

We need the following lemma.

Lemma 5. Let (c_j) be a submultiplicative sequence of positive numbers such that $\lim_{j\to\infty} c_j^{1/j} = 1$. Let $0 < r < r' < 1$. Then there exists $k \in \mathbb{N}$ with the following property: if (d_j) is a submultiplicative sequence satisfying

$$0 \le d_j \le c_j \qquad (j=1,2,\dots)$$

and

$$d_j < r^j c_j \qquad (1 \le j \le k)$$

then

$$d_j < r'^j c_j \qquad (j=1,2,\dots).$$

Proof. Choose r'', $r < r'' < r'$. Find k_0 such that $c_{k_0} < \left(\frac{r''}{r}\right)^{k_0}$.

Let $M = \max\{1, c_1, \dots, c_{k_0-1}\}$ and $L = \frac{M}{r''^{k_0}}$. Then there exists $k \ge k_0$ such that $Lr''^k < r'^k$.

Let (d_j) be a submultiplicative sequence satisfying $d_j \le c_j \quad (j\in\mathbb{N})$ and $d_j < r^j c_j \quad (1 \le j \le k)$. For $1 \le j \le k$ we have $d_j < r^j c_j < r'^j c_j$.

Let $j > k$. Then $j = sk_0 + z$ for some $s \in \mathbb{N}$ and $z \in \{0,1,\dots,k_0-1\}$. Then

$$d_j \le d_{k_0}^s \cdot d_z < (r^{k_0}c_{k_0})^s \cdot c_z < M\cdot r''^{k_0 s} = \frac{Mr''^j}{r''^z} \le Lr''^j < r'^j \le r'^j c_j.$$

□

Theorem 6. Let H be an infinite-dimensional Hilbert space. Let $S \in B(H)$ satisfy $r_e(S) \ne 0$. Then there exists a compact operator $K \in B(H)$ such that

$$\|(S+K)^j\| = \|S^j\|_e$$

for all $j = 1, 2, \dots$.

Proof. Without loss of generality we may assume that H is separable. Indeed, there exists a decomposition $H = \bigoplus_{\nu \in J} H_\nu$ such that all the subspaces H_ν are separable and reducing for S. Write $S_\nu = P_{H_\nu} S P_{H_\nu}$.

For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there are only finitely many $\nu \in J$ such that $\|S_\nu^n\| > \|S^n\|_e + \varepsilon$. So there are only countable many $\nu \in J$ such that $\|S_\nu^n\| > \|S^n\|_e$. Hence there exists a countable subset $J_0 \subset J$ such that $\|S_\nu^n\| \le \|S^n\|_e$ for all $\nu \notin J_0$ and $n \in \mathbb{N}$. Let $H_0 = \bigoplus_{\nu \in J_0} H_\nu$. Then H_0 is a separable subspace reducing for T and $\|P_{H \ominus H_0} S^n P_{H \ominus H_0}\| \le \|S^n\|_e$ for all $n \in \mathbb{N}$. So we may consider only the operator $P_{H_0} S P_{H_0}$.

Without loss of generality we may assume that $r_e(S) = 1$. Fix a sequence (r_n) such that $0 = r_0 < r_1 < \cdots < 1$ and $\lim_{n \to \infty} r_n = 1$.

Consider the submultiplicative sequence $c_n = \|S^n\|_e$. Then $1 = r_e(S) = \lim_{j \to \infty} \|S^j\|_e^{1/j} = \lim_{j \to \infty} c_j^{1/j}$.

For $n = 1, 2, \dots$, let k_n be the number constructed in Lemma 5 for the sequence (c_j) and numbers r_n, r_{n+1}. Thus, if (d_j) is a submultiplicative sequence satisfying $d_j \le c_j$ for all j and $d_j < r_n^j c_j \quad (1 \le j \le k_n)$ then $d_j < r_{n+1}^j c_j$ for all $j \in \mathbb{N}$.

Construct inductively a sequence (T_n) of operators such that

$$\|T_n^j\| < r_n^j \|S^j\|_e \qquad (j = 1, \dots, k_n),$$

$$\|T_{n+1} - T_n\| \le (r_{n+1} - r_n)\|S\|$$

and

$$T_n - r_{n-1} S \in \mathcal{F}(H).$$

Set $T_1 = 0$. Suppose that $n \ge 1$ and $T_1, \dots, T_n \in B(H)$ with the above properties have already been constructed. Since $\|T_n^j\| < r_n^j \|S^j\|_e \quad (1 \le j \le k_n)$, we have $\|T_n^j\| < r_{n+1}^j \|S^j\|_e$ for all $j \in \mathbb{N}$.

By Theorem 4 for the operator $r_{n+1} S$ and the pair of numbers $\frac{r_{n-1}}{r_{n+1}}, \frac{r_n}{r_{n+1}}$, there exists $T_{n+1} \in B(H)$ such that

$$\|T_{n+1} - T_n\| \le (r_n - r_{n-1})\|S\|,$$

$$T_{n+1} - r_n S \in \mathcal{F}(H)$$

and

$$\|T_{n+1}^j\| < r_{n+1}^j \|S^j\|_e \qquad (1 \le j \le k_{n+1}).$$

Let (T_n) be the sequence of operators satisfying the above properties. Clearly (T_n) is a Cauchy sequence. Let T be the norm-limit of the sequence (T_n) and $K = T - S$. For each $j \in \mathbb{N}$ we have

$$\|T^j\| = \lim_{n \to \infty} \|T_n^j\| \le \lim_{n \to \infty} r_n^j \|S^j\|_e \le \|S^j\|_e.$$

Moreover,

$$K = \lim_{n\to\infty} T_n - S = \lim_{n\to\infty} (T_n - r_{n-1}S),$$

where $T_n - r_{n-1}S \in \mathcal{F}(H)$ for all n. So K is a compact operator and $\|T^j\| = \|S^j\|_e$ for all j. □

Problem 7. Is the statement of Theorem 6 true for Riesz operators?

Recall that $T \in B(H)$ is called power bounded if $\sup_n \|T^n\| < \infty$. An operator $S \in B(H)$ is called essentially power bounded if $\sup_n \|S^n\|_e < \infty$.

Essentially power bounded operators may serve as a source of interesting examples of Hilbert space operators. Note that for example the Read-type operator constructed by S. Grivaux and M. Roginskaya [GR] is of this class.

Theorem 6 has the following simple consequence.

Corollary 8. An operator $S \in B(H)$ is essentially power bounded if and only if $S = T + K$ for some power bounded operator $T \in B(H)$ and a compact operator $K \in B(H)$.

Proof. If $\sup_n \|T^n\| < \infty$ and $K \in B(H)$ is compact, then $\|(T + K)^n\|_e \le \|T^n\|$ for all n, and so $T + K$ is essentially power bounded.

Let $S \in B(H)$ be essentially power bounded. If $r_e(S) \ne 0$ then $S = T + K$ for some compact operator K and an operator T satisfying $\|T^n\| = \|S^n\|_e$ for all $n \in \mathbb{N}$. So T is power bounded.

If $r_e(s) = 0$, then $S = T + K$ for some $K \in \mathcal{K}(H)$ and a quasinilpotent $T \in B(H)$ by the West decomposition. Clearly then T is power bounded. □

References

[CLSW] C.K. Chui, D.A. Legg, R.R. Smith, J.D. Ward: *On a question of Olsen concerning compact perturbations of operators*, Michigan Math. J. 24 (1977), 119–127.

[CSSW] C.K. Chui, P.W. Smith, R.R. Smith, J.D. Ward: *L-ideals and numerical range preservation*, Illinois J. Math. 21 (1977), 365–373.

[GR] S. Grivaux, M. Roginskaya: *On Read's type operators on Hilbert spaces*, Int. Mat. Res. Not. IMRN 2008.

[M] V. Müller: *The joint essential numerical range, compact perturbations, and the Olsen problem*, Studia Math. 197 (2010), 275–290.

[O1] C.L. Olsen: *A structure theorem for polynomially compact operators*, Amer. J. Math. 93 (1971), 686–698.

[O2] C.L. Olsen: *Norms of compact perturbations of operators*, Pacific J. Math. 68 (1977), 209–228.

[OP] C.L. Olsen, J.K. Plastiras: *Quasialgebraic operators, compact perturbations, and the essential norm*, Michigan Math. J. 21 (1974), 385–397.

[S] J.G. Stampfli: *Compact perturbations, normal eigenvalues and a problem of Salinas*, J. London Math. Soc. 2 (1974), 165–175.

[W] T.T. West: *The decomposition of Riesz operators*, Proc. London Math. Soc. 16 (1966), 737–752.

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