The method of relative entropies and the Navier-Stokes-Fourier system

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Relative entropy (energy)

DYNAMICAL SYSTEM:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = A(t, u(t)),\ u(t) \in X,\ u(0) = u_0$$

RELATIVE ENTROPY:

$U : t \mapsto U(t) \in Y$ a "trajectory" in the phase space $Y \subset X$

$$\mathcal{E}\left\{ u(t) \middle| U(t) \right\},\ \mathcal{E} : X \times Y \to R$$

- $\mathcal{E}\left\{u\ |U\right\}$ is a "distance" between u, and U, meaning $\mathcal{E}(u,U) \geq 0$ and $\mathcal{E}\left\{u|U\right\} = 0$ only if $u = U$
- $\mathcal{E}\left\{u(t)|\tilde{U}\right\}$ is a Lyapunov function provided $\tilde{U}$ is a equilibrium, meaning

$$A(t,\tilde{U}) = 0 \text{ for all } t.$$

$$t \mapsto \mathcal{E}\left\{u(t)\Big|\tilde{U}\right\} \text{ is non-increasing}$$

- $$\mathcal{E}\left\{u(\tau)\Big|U(\tau)\right\} \leq \mathcal{E}\left\{u(s)\Big|U(s)\right\} + c(T)\int_s^\tau \mathcal{E}\left\{u(t)\Big|U(t)\right\}\,\mathrm{d}t$$

  if U is a solution of the same system (ranging in a "better" space) Y

Applications

- STABILITY OF EQUILIBRIA. Any solution ranging in X stabilizes to an equilibrium belonging to Y (to be proved!)
- WEAK-STRONG UNIQUENESS. Solutions ranging in the "better" space Y are unique among solutions in X.
- SINGULAR LIMITS. Stability and convergence of a family of solutions u_ε corresponding to A_ε to a solution $U = u$ of the limit problem with generator A.

EQUATION OF CONTINUITY:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

MOMENTUM BALANCE:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\mathbf{u}|_{\partial\Omega} = 0, \ \text{or } [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

ENTROPY PRODUCTION:

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

TOTAL ENERGY CONSERVATION:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \ \mathrm{d}x = 0$$

Constitutive relations

Gibbs' relation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \frac{1}{\varrho}$$

Newton's rheological law:

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law:

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Entropy production rate:

$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Equilibrium (static) solutions

Equilibrium solutions minimize the entropy production:

$$\mathbf{u} \equiv 0, \ \vartheta \equiv \overline{\vartheta} > 0 \text{ a positive constant}$$

Static problem:

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x F$$

Total mass and energy are constants of motion:

$$\int_\Omega \tilde{\varrho} \ \mathrm{d}x = M_0, \ \int_\Omega \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) - \tilde{\varrho} F \ \mathrm{d}x = E_0$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho,\vartheta) - \frac{\partial H_{\overline{\vartheta}}(\tilde\varrho,\overline{\vartheta})}{\partial \varrho}(\varrho - \tilde\varrho) - H_{\overline{\vartheta}}(\tilde\varrho,\overline{\vartheta}) \right)\ \mathrm{d}x$$

$$+ \int_\Omega \sigma\ \mathrm{d}x = 0$$

Ballistic free energy:

$$H_{\overline{\vartheta}}(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \overline{\vartheta} s(\varrho,\vartheta)\Big)$$

Thermodynamic stability

Positive compressibility:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

Positive specific heat at constant volume:

$$\frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Coercivity of the ballistic free energy:

- $\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$ convex
- $\vartheta \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta)$ attains its global minimum (zero) at $\vartheta = \overline{\vartheta}$

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right) \, \mathrm{d}x$$

RELATIVE ENTROPY INEQUALITY

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle| r,\Theta,\mathbf{U}\right)\right]_{t=0}^{\tau}$$

$$+\int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho,\vartheta,\mathbf{u},r,\Theta,\mathbf{U})\,\mathrm{d}t$$

for any $r>0$, $\Theta>0$, $\mathbf{U}$ satisfying relevant boundary conditions

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Calculations

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 \ \mathrm{d}x = \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho\mathbf{u}\cdot\mathbf{U} + \frac{1}{2}\varrho|\mathbf{U}|^2 \ \mathrm{d}x$$

$$\int_\Omega H_\Theta(\varrho,\vartheta) \ \mathrm{d}x = \int_\Omega \left(\varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)\right) \ \mathrm{d}x$$

$$\int_\Omega \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) \ \mathrm{d}x$$

Applications

- UNCONDITIONAL STABILITY OF THE EQUILIBRIUM SOLUTIONS. Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for $t \to \infty$.
- WEAK-STRONG UNIQUENESS. Weak and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions.
- SINGULAR LIMIT IN THE INCOMPRESSIBLE, INVISCID REGIME. Solutions of the Navier-Stokes-Fourier system converge in the limit of low Mach and high Reynolds and Péclet number to the Euler-Boussinesq system.

Scaled Navier-Stokes-Fourier system

EQUATION OF CONTINUITY

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

BALANCE OF MOMENTUM

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}$$

ENTROPY PRODUCTION

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Target system

INCOMPRESSIBILITY

$$\mathrm{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

BASIC ASSUMPTION

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\rm max})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

NAVIER'S COMPLETE SLIP CONDITION

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Convergence

$$b > 0, \ 0 < a < \frac{10}{3}$$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \varrho_\varepsilon(t, \cdot) - \overline{\varrho} \|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \text{ in } \boxed{L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3))}$$

$$\text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega; R^3))$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \text{ in } \boxed{L^\infty_{\rm loc}((0,T]; L^q_{\rm loc}(\Omega; R^3)), \ 1 \leq q < 2},$$

$$\text{and weakly-(*) in } L^\infty(0,T; L^2(\Omega))$$

Uniform bounds

The uniform bounds independent of ε are obtained by means of the choice

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality:

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; R^3)} \leq c$$

Linearization

$$\varepsilon \partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)} + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \varepsilon \mathbf{f}_1$$

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+ \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = \varepsilon f_2$$

Main idea of the proof

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Dispersion of acoustic waves

$-\Delta_N$ Neumann Laplacian

$$\partial^2_{t,t}\Phi - \omega\Delta_N\Phi = 0$$

Hypotheses imposed on Ω

- Limiting absorption principle. The operator Δ_N satisfies the limiting absorption principle in Ω:

$$\varphi \circ [-\Delta_N^{-1} - \lambda \pm \mathrm{i}\delta]^{-1} \circ \varphi, \ \varphi \in C_c^\infty(\Omega) \text{ bounded in } L^2(\Omega)$$

  for λ belonging to compact subintervals of $(0, \infty)$, $\delta > 0$.
- There is a compact set B such that Δ_N satisfies the Strichartz estimates on $D = \Omega \cup B$.
- The operator Δ_N satisfies the local energy decay.

Strichartz estimates and local energy decay

$$\|\Phi\|_{L^p(R;L^q(D))} \leq c\left(\|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)}\right)$$

$$2 \leq q < \infty,\ \frac{2}{p} \leq \left(1 - \frac{2}{q}\right),\ \gamma = \frac{3}{2} - \frac{3}{q} - \frac{1}{p}$$

$$\int_{-\infty}^{\infty} \left(\|\chi\Phi(t,\cdot)\|^2_{H^\gamma(D)} + \|\chi\partial_t\Phi(t,\cdot)\|^2_{H^{\gamma-1}(D)}\right)\ \mathrm{d}t$$

$$\leq c\left(\|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)}\right),$$

$\chi \in C^\infty_c(D)$.