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Outward pointing inverse Preisach operators

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Abstract

In some recent papers, the concept of outward pointing operators has been used as a tool within the mathematical investigation of equations involving hysteresis operators and has been considered for the stop operator, the play operator, Prandtl-Ishlinskiĭ operators and Preisach operators. Now, for inverse Preisach operators conditions will be discussed which ensure that the outward pointing property is satisfied. As an application of the theory, a stability result for solutions of a P.D.E. with hysteresis appearing in the context of electromagnetic processes is derived.
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1. Outward pointing operators

We deal with mappings $\mathscr{H}:\mathscr{C}(\mathbb{R}_+)\to\mathscr{C}(\mathbb{R}_+)$, where $\mathscr{C}(\mathbb{R}_+)$ denotes the space of continuous functions $u:\mathbb{R}_+\to\mathbb{R}$, $\mathbb{R}_+=[0,\infty)$, endowed with the family of seminorms

$$|u|_{[0,t]}=\max\{|u(s)|;0\leqslant s\leqslant t\}.$$

We start with the following definitions.

Definition. A mapping $\mathscr{H}:\mathscr{C}(\mathbb{R}_+)\to\mathscr{C}(\mathbb{R}_+)$ is said to be:

(i) *pointing outwards with bound h in the δ-neighbourhood of* $[A,B]$ *for initial values in* $[a,b]$, if for every $t\geqslant 0$ and every $u\in\mathscr{C}(\mathbb{R}_+)$ such that

$$u(0)\in[a,b],\quad u(s)\in(A-\delta,B+\delta)\quad\forall s\in[0,t], \tag{1}$$

we have

$$\begin{cases}(\mathscr{H}[u](t)-h)(u(t)-B)^+\geqslant 0,\\(\mathscr{H}[u](t)+h)(u(t)-A)^-\leqslant 0\end{cases} \tag{2}$$

for some given values of $\delta>0$, $h\geqslant 0$, $A\leqslant a\leqslant b\leqslant B$, where $z^+=\max\{z,0\}$ and $z^-=\max\{-z,0\}$ for $z\in\mathbb{R}$ denote the positive and negative part of z, respectively.

(ii) *strongly pointing outwards for all bounds* if and only if for all $a\leqslant b$, all $\delta>0$ and all $h\geqslant 0$ one can find some $A\leqslant a$ and $B\geqslant b$ such that $\mathscr{H}$ is pointing outwards with bound h in the δ-neighbourhood of $[A,B]$ for initial values in $[a,b]$.

Condition (2) can be explained as follows: if the value of u at time t exceeds the limits $[A,B]$ and the values of u in the past remained in a δ-neighbourhood of $[A,B]$, then $\mathscr{H}[u](t)$ "points outwards" with respect to the relative position of $u(t)$ and the interval $[A,B]$, that is

$$\begin{cases}u(t)>B\Rightarrow\mathscr{H}[u](t)\geqslant h\geqslant 0,\\u(t)<A\Rightarrow\mathscr{H}[u](t)\leqslant -h\leqslant 0.\end{cases} \tag{3}$$

For more details on the concept of outward pointing operators, see for example [1–4].

2. The Preisach operator

We denote by $W^{1,\infty}_C(\mathbb{R}_+)$ the space of Lipschitz continuous functions $\lambda:\mathbb{R}_+\to\mathbb{R}$ with compact support in $\mathbb{R}_+$ and introduce the set

$$\Lambda=\{\lambda\in W^{1,\infty}_C(\mathbb{R}_+):|\lambda'(r)|\leqslant 1\text{ a.e.}\}.$$

The play operator $\wp_r[\lambda,u]$ generates for every $t\geqslant 0$ a continuous state mapping $\Pi_t:\Lambda\times\mathscr{C}(\mathbb{R}_+)\to\Lambda$ which with each $(\lambda,u)\in\Lambda\times\mathscr{C}(\mathbb{R}_+)$ associates the state $\lambda_t\in\Lambda$ at time

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t. We recall that, following [5], we define the value of the play first for piecewise monotone functions u defined on a bounded time interval $[0,T]$ recurrently in each monotonicity interval, i.e. for $\lambda \in \Lambda$

$$\lambda_0(r) = \min\{u(0)+r, \max\{u(0)-r, \lambda(r)\}\}$$

and for each monotonicity interval $[t_*, t^*] \subset [0,T]$ of u

$$\lambda_t(r) = \min\{u(t)+r, \max\{u(t)-r, \lambda_{t_*}(r)\}\}.$$

By density, this definition is then extended to the whole space $\mathscr{C}(\mathbb{R}_+)$.

Now, consider two given functions $g : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ so that, if we set

$$\Psi(r,s) = \int_0^s \psi(r,\sigma)\,\mathrm{d}\sigma,$$

the following assumptions are satisfied:

Assumption 1. (i) The function g is continuous and nondecreasing;

(ii) the function ψ is locally integrable and nonnegative in $\mathbb{R}_+ \times \mathbb{R}$, $\Psi(r,0) = 0$ for a.e. $r > 0$.

For $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$, $\lambda \in \Lambda$ and $t \geqslant 0$ put

$$\mathscr{H}[\lambda,u](t) = g(u(t)) + \int_0^\infty \Psi(r, \wp_r[\lambda,u](t))\,\mathrm{d}r. \tag{4}$$

The mapping $\mathscr{H} : \Lambda \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+) \to \mathscr{C}(\mathbb{R}_+)$ is called *Preisach operator*. We also introduce the *Preisach initial loading curve* $\Phi : \mathbb{R} \to \mathbb{R}$ by the formula

$$\Phi(v) = g(v) + \int_0^{|v|} \Psi(r, v - r\,\mathrm{sign}(v))\,\mathrm{d}r$$
$$= g(v) + \begin{cases} \int_0^v \int_0^{v-r} \psi(r,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r & \text{if } v \geqslant 0, \\ \int_0^{-v} \int_0^{v+r} \psi(r,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}r & \text{if } v < 0. \end{cases} \tag{5}$$

Moreover let us set

$$A(r,u(0)) := \max\{u(0)-r, \min\{u(0)+r, \lambda(r)\}\}$$

and introduce the following initial value mapping

$$\mathscr{A} : \mathbb{R} \to \mathbb{R}$$
$$u(0) \mapsto g(u(0)) + \int_0^\infty \psi(r, A(r,u(0)))\,\mathrm{d}r \tag{6}$$

which with the initial input value $u(0)$ associates the initial output value $\mathscr{H}[\lambda,u](0)$.

Let us recall the following result (see Ref. [6, Section II.3, Theorem 3.17], see also Ref. [7, Theorem 5.8]).

Theorem 2. *Let Assumption* 1 *be fulfilled and let* $\lambda \in \Lambda$ *and* $\beta > 0$ *be given. Then the operator* $\beta I + \mathscr{H}[\lambda,\cdot]$ *where* I *is the identity operator and* $\mathscr{H}[\lambda,\cdot]$ *is the Preisach operator introduced in Eq.* (4) *is invertible and its inverse is Lipschitz continuous and clockwise oriented. Moreover the initial value mapping* $\beta I + \mathscr{A}$ *is increasing and bi-Lipschitz.*

From now on we set, for $\lambda \in \Lambda$ and $u \in \mathscr{C}(\mathbb{R}_+)$

$$\mathscr{G}[\lambda,u](t) := (I + \mathscr{H}[\lambda,\cdot])^{-1}(u)(t). \tag{7}$$

3. Outward pointing inverse Preisach

The main result of the section is the following.

Theorem 3. *Let* $\lambda \in \Lambda$ *and Assumption* 1 *be satisfied. Then* $\mathscr{G}[\lambda,\cdot]$ *is strongly pointing outwards for all bounds.*

Proof. Let us fix some $a \leqslant b$, some $\delta > 0$ and some $h \geqslant 0$. We check that $\mathscr{G}[\lambda,\cdot]$ satisfies (3) for suitable A and B.

First set $f_H := I + \mathscr{A}$, where $\mathscr{A}$ is the initial value mapping for the Preisach operator (4) introduced in Eq. (6). Let us set

$$R_0 := \max\{K, |f_H^{-1}(a)|, |f_H^{-1}(b)|\},$$

where $K > 0$ is such that $\lambda(r) = 0$ for all $r \geqslant K$. Now set

$$R := \max\{R_0, h+\delta\}. \tag{8}$$

We define

$$B := h + g(h) + \iint_{\mathscr{T}_+} \psi(r,v)\,\mathrm{d}r\,\mathrm{d}v$$

and

$$A := -h + g(-h) - \iint_{\mathscr{T}_-} \psi(r,v)\,\mathrm{d}r\,\mathrm{d}v,$$

where

$$\mathscr{T}_+ := \{(r,v) : 0 \leqslant r \leqslant R; 0 \leqslant v \leqslant \min\{h+r, R-r\}\},$$

$$\mathscr{T}_- := \{(r,v) : 0 \leqslant r \leqslant R; 0 \geqslant v \geqslant \max\{-h-r, -R+r\}\}.$$

We will show that $\mathscr{G}[\lambda,\cdot]$ is pointing outwards with bound h in the δ-neighbourhood of $[A,B]$ for initial values in $[a,b]$.

Consider some $u \in \mathscr{C}(\mathbb{R}_+)$ and some $t \geqslant 0$ with

$$u(0) + \mathscr{H}[\lambda,u](0) \in [a,b] \tag{9}$$

and

$$A - \delta \leqslant u(s) + \mathscr{H}[\lambda,u](s) \leqslant B + \delta \quad \forall s \in [0,t]. \tag{10}$$

In view of Eq. (3), we have to show that the following implications hold

$$\begin{cases} u(t) + \mathscr{H}[\lambda,u](t) > B \Rightarrow u(t) \geqslant h, \\ u(t) + \mathscr{H}[\lambda,u](t) < A \Rightarrow u(t) \leqslant -h. \end{cases} \tag{11}$$

First we remark that condition (9) implies that $u(0) \in [f_H^{-1}(a), f_H^{-1}(b)]$.

From Eqs. (8) to (10) it follows:

$$u_{\max} := \max_{s \in [0,t]} u(s) \leqslant R.$$

By virtue of Lemma 3.3. in Ref. [2], this in turn implies for all $s \in [0,t]$ that

$$\wp_r[\lambda,u](s) \leqslant \min\{u(s)+r, R-r\}. \tag{12}$$

At this point, assuming $u(t) < h$, then by Eq. (12) we have that $u(t) + \mathscr{H}[\lambda,u](t) \leqslant B$ which we wanted to prove.

The second implication in Eq. (11) is analogous. This yields the thesis. □

4. Asymptotic behaviour

The main motivation for the result obtained in Section 3 is the study of the asymptotic behaviour of the following P.D.E. containing the hysteresis operator $\mathscr{G}$ given by Eq. (7) (omitting the dependence on λ)

$$\varepsilon u_{tt} + \sigma u_t - \Delta(\gamma u_t + \mathscr{G}(u)) = f \quad \text{in } \Omega \times (0, T), \tag{13}$$

where Ω is an open bounded set of $\mathbb{R}^2$ and u represents the unique nonzero component of the magnetic induction $\mathbf{B} = (0, 0, u)$ (actually we consider for simplicity planar waves and this leads to the scalar character of the model equation). Moreover Δ is the Laplace operator, ε is the electric permittivity, σ is the electric conductivity, γ is a suitable relaxation parameter and f is given.

This model equation appears in the context of electromagnetic processes. In particular it can be obtained by coupling the Maxwell equations, the Ohm law and the following constitutive relation between the magnetic field and the magnetic induction

$$H = \mathscr{G}(u) + \gamma u_t,$$

which is a rheological combination in series of a ferromagnetic element with hysteresis and a conducting solenoid filled with a paramagnetic core. The model is thermodynamically consistent because $\mathscr{G}$ is clockwise oriented.

It is possible to introduce a weak formulation for Eq. (13) (see Ref. [8, Problem 4.1]) which can be in turn interpreted as

$$\begin{cases} \varepsilon A^{-1} u_{tt} + \sigma A^{-1} u_t + \gamma u_t + \mathscr{G}(u) = A^{-1} f, \\ A^{-1} u_{|_{t=0}} = A^{-1} u_0, \quad A^{-1} u_{t|_{t=0}} = A^{-1} v_0, \end{cases} \tag{14}$$

a.e. in $L^2(\Omega)$, where the operator A^{-1} is the inverse of the Laplace operator $-\Delta u$ associated with the homogeneous Dirichlet boundary conditions.

It can be proved by means of a technique based on the contraction mapping principle that problem (14) admits a unique solution $u \in H^1(0, T; L^2(\Omega))$ (see Ref. [8, Section 4.3]). Moreover by suitable approximation arguments it turns out that actually

$$u \in H^2(0, T; L^2(\Omega)).$$

The main result of the section is the following.

Theorem 4. *Let Assumption* 1 *hold and let* $\mathscr{G}$ *be given by Eq.* (7) *for some fixed* $\lambda \in \Lambda$. *Consider the following assumptions on the initial data*

$$u_0 \in H^1(\Omega), \quad v_0 \in H^1(\Omega), \tag{15}$$

and on the given function f

$$A^{-1} f \in L^2(0, \infty; L^2(\Omega))$$

$$A^{-1} f(0) \in H^1(\Omega)$$

$$(A^{-1} f)_t \in L^2(0, \infty; L^2(\Omega)) \tag{16}$$

$$|A^{-1} f(t)|_{L^\infty(\Omega)} \leqslant k \quad \forall t \geqslant 0.$$

Then problem (14) *admits a unique solution*

$$u \in L^\infty(\Omega \times (0, \infty)) \cap \mathscr{C}(\bar{\Omega} \times [0, \infty)) \tag{17}$$

such that, if we set $w := A^{-1} u$, *we have the following regularity for the solution*

$$w_t \in L^\infty(\Omega \times (0, \infty)) \cap \mathscr{C}(\bar{\Omega} \times [0, \infty)),$$

$$\nabla w_t, \nabla w_{tt} \in L^\infty(0, \infty; L^2(\Omega)), \tag{18}$$

$$\nabla w_t, \Delta w_t, \nabla w_{tt}, \Delta w_{tt} \in L^2((0, \infty) \times \Omega).$$

Moreover we have

$$\lim_{t \to \infty} \int_\Omega (|\mathscr{G}(u)|^2 + |u_t|^2 + |\nabla w_{tt}|^2)\, dx = 0. \tag{19}$$

Proof. The regularity of the solution (18) and the asymptotic convergence result (19) can be obtained by testing the first equation of problem (14), respectively, by u_t and u_{tt} in the scalar product of $L^2(\Omega)$. The uniform bound (17) for the solution u can be instead achieved using Theorem 3, i.e. exploiting the outward pointing properties of the operator $\mathscr{G}$. □

References

[1] O. Klein, Appl. Math. 49 (4) (2004) 309.
[2] O. Klein, P. Krejčí, Nonlinear Anal. Real World Appl. 4 (2003) 755.
[3] O. Klein, P. Krejčí, Phys. B 343 (2004) 53.
[4] M. Brokate, O. Klein, P. Krejčí, Phys. B 372 (2006) 5.
[5] M.A. Krasnosel'skiĭ, A.V. Pokrovskiĭ, Springer, Berlin, 1989, Russian edition: Nauka, Moscow, 1983.
[6] P. Krejčí, Gakkotosho, Tokyo, 1996.
[7] M. Brokate, A. Visintin, J. Reine Angew. Math. 402 (1989) 1.
[8] M. Eleuteri, Nonlinear Anal. Real World Appl. 8 (2007) 1494.