INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

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Preprint No. 14-2014

PRAHA 2014

Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We make a comparison among quantization schemes for conformal Toda models.

Keywords Conformal Toda models · Quantum groups · Heisenberg operators

1 Introduction

In this review we discuss various approaches to the construction of the Toda Heisenberg feld operator equatons and their solutions. In the case of the non-periodic boundary condition we start with the light-cone formalism which was applied in [12] to the quantization of the conformal Toda model. Then we remind results obtained in [8] in frames of the perturbative Yang–Feldman approach [14] and its coincidence with the approach of [9] quantum group based construction. For the periodic boundary condition case, we recall the quantizaion of the Liouville field operators in the sense of [3], [4]. Finally we compare results in the above mentioned constructions in the most simple case of conformal Toda model, i.e., the quantum Liouville equation.

2 Conformal Toda models: classical region

Let M be the manifold $\mathbf{R}^2$ or $\mathbf{C}^1$ with the standard coordinates $z^{\pm}$. For $\mathbf{C}^1$ we assume $z^- = (z^+)^*$. Let G be a complex simple Lie group of rank r with the Lie algebra [7] $\mathcal{G}$ endowed with the principal gradation. In this, in the decomposition $\mathcal{G} = \oplus_{m \in \mathbf{Z}} \mathcal{G}_m$ the subspace $\mathcal{G}_0$ is abelian. Denote by G_0 and $G_\pm$ the subspaces corresponding to $\mathcal{G}_0$ and $\oplus_{m>1} \mathcal{G}_{\pm m}$, respectively. Denote by h_i and $x_{\pm i}$ the Cartan and Chevalley generators of $\mathcal{G}$, i.e., for its principal

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gradation, the elements of $\mathcal{G}_0$ and $\mathcal{G}_{\pm 1}$, respectively, satisfying the defining relations

$$[h_i, h_j] = 0, \; [h_i, x_{\pm j}] = \pm k_{ij} x_{\pm j}, \quad [x_{+i}, x_{-j}] = \delta_{ij} h_i, \quad 1 \le i, j \le r, \qquad (1)$$

where k is the Cartan matrix of $\mathcal{G}$. There are also the Serre relations.

Conformal abelian Toda fields $\phi = \sum_{i=1}^{r} h_i \phi_i$, satisfy the equations

$$\partial_+ \partial_- \phi + \frac{4\eta^2}{\beta} \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta \alpha_i \cdot \phi} = 0, \qquad (2)$$

with some coupling constant β and a length scale factor η. Their general solution written in the [10] in a holomorphically factorisable form given by

$$e^{-\beta \lambda_j \cdot \phi} = < \Lambda_j | \gamma_+^{-1} \mu_+^{-1} \mu_- \gamma_- | \Lambda_j >, \qquad (3)$$

where $\gamma_\pm(z_\pm)$ and $\mu_\pm(z_\pm)$ are holomorphic and anti-holomorphic mappings $M \to G_0$ and $M \to G_\pm$, respectively; $|\Lambda_i>$ is the highest vector of the i-th fundamental representation of $\mathcal{G}$. Moreover, the mappings $\mu_\pm(z^\pm)$ satisfy the initial value problem

$$\partial_\pm \mu_\pm = \mu_\pm \kappa_\pm,$$

where $\kappa_\pm$ realize mappings $M \to \mathcal{G}_{\pm 1}$, i.e., $\kappa_\pm(z^\pm) = \sum\limits_{i=1}^{r} \Psi_{\pm i}^{(0)} \cdot x_{\pm i}$, $\Psi_{\pm j}^{(0)} = m_j \exp\left(\mp \beta \sum\limits_{i=1}^{r} k_{ji} \phi_{\pm i}^{(0)} \right)$.

3 Light-cone quantization

The conformal Toda field theory was quantized in the light-cone formalism in [12], [5]. Introduce a set of r scalar fields subject to the canonical commutation relations

$$[\phi_i(z^+, z^-), \phi_j(\tilde{z}^+, z^-)] = -1/4 i\hbar \delta_{ij} \epsilon(z^+ - \tilde{z}^+),$$
$$[\phi_i(z^+, z^-), \phi_j(z^+, \tilde{z}^-)] = -1/4 i\hbar \delta_{ij} \epsilon(z^- - \tilde{z}^-),$$
$$[\phi_i(z^+, z^-), \phi_i(\tilde{z}^+, \tilde{z}^-)] = 0,$$

when $(z^+ - \tilde{z}^+)(z^- - \tilde{z}^-) < 0$, ($\epsilon(z)$ is a standard sign function). Then one takes

$$\partial_+ \partial_- (\beta \Phi_i) + \alpha^2 \sigma \sum_{j=1}^{r} k_{ij} : e^{\beta \Phi_i} := 0, \qquad (4)$$

as the quantum Toda equation for the Heisenberg field operators. Here α, β are constants, $\Phi_i = \sum\limits_{s=1}^{r} M_{is} \phi_s$, and columns denote the normal ordering in the light-cone $I_\tau^\pm = \left\{ z^\pm = \frac{t}{2}, z^\mp \ge \frac{t}{2} \right\}$, with respect to associated Fock space oscillator operators $a_j(p) = \int\limits_{I_\tau^+} dz^- e^{i\mathbf{p}\cdot\mathbf{x}} i\hat{\partial}_- \phi_i + \int\limits_{I_\tau^-} dz^+ e^{i\mathbf{p}\cdot\mathbf{x}} i\hat{\partial}_+ \phi_i$, where

$a\hat{\partial}_\pm b = a(\partial_\pm b) - (\partial_\pm a)b$. Then the exponential of the field $m^{-\frac{\hbar\beta^2}{2\pi}} : e^{\beta\Phi_i}$: does not depend on the mass m. The equation (4) can be rewritten as the zero curvature condition

$$[\partial_+ + \omega_+, \partial_- + \omega_-] = 0, \tag{5}$$

on the quantum Lax pair operators containing both non-commuting Heisenberg field operators ψ_i and Lie algebra generators $x_{\pm i}$, h_i

$$\omega_+ = \partial_+\psi \cdot h + \alpha \sum_{i=1}^{r} x_{+i} + \zeta\Omega, \quad \omega_- = -\alpha\sigma \sum_{i=1}^{r} : e^{\beta\Phi_i} : x_{-i}, \tag{6}$$

where $\psi \cdot h = \sum_{i=1}^{r} \psi_i h_i$, and $\psi_i = \beta \sum_{j=1}^{r} k_{ij}^{-1}\Phi_j$. To make the left hand side of (5) equal to zero we have to put $\zeta = -1/4i\hbar\beta^2\delta(0)$. Then one can construct a solution to (4) using properties of the highest weight representation of Lie algebras. In this formalism, the Heisenberg field operator solution is of the form

$$: e^{-\psi_j(z^+,z^-)} := \quad < \lambda_i| \quad g(z^+,0) \cdot g^{-1}(0,z^-) \quad |\lambda_i > \quad : e^{-\psi_j(z^+,0)} :,$$

where $P = (z^+, z^-)$ is an arbitrary point inside the light cone and $(z^+, 0)$, $(0, z^-)$ are points on $I^\pm_\tau$ with $z_\pm$-coordinates same as at point P. Evaluating $g(z^+, 0)$, $g^{-1}(0, z^-)$ elements one finally finds

$$\begin{aligned} : e^{-\psi_j(z^+,z^-)} : \quad =: e^{-\psi_j(z^+,0)} :< \lambda_i| \left(Texp \int_{(0,0)}^{(z^+,0)} dz^+ B_+ \right) \cdot u : e^{-\psi(0,0)\cdot h} : \\ \times \left(\bar{T}exp \int_{(0,0)}^{(z^+,0)} dz^- \omega_- \right) |\lambda_i > \quad : e^{-\psi_j(0,z^-)} : \cdot e^{\frac{1}{2}\hbar\beta^2 \Delta(0) k_{ij}^{-1}}, \end{aligned} \tag{7}$$

where $B_+ = -\alpha \sum_{j=1}^{r} x_{+i} : e^{\beta\Phi_j} : \ e^{\hbar\beta^2\Delta(0)}$, and $\Delta(\mathbf{x}) = \int_{-\infty}^{+\infty} \frac{dp}{4\pi\omega} e^{-i\mathbf{p}\cdot\mathbf{x}}$. A regularization $\Delta(\mathbf{x}) = \int_{p_-}^{p_+} \frac{dp}{4\pi\omega} e^{-i\mathbf{p}\cdot\mathbf{x}}$ with some p_-, p_+ is implied in B_+. The T-exponents in the last formula denote $z^\pm$ ordering on the light-cone.

4 Yang–Feldman perturbative approach

It has been observed in [9] that the explicit expressions for the conformal Toda Heisenberg operators (satisfying the equations of the form (2)) obtained via of the Yang–Feldman perturbation procedure [8], [14] exactly, order by order, coincide with the expression for the general solution of the form (3) constructed by means of corresponding quantum group. We end up with the exact explicit expressions for the Heisenberg operators $\phi_i^{(q)}$ as finite series in terms of the free fields operators ${}_q\phi_i^{(0)}$ satisfying the canonical commutation relations. Here, in the same way as for the classical solutions, the number of terms in the series is precisely equal to the dimension of the i-th fundamental representation of corresponding Lis algebra $\mathcal{G}$.

In the Yang–Feldman formalism [11], [8] the expression for the exponential of the Heisenberg field operator in the n-th order is

$$\left(e^{-\phi_\alpha}\right)_{(n)} = \frac{1}{(i\hbar)^n} \int\limits_{-\infty}^{+\infty} d\mathbf{z}\theta(z-z_1)...\theta(z_{n-1}-z_n) \sum_{\substack{P(k_1,\\ ...,k_n)}} \left[e^{-\phi^{(0)}_\alpha}, V_{1k_1}, ..., V_{nk_n}\right],$$

where $d\mathbf{z} := dz_1...dz_n$, $\theta(z-z') = \theta(z^+ - z^{+'})\,\theta(z^- - z^{-'})$ is a usual step function, $V_{ij} = \sum\limits_{\alpha=1}^{r} 2w_e^{\psi^{(0)}_{+\alpha}(z^{+i})} e^{\psi^{(0)}_{-\alpha}(z^{-j})}$, $\psi^{(0)}_{\pm\alpha} = \sum\limits_{\beta=1}^{r} k_{\alpha\beta}\phi^{(0)}_{\pm\beta}$, $\phi^{(0)}_{\pm\alpha} = \sum\limits_{\beta=1}^{r} k^{-1}_{\alpha\beta}\psi^{(0)}_{\pm\beta}$, $[A,B,...,C] = [[...[A,B],...],C]$, and $P(k_1,...,k_n)$ denotes ... Here $\phi^{(0)}_{\pm\alpha}$ are asymptotic fields satisfying canonical commutation relations

$$\left[\phi^{(0)}_{\pm i}(z^\pm), \phi^{(0)}_{\pm j}(z^{\pm'})\right] = \tfrac{\hbar}{4i} k^{-1}_{ij} w^{-1}_j \epsilon(z^\pm - z^{\pm'}), \quad \left[\phi^{(0)}_{\pm i}(z^\pm), \phi^{(0)}_{\mp j}(z^{\mp'})\right] = 0,$$

$$\left[\psi^{(0)}_{\pm i}(z^\pm), \psi^{(0)}_{\pm j}(z^{\pm'})\right] = \tfrac{\hbar}{4i} \hat{k}^{-1}_{ij} w^{-1}_j \epsilon(z^\pm - z^{\pm'}),$$

$$\left[\psi^{(0)}_{\pm i}(z^\pm), \phi^{(0)}_{\mp j}(z^{\mp'})\right] = \tfrac{\hbar\delta_{ij}}{4iw_j} \epsilon(z^\pm - z^{\pm'}), \quad \hat{k}_{ij} = k_{ij} w^{-1}_j.$$

The three first orders in the Yang–Feldman formula are

$$\left(e^{-\phi_\alpha}\right)_{(0)} = e^{-\phi^{(0)}_\alpha}, \quad \left(e^{-\phi_\alpha}\right)_{(1)} = e^{-\phi^{(0)}_\alpha} \frac{2w_\alpha}{i\hbar}\left[1 - e^{-\frac{i\hbar}{2w_\alpha}}\right] \Phi^+_\alpha \Phi^-_\alpha, \tag{8}$$

$$\begin{gathered}\left(e^{-\phi_\alpha}\right)_{(2)} = e^{-\phi^{(0)}_\alpha} \tfrac{2w_\alpha}{i\hbar}\left[1 - e^{-\frac{i\hbar}{2w_\alpha}}\right] \sum_{\gamma\neq\alpha} \tfrac{2w_\alpha}{i\hbar}\left[1 - e^{\frac{i\hbar}{2}\hat{k}_{\alpha\gamma}}\right] \Phi^+_{\alpha\gamma}\Phi^-_{\alpha\gamma}, \\ \Phi^\pm_{\alpha_1,...,\alpha_m}(z^\pm) = \int_{-\infty}^{z^\pm} dz^\pm_1 e^{\psi^{(0)}_{\alpha_1}(z^\pm_1)} \int_{-\infty}^{z^\pm_1} dz^\pm_2 e^{\psi^{(0)}_{\alpha_2}(z^\pm_2)} ... \int_{-\infty}^{z^\pm_{m-1}} e^{\psi^{(0)}_{\alpha_m}(z^\pm_m)},\end{gathered} \tag{9}$$

e.g.,

$$\Phi^\pm_\alpha(z^\pm) = \int_{-\infty}^{z^\pm} dz^\pm_1 e^{\psi^{(0)}_{\alpha_1}(z^\pm_1)}, \tag{10}$$

$$\Phi^\pm_{\alpha_1,\beta}(z^\pm) = \int_{-\infty}^{z^\pm} dz^\pm_1 e^{\psi^{(0)}_{\alpha_1}(z^\pm_1)} \int_{-\infty}^{z^\pm_1} dz^\pm_2 e^{\psi^{(0)}_{\beta}(z^\pm_2)}. \tag{11}$$

5 Solution construction based on quantum group

Now take the general solution (3) to the conformal Toda equations (2) and substitute [9] the group elements and representation vectors in the right hand side by quantum group elements and representation vectors correspondingly. In particular, substitute the mappings $\kappa_\pm$ with ${}_q\kappa_\pm(z^\pm) = \sum\limits_{i=1}^{r} {}_q\Psi^{(0)}_{\pm i} \cdot x^q_{\pm i}$. Here ${}_q\Psi^{(0)}_{\pm j} = m_j \exp\left(\mp\beta \sum\limits_{i=1}^{r} k_{ji\,q}\phi^{(0)}_{\pm i}\right)$. Here as before ${}_q\phi^{(0)}$ are operators of the asymptotic fields, i.e., fields in absence of interactions. ${}_q\Psi^{(0)}_{\pm j}$ is contains elements $x^q_{\pm i}$ that belong to the quantized universal enveloping algebra $U_q(\mathcal{G})$ for a Lie algebra $\mathcal{G}$. Instead of (1) the quantum group $U_q(\mathcal{G})$ has the commutation relations (e.g., in Jimbo–Drinfeld form [6], [1])

$$[h^q_i, h^q_j] = 0, \quad [h^q_i, x^q_{\pm j}] = \pm k_{ij} x^q_{\pm j}, \quad [x^q_i, x^q_j] = \delta_{ij} \frac{q_i^{h^q_i} - q_i^{-h^q_i}}{q_i - q_i^{-1}}, \tag{12}$$

where q_i is defined as $q_i = e^{d_i \hbar}$ in terms of the Planck constant $\hbar$ and coprime integers d_i such that $d\ k$ is a symmetric matrix. There are also analogues of the Serre relations.

In place of (3) we get a similar expression for the exponential of a Heisenberg field operator solution

$$e^{-\beta\lambda^{(q)}_j \cdot \phi^{(q)}} = {}_q < \Lambda_j|_q \gamma^{-1}_+ {}_q \cdot \mu^{-1}_+ \cdot {}_q\mu_- \cdot {}_q\gamma_- |\Lambda_j >_q, \tag{13}$$

As in the classical case the group elements ${}_q\mu_\pm$ satisfy $\partial_{\pm q}\mu_\pm = {}_q\mu_\pm \cdot {}_q\kappa_\pm$. In this approach it is easy to find the first three order terms of (13):

$$\left(e^{-{}_q\phi_\alpha}\right)_{(0)} = {}_q < \alpha| e^{-\sum\limits_{\beta=1}^{r} h^q_\beta \left({}_q\phi^{(0)}_{+\beta} + {}_q\phi^{(0)}_{-\beta}\right)} |\alpha >_q = e^{-\left({}_q\phi^{(0)}_{\alpha +} + {}_q\phi^{(0)}_{\alpha -}\right)} = e^{-{}_q\phi^{(0)}_\alpha}, \tag{14}$$

$$\begin{aligned} \left(e^{-{}_q\phi_\alpha}\right)_{(1)} &= -e^{-{}_q\phi^{(0)}_{+\alpha}} \sum_{\theta, p=1}^{r} \Phi^+_\theta \Phi^-_p \cdot {}_q < \alpha| x^q_{+\theta}\ x^q_{-p} |\alpha >_q \ e^{-{}_q\phi^{(0)}_{-\alpha}} \\ &= e^{-{}_q\phi^{(0)}_\alpha} \tfrac{2w_\alpha}{i\hbar} \left[1 - e^{-\frac{i\hbar}{2w_\alpha}}\right] \Phi^+_\alpha \Phi^-_\alpha, \end{aligned} \tag{15}$$

$$\begin{aligned} \left(e^{-{}_q\phi_\alpha}\right)_{(2)} &= -e^{-{}_q\phi^{(0)}_{+\alpha}} \sum_{\theta,\tau,p,q=1}^{r} {}_q < \alpha| x^q_{+\theta}\ x^q_{+\tau}\ x^q_{-p}\ x^q_{-q}\ |\alpha >_q \Phi^+_{\theta\tau} \widetilde{\Phi}^-_{pq} \quad e^{-{}_q\phi^-_\alpha} \\ = e^{-{}_q\phi^{(0)}_{+\alpha}} &\sum_\theta e^{\frac{i\hbar \hat{k}_{\theta\alpha}}{4}} \tfrac{2w_\alpha}{i\hbar} \sinh\left(\tfrac{i\hbar}{4w_\alpha}\right) \tfrac{2w_\theta}{i\hbar} \sinh\left(\tfrac{i\hbar}{4}\left(2\delta_{\alpha\theta} - k_{\alpha\theta}\right)\right) \Phi^+_{\theta\alpha} \Phi^-_{\theta\alpha} e^{-{}_q\phi^{(0)}_{-\alpha}}, \end{aligned} \tag{16}$$

$$\widetilde{\Phi}^-_{pq} = \int^{z^-} dz^-_1 \int^{z^-} dz^-_2 \theta(z^-_2 - z^-_1) e^{q\psi^-_p} e^{q\psi^-_p} \ldots = e^{-\frac{\hbar}{4i}\hat{k}_{pq}} \Phi^-_{pq},$$

and $\Phi^\pm_\alpha$, $\Phi^\pm_{\alpha\beta}$ is the same as in (10), (11). One can see that (14)–(16) are the same as (8)–(9) correspondingly.

6 Solution construction for quantum conformal Toda model

In order to explain and compare formulae given in [9], one considers the quantum group construction of operator solutions to the conformal Toda equations. Recall the light-cone quantization of the conformal Toda field theory given in [12]. The way one derives solutions (3) in classical region for the conformal Toda model (2) can be modified to get solutions in the quantum group case. The zero-curvature condition (5) implies the gradient form of ${}_qA_\pm$.

The pure gauge ${}_qA_\pm = {}_qg^{-1}\partial_\pm({}_qg)$ therefore implies $\partial_{\pm q}g^{-1} = -{}_qA_{\pm q}g^{-1}$, $\partial_{\pm q}g = {}_qg_qA_\pm$, (here ${}_qA_\pm = \eta \sum_{i=1}^{r} \phi^0_{\pm i}(x_{+i}\hat{\otimes}...)$ (i.e., the same (in $x_{\pm i}$-part) as in [9]). Then the properties of the fundamental highest weight representation of a quantum group implies that

$$\partial_-\left({}_q < \Lambda_i|e^{-\beta\phi(z^+,\widetilde{z}^-)\cdot h}{}_qg^{-1}\right) =< \Lambda_i|e^{-\beta\Lambda\cdot\phi(z^+,\widetilde{z}^-)}\left(-{}_qA_{-q}g^{-1}\right) = 0,$$

and $\partial_+({}_qg|\Lambda_i >_q) = {}_qg_qA_+|\Lambda_i >_q = 0$. Here $|\Lambda_i >_q$ are vectors in the fundamental highest weight representation of the $U_q(\mathcal{G})$. Therefore we derive that

$${}_q < \Lambda_i|e^{-\beta\phi(z^+,\widetilde{z}^-)\cdot h} \quad {}_qg^{-1}(z^+,\widetilde{z}^-) \quad \cdot {}_qg(\widetilde{z}^+,z^-)|\Lambda_i >_q,$$

does not depend on $\widetilde{z}^\pm$ and resultes in $e^{-\beta\Lambda\cdot\phi(z^+,z^-)}$ when $\widetilde{z}^\pm = z^\pm$. Thus for a point $p = (z^+,z^-)$ and two arbitrary points $(z^+,0)$, $(0,z^-)$ on the null rays, we have

$$e^{-\beta\Lambda\cdot\phi(z^+,z^-)} = {}_q < \Lambda_i|e^{-\beta\phi(z^+,0)\cdot h}{}_qg^{-1}(z^+,0)\cdot{}_qg(0,z^-)|\Lambda_i >_q . \tag{17}$$

The quantum Gauss decomposition [2] can be represented as ${}_qg(z^+,z^-) = {}_q\mu_+(z^+){}_q\nu_-(z^-){}_q\gamma_{0+}$, so that ${}_qg^{-1}(z^+,z^-) = {}_q\gamma^{-1}_{0+}{}_q\nu^{-1}_-(z^-){}_q\mu^{-1}_+(z^+)$, and ${}_qg^{-1}(z^+,0) = {}_q\gamma^{-1}_{0+}{}_q\mu^{-1}_+(z^+)$, and elements ${}_qg(z^+,0)$ and ${}_qg^{-1}(z^-,0)$ are just ${}_q\mu^{-1}_+(z^+)$, ${}_q\mu_-(z^-)$ elements of the modified Gauss decomposition [2] (same for ${}_qg(0,z^-)$). All that gives us a solution to the quantum conformal Toda equations (2). Note also that solutions of an inhomogeneous equation (line conformal Toda equations) with fixed $z^\mp$ dependence satisfy the homogeneous equation, namely $\partial_+\partial_-\phi(z^+,0) = \partial_+\partial_-\phi(0,z^-) = 0$, i. e., we retain the formula for the general solution to the quantized conformal Toda equation given in [8].

7 Quantization scheme for periodic boundary case

An alternative construction for the quantum Liouvillle exponentials was elaborated in the papers [3], [4]. The final expression for the quantum Liouvillle exponential was found in this approach in a form similar to the classical solution in [10], [11]

$$e^{-J\alpha_-\Phi(z_+,z_-)} =< J,J|\; E^{(J)}_{q+}(\eta_+(z^+),J_+)\; E^{(J)}_{q-}(\eta_-(z^-),J_-)\;|J,J>,$$

$$E^+_q = \sum_{J+M=0}^{\infty} e^{ih(J+M)}(-1)^{J+M}\eta^{(J)}_{+,-M}\frac{(J_+)^{J+M}}{[J+M]!},$$

$$E^-_q = \sum_{J+M=0}^{\infty}\eta^{(J)}_{-,M}\frac{(J_-)^{J+M}}{[J+M]!}, \quad \eta^{(J)}_{\mp,\mp M} = \xi^{(J)}_{\mp,\mp M}\begin{pmatrix} 2J \\ J\mp M\end{pmatrix}_q^{-\frac{1}{2}},$$

$$\begin{pmatrix} A \\ B\end{pmatrix}_q = \frac{[A]!}{[B]![A-B]!}, \quad [A]! = \prod_{k=1}^{n}[k], \quad [k] = \frac{\sin(hx)}{\sin(h)},$$

$$\xi^{(J)}_{\pm,M}(\sigma) = \sum_{-J\le m\le J}|J,\omega)^m_{\pm M}\psi^{(J)}_m(\sigma),$$

$$|J,\omega)_M^m = \left(e^{-ih\frac{a_2}{a_1}}\right)_q^{J+M} k_J \begin{pmatrix} 2J \\ J+M \end{pmatrix}_q^{\frac{1}{2}} e^{\frac{ihm}{2}}$$
$$\times \sum_{\frac{1}{2}(J_M+m-s)\in\mathbf{Z}} e^{ihs(\omega+m)} \begin{pmatrix} J-M \\ \frac{1}{2}(J-M+m-s) \end{pmatrix}_q \begin{pmatrix} J+M \\ \frac{1}{2}(J+M+m+s) \end{pmatrix}_q ,$$
$$\psi_m^{(J)} = \psi^{J-m,J+m}, \quad \psi^{\mu,\nu} \sim \psi_1^\mu \psi_2^\nu \sim N^{(1)}\left(e^{\mu\sqrt{\frac{h}{2\pi}}\phi_1}\right) N^{(2)}\left(e^{\mu\sqrt{\frac{h}{2\pi}}\phi_2}\right),$$

and ψ_j, $j=1,2$ are two independent solutions of Schrodinger equation $-\psi_j''+T\psi_j=0$, associated to the Liouvillle equation which implies $\partial_-\left(e^{-\frac{\phi}{2}}\partial_+ e^{\frac{\phi}{2}}\right)=0$. Here the normal ordering is defined as

$$N^{(j)}(e^{\gamma\phi_j(z)}) = e^{\gamma q_0^{(j)}} e^{-iz(\pm\omega+\frac{1}{2})\frac{h}{2\pi}} exp\left(\gamma\sum_{n<0}\frac{p_n^{(j)}}{n}e^{-inz}\right) exp\left(\gamma\sum_{n>0}\frac{p_n^{(j)}}{n}e^{-inz}\right),$$

where $\phi_j(z) = q_0^{(j)} + p_0^{(j)} + i\sum_{n<0}\frac{e^{-inz}}{n}p_n^{(j)}$. For the case $J=\frac{1}{2}$ the general formula gives $e^{-J\alpha_-\Phi(z^+,z^-)} = (a_1\kappa_1\kappa_2 - a_1\kappa_1\kappa_2)N^{(1)}N^{(2)}...$ where κ_i are some infinite series containing h. We have then $\psi_j = d_j N^{(j)}(e^{\gamma\phi_j})$. Let's put $\phi_2=0$ and $\phi_1=\phi_0^+$, $\phi_1=\phi_0^-$. Then we have

$$e^{-J\alpha_-\Phi(z,\bar{z})} = \sigma_1 + \sigma_2 N^{(1)}\left(e^\gamma\right) + \sigma_3 N^{(1)}\left(e^\gamma\right) + \sigma_4 N^{(1)}\left(e^\gamma\right) N^{(1)}\left(e^\gamma\right), \quad (18)$$

which is very close to the formulas for the quantum Liouville exponential in [9] and [12], [5] formulations in appropriate limits.

8 Comparison of approaches

In the subsection 3 we recalled the light-cone quantization procedure applied in [12] to the conformal Toda field theory. In this approach the Lax operators (6) corresponding to the quantum Heisenberg field operator equations (4) contain elements of ordinary Lie algebra, but, as it was shown in [5], this construction possesses an underlying quantum group structure. This case gives as an example of the Gursa problem sinse the exponent of the field operator depends on its value on the characteristics. Finally one obtains a solution (17) to the equation (4) which is of the form given in [10] in the classical case. At the same time this way of Lax pair construction for the quantum conformal Toda has some obvious drawbacks. One may think of the Lax pair (6) as a result of some gauge transformation sending the symmetric form of Lax operators $A_\pm = u_\pm h + f_\pm x_\pm$ (where $u_\pm$, $f_\pm$ are operators) to (6) with a constant operator near $\sum_{i=1}^{r} x_{+i}$. Though infinite constant ξ in (6) does not appear in the final expression for the solution (7) still it looks very discouraging in the operator form.

Now we would like to compare the solution in the most simple case of $\mathcal{G}=sl_2$ which corresponds to the quantum Liouville equation $\partial_+\partial_-(\beta\Phi)+$

$2\alpha^2\sigma : e^{\beta\Phi} := 0$. The final expression for the exponent of the Liouville field in the construction of [12], [5] has the form

$$: e^{-\frac{\beta}{2}\Phi(z^+,z^-)} :=: e^{-\frac{\beta}{2}\Phi(z^+,0)} : \left(: e^{\frac{\beta}{2}\Phi(0,0)} : e^{\frac{\hbar\beta^2}{2}\Delta(0)} + \alpha^2\sigma \left(\int\limits_{(0,0)}^{(z^+,0)} : e^{\beta\Phi} : dz^+ \right) \right.$$
$$\left. \times : e^{-\frac{\beta}{2}\Phi(0,0)} : \left(\int\limits_{(0,0)}^{(0,z^-)} : e^{\beta\Phi} : dz^- \right) e^{\frac{3\hbar\beta^2}{2}\Delta(0)} \right) \Bigg) : e^{-\frac{\beta}{2}\Phi(0,z^-)} : .$$

If we get rid of the normal ordering in the last formulausing the fact that $e^{\beta\Phi} =: e^{\beta\Phi} : e^{\hbar\beta^2\Delta(0)}$, then we get the expression

$$e^{-\frac{\beta}{2}\Phi(z^+,z^-)} = e^{-\frac{\beta}{2}\Phi(z^+,0)} \left(e^{-\frac{\beta}{2}\Phi(0,0)} + \alpha^2\sigma \left(\int\limits_{(0,0)}^{(z^+,0)} e^{\beta\Phi(\widetilde{z}^+,0)} d\widetilde{z}^+ \right) \cdot e^{-\hbar\beta^2\Delta(0)} \right.$$
$$\left. \times e^{-\frac{\beta}{2}\Phi(0,0)} \cdot \left(\int\limits_{(0,0)}^{(0,z^-)} e^{\beta\Phi(0,\widetilde{z}^-)} d\widetilde{z}^- \right) e^{-\frac{\beta}{2}\Phi(0,z^-)} \right)$$
$$= e^{-\frac{\beta}{2}\Phi(z^+,0)} \cdot e^{-\frac{\beta}{2}\Phi(0,0)} \cdot e^{-\frac{\beta}{2}\Phi(0,z^-)} \cdot \left(1 + \alpha^2\sigma e^{-\hbar\beta^2\delta(0)} \Phi^+\Phi^- \right).$$

Thus we see that the quantum Toda exponent is expressed through its values at points on characteristics. Therefore this is an example of the Gursat problem.

In the approach of [9] one deals with the quantum exponents depending on asymptotic fields which are in fact solutions fo the homogeneous operator equation $\partial_+\partial_-\phi = 0$. Then in the Liouville case using (14)–(15) we obtain

$$\left(e^{-\phi}\right)_{(0)} = e^{-{}_q\phi^{(0)}},\ \left(e^{-\phi}\right)_{(1)} = -e^{-{}_q\phi^{(0)}} \frac{2w}{i\hbar} \left[1 - e^{-\frac{i\hbar}{2w}} \right] \Phi^+\Phi^-,\ \left(e^{-\phi}\right)_{(2)} = 0.$$

Thus the series terminates after the first order. It is not a surprise that the formulas (8)–(9) and (14)–(16) coincide in [12] and [9] approaches. The reason is that the elements of the quantum group are used in (13) and give extra multipliers in the first order due to the non-linear right hand side of the commutation relation (12) of the x^q_+, x^q_- generators of the quantum group.

One can also try to construct the Lax pair for the quantum conformal Toda model equation (4) in the same as in the approach of [12] but using generators of the quantized universal enveloping algebra. Indeed, consider the following pair of operators

$$ {}_q\omega_+ = \partial_+\psi \cdot h^q + \alpha \sum_{i=1}^{r} x^q_{+i} + \zeta\Omega_q, \quad {}_q\omega_- = -\alpha\sigma \sum_{i=1}^{r} : e^{\beta\Phi_i} : x^q_{-i}, \qquad (19)$$

where h^q_i, $x^q_{\pm i}$ are generators of the quantum group. Here $\Omega_q \in U_q(\mathcal{G})$ is chosen in such a way that it would kill higher h-terms in the zero curvature condition (5). Thus the zero curvainteractions. ture condition would give us (4). The problem is to compute Ω_q precisely. Then one could find a solution to (4) using the Lax pair (19) which comply with the solution in [9] solution under an appropriate limit.

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