INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

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Alexander Zuevsky

Preprint No. 15-2014

PRAHA 2014

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Alexander Zuevsky

Institute of Mathematics, Academy of Sciences of the Czech Republic

E-mail: `zuevsky@yahoo.com`

Abstract. We derive equations for the homogeneous higher grading affine Toda models and propose corresponding solutions. The main example of the homogeneous higher grading sine–Gordon equation is discussed.

1. Introduction

The Lie-algebraic way to construct non-linear exactly solvable models in classical regions is very well known and elaborated [10]. Applying the zero-curvature conditions on elements of connection containing Lie algebra generators in appropriate grading subspaces, we obtain systems of equations of motion associated to a specific Lie algebra. In [4] the higher grading generalization to the conformal affine Toda models was considered. Elements of the higher (then number one) grading subspaces are taking into account while connection elements are constructed. The main example of [4] is the principal grading case. In this paper we consider an alternative, the homogeneous grading case. We derive the systems of equations generalizing the case of the sine–Gordon equation and provide quantum group solutions.

2. Homogeneous higher grading generalization of the affine Toda model

We start with the equations (23–26) of [4] (see subsection 6.2 in Appendix). Consider the case $l = 1$. In the principal grading we obtain from (23) the sin–Gordon equation. Recall that In the homogeneous grading of $\widetilde{\mathcal{G}}$ the grading subspaces are $\widetilde{\mathcal{G}}_n = \left\{H^n, E^n_\pm\right\}$. We take

$$E_1 = E^1_+ + E^1_-, \quad E_{-1} = E^{-1}_+ + E^1_-. \tag{1}$$

Consider a particular case when we parameterize the group element b as

$$b = e^{\phi H^0}. \tag{2}$$

Then, substituting (1) and (2) into (23–25) we get the following system of equations

$$\partial_\pm \phi = e^\eta \left(e^{-2\phi} - e^{2\phi}\right), \quad \partial_\pm \nu = -e^\eta \left(e^{2\phi} + e^{-2\phi}\right), \quad \partial_\pm \eta = 0,$$

i.e., in the first equation is again the sine–Gordon equation. The solution to the field ϕ is then the standard classical solution (28), [12] (see subsection 6.2 of Appendix).

Now consider the case $l = 2$. The equations corresponding to the principal grading can be found in [4]. Here again we take $b = e^{\phi H^0}$ though it this is not the most general choice of the group element parameterization since it does not contain dependence on the $E^0_\pm$ elements from $\widehat{\mathcal{G}}$, i.e., we have send corresponding fields near those generators to zero. Let us also put

$$F_1^+ = \kappa^+ H^1 + f_+^+ E_+^1 + f_-^+ E_-^1, \quad F_1^- = \kappa^- H^1 + f_+^- E_+^{-1} + f_-^- E_-^{-1}.$$

Then the system (23–25) gives the following system of equations:

$$\begin{array}{l} \partial_\pm \phi = e^\eta \left(e^{-2\phi} - e^{2\phi}\right) + e^\eta \left(f_-^+ f_+^- e^{-2\phi} - f_-^- f_+^+ e^{2\phi}\right), \\ \partial_\pm \nu = 2e^\eta \left(e^{2\phi} - e^{-2\phi}\right) + e^\eta \left(-2\kappa^+\kappa^- + f_-^- f_+^+ e^{2\phi} - f_-^+ f_+^- e^{-2\phi}\right),\ \partial_\pm \eta = 0 \end{array} \quad (3)$$

$$\begin{array}{l} \partial_+ \kappa^- = -e^\eta \left(f_-^+ e^{-2\phi} - f_+^+ e^{2\phi}\right), \quad \partial_- \kappa^+ = e^\eta \left(f_-^- e^{-2\phi} - f_+^- e^{2\phi}\right), \\ \partial_+ f_+^- = 2e^\eta \kappa^+,\ \partial_- f_+^+ = -2e^\eta \kappa^-,\ \partial_+ f_-^- = -2e^\eta \kappa^+,\ \partial_- f_-^+ = 2e^\eta \kappa^-, \\ e^{2\phi} f_+^+ \kappa^- = \kappa^+ f_+^-, \quad e^{-2\phi} f_-^+ \kappa^- = \kappa^+ f_-^-. \end{array} \quad (4)$$

The formal general solution to (3–4) age given in [4]:

$$e^{-\phi} = e^{\phi_0^+ - \phi_0} \frac{\langle \Lambda_1 | \mu_+^{-1} \mu_- | \Lambda_1 \rangle}{\langle \Lambda_0 | \mu_+^{-1} \mu_- | \Lambda_0 \rangle^{m_1}}, \quad (5)$$

for the ϕ field and for the $F_1^\pm$ elements

$$\langle i | F_i^+ | i; i \rangle = e^{k_{i1}\left(\phi_{0i}^- - \phi_{0i}\right)} \partial_+ \left(\langle i | \gamma_0^- b \left(\gamma_0^+\right)^{-1} \mu_+^{-1} \mu_- | i; i \rangle \right).$$

In the homogeneous grading case, taking into account the parameterization of b element, we get

$$\langle 1 | F_i^+ | 1; 1 \rangle = e^{2\left(\phi_0^- - \phi_0\right)} \partial_+ \left(\langle 1 | e^{\phi_0^- H^0} e^{\phi H^0} e^{-\phi_0^+ H^0} \mu_+^{-1} \mu_- | 1; 1 \rangle \right).$$

Here we have made use of the properties of the i-th fundamental representation (corresponding to the homogeneous grading) of the Lie algebra $\widehat{sl}_2$. Thus, using (5) we get

$$\langle 1 | F_i^+ | 1; 1 \rangle = e^{2\left(\phi_0^- - \phi_0\right)} \partial_+ \left(\langle 1 | \mu_+^{-1} \mu_- | 1; 1 \rangle \cdot \frac{\langle \Lambda_0 | \mu_+^{-1} \mu_- | \Lambda_0^{m_1} \rangle}{\langle \Lambda_1 | \mu_+^{-1} \mu_- | \Lambda_1 \rangle} \right).$$

3. Dirac equations

Let's switch notations similarly to Dirac field components, i.e., $\psi_R = f_+^+$, $\psi_L = f_+^-$, $\widetilde{\psi}_R = f_-^+$, $\widetilde{\psi}_L = f_-^-$. Now use the extra conditions (4), substituting them into (3). Then we see that the second summands in the first two formulae in (3) vanish, i.e., the final equations are

$$\partial_+\partial_- \phi = e^\eta \left(e^{-2\phi} - e^{2\phi}\right), \quad \partial_+\partial_- \nu = 2e^\eta \left(e^{2\phi} - e^{-2\phi}\right) - 2e^\eta \kappa^+\kappa^-, \quad \partial_+\partial_- \eta = 0, \quad (6)$$

i.e., equations (6) do not differ much from the corresponding equations with $l = 1$.

Now suppose that $\eta = \eta_0 = const$. Then substitute the last four equations of (**??**) on $f = \psi$ fields into the first two on $\kappa^\pm$ fields. Then we get

$$\partial_+\partial_- \psi_R = 2e^{2\eta_0} \left(\widetilde{\psi}_R e^{-2\phi} - \psi_R e^{2\phi}\right), \quad \partial_+\partial_- \widetilde{\psi}_R = -2e^{2\eta_0} \left(\widetilde{\psi}_R e^{-2\phi} - \psi_R e^{2\phi}\right), \quad (7)$$

$$\partial_+\partial_- \psi_L = 2e^{2\eta_0} \left(\widetilde{\psi}_L e^{-2\phi} - \psi_L e^{2\phi}\right), \quad \partial_+\partial_- \widetilde{\psi}_L = -2e^{2\eta_0} \left(\widetilde{\psi}_L e^{-2\phi} - \psi_L e^{2\phi}\right), \quad (8)$$

that can be rewritten as

$$\partial_+\partial_- \omega_R = 0,\ \partial_+\partial_- \tau_R = 2e^{2\eta_0} \omega_R \left(e^{-2\phi} - e^{2\phi}\right),\ \partial_+\partial_- \omega_L = 0,\ \partial_+\partial_- \tau_L = 2e^{2\eta_0} \omega_L \left(e^{-2\phi} - e^{2\phi}\right), \quad (9)$$

where $\omega_{R,L} = \psi_{R,L} + \widetilde{\psi}_{R,L}$, $\tau_{R,L} = \psi_{R,L} - \widetilde{\psi}_{R,L}$. The upshot is that using such a parametrization $b = e^{\phi H^0}$ we arrive at three systems of sine–Gordon like systems when η is a constant.

3.1. $l=2$. The general case

Let's consider such $b\in G_0$ that involves all generators of the $\mathcal{G}_0$ in the homogenous gradation. Take for instance $b=e^{\phi_+E^0_+}e^{\phi H^0}e^{\phi_-E^0_-}$, Then for $l=2$ the equations are

$$\partial_+\left(\partial_-bb^{-1}\right)+\partial_+\partial_-\nu C=e^{2\eta}\left[E_{-2},bE_2b^{-1}\right]+\left[F_1^-,bF_1^+b^{-1}\right], \tag{10}$$

$$\partial_-F_1^+=e^{\eta}\left[E_2,b^{-1}F_1^-b\right],\quad \partial_+F_1^-=-e^{\eta}\left[E_{-2},bF_1^+b^{-1}\right],\quad \partial_+\partial_-\eta=0, \tag{11}$$

where $F_1^+=\kappa^+H^1+f_+^+E_+^1+f_-^+E_-^1$, $F_1^-=\kappa^-H^1+f_+^-E_+^{-1}+f_-^-E_-^{-1}$. Let us take $\kappa^+=\kappa_-=0$. Then we have

$$\begin{aligned}
&\partial_+\left(\partial_-\phi+\phi_+\partial_-\phi_-\right)=e^{2\eta}\left[e^{-2\phi}(1-\phi_-^2)-e^{2\phi}-2\phi_+\phi_-)\right]\\
&\qquad\qquad +e^{\eta}\left[e^{-2\phi}(f_-^+-f_+^+\phi_-^2)(f_+^-+f_-^-\phi_+^2)-2f_+^+f_-^-\phi_-\phi_+-f_+^+f_-^-e^{2\phi}\right],\\
&\partial_+\partial_-\nu=2e^{2\eta}\left[-e^{-2\phi}(1-\phi_-^2)-e^{2\phi}-2\phi_+\phi_-)\right]\\
&\qquad\qquad +e^{\eta}\left[e^{-2\phi}(f_-^+-f_+^+\phi_-^2)(f_+^-+f_-^-\phi_+^2)-2f_+^+f_-^-\phi_-\phi_+-f_+^+f_-^-e^{2\phi}\right],\\
&\partial_+(\partial_-\phi_+-2(\partial_-\phi)\phi_+-\phi_+^2(\partial_-\phi_-)e^{-2\phi})=e^{2\eta}\left[2\phi_+e^{-2\phi}(1-\phi_-^2)-2\phi_-\right]\\
&\qquad\qquad +e^{\eta}\left[-2f_+^-(e^{-2\phi}(f_-^+-f_+^+\phi_-^2)\phi_+-\phi_-f_+^+\right],\\
&\partial_+(\partial_-\phi_-e^{-2\phi})=e^{2\eta}\left[2\phi_+e^{-2\phi}(1-\phi_-^2)-\phi_-\right]+e^{\eta}\left[2f_+^+(e^{-2\phi}(f_-^+-f_+^+\phi_-^2)\phi_+-\phi_-f_+^+\right],\\
&\partial_+\partial_-\eta=0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\partial_+f_+^-=e^{\eta}\left[-2(-f_+^+\phi_-+e^{-2\phi}(f_+^+\phi_-^2+f_-^+\phi_+\right],\ \partial_+f_-^-=e^{\eta}\left[2(-f_+^+\phi_-+e^{-2\phi}(f_+^+\phi_-^2+f_-^+\phi_+\right]\\
&e^{-2\phi}(f_+^+\phi_-^2+f_-^+)(1+\phi_+^2)-2f_+^+\phi_-\phi_+-f_+^+e^{2\phi}=0
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\partial_-f_+^+=e^{\eta}\left[-2(-\phi_-e^{-2\phi}(f_+^--\phi_+^2f_-^-)-\phi_+f_-^-\right],\partial_-f_-^+=\left[2(-\phi_-e^{-2\phi}(f_+^--\phi_+^2f_-^-)-\phi_+f_-^-\right],\\
&-e^{-2\phi}(f_+^--\phi_+^2f_-^-)(1+\phi_-^2)e^{2\phi}f_-^-+\phi_-\phi_+f_-^-=0.
\end{aligned} \tag{14}$$

When $\kappa^+\neq 0$ and $\kappa^-\neq 0$ the system of equations is more complicated.

4. Solitonic solutions from general solutions

In [12] it was shown how to extract solitonic solutions from the formal general solutions of the affine Toda field equations. Let's take $\gamma_0^\pm=1$ in (28) to be a constant function. Then the mappings $\mu_\pm$ are $\mu_\pm=\mu_\pm^0e^{z_\pm\mathcal{E}_\pm}$ with $\mu_\pm^0$ being some fixed mappings independent of $z_\pm$. Next take $\tilde{\mathcal{E}}_\pm$ in (30) as $\mathcal{E}_\pm\equiv E_{\pm l}+\sum_{N=1}^{l-1}c_N^\pm E_{\pm N}$ where $E_\pm$ are elements of a Heisenberg subalgebra of $\widehat{\mathcal{G}}$, namely $[\mathcal{E}_+,\mathcal{E}_-]=\Omega C$. One can consider principal of homogeneous Heisenberg subalgebras for that purpose. In this paper we only deel with the principal case while the homogeneous case will be discussed elsewhere. Thus, we arrive at a special solution to (27)

$$e^{-\beta\lambda_i\cdot\phi}=e^{-\beta\lambda_i\cdot\phi_0}\frac{{}^{(1)}\langle\lambda_i|e^{x_\pm\mathcal{E}_\pm}\mu^0e^{x_\pm\mathcal{E}_\pm}|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|e^{x_\pm\mathcal{E}_\pm}\mu^0e^{x_\pm\mathcal{E}_\pm}|\lambda_0\rangle^{(1)}\right)^{m_j}}. \tag{15}$$

In order to compute these solutions explicitly we have to remove $\mathcal{E}_\pm$-dependence from (15) moving $\mathcal{E}_+$ to the right and $\mathcal{E}_-$ to the left . Then we should find such a $\mu_0=\prod_{i=1}^N e^{\mathcal{V}_i}$ so that

$\mathcal{V}_i$ would be eigenvectors with respect to the adjoint action of $\mathcal{E}_\pm$, i.e., $[\mathcal{E}_\pm, \mathcal{V}_i] = \omega_\pm^{(i)} \mathcal{V}_i$. Then it turns out [12] that resulting expressions provide us with solitonic solutions to the equations under considerations while parameters $\omega_\pm^{(i)}$ characterize solitons.

5. Quantum group soliton solution for sine–Gordon in homogeneous grading

As in [11] one can show that the affine Toda models are co-invariant with respect to the light-cone quantization. Namely, the equation of motion are preserved in form though a standard normal ordering has to be introduced as well as some infinite constant comming from quantum versions of Lax pair to generate equations using Lie algebra elements in quantum case. At the same time infinite constants do not appear in final formal solutions to the light-cone quantized versions of equations. In order to find quantum solutions, one has to replace [3], [8], [9] group elements as well as state vectors formal general solutions by their quantum group counterparts.

In this subsection we write examples of quantum group solutions to the quantized affine Toda model in the specific case of the higher grading sine–Gordon equation (the cases $l = 1, 2, 3$). Recall [13], that the homogeneous grading subspaces of $U_q(\widehat{sl}_2)$ are ${}_q\widehat{\mathcal{G}}_0 = \{K, \gamma, x_0^+, x_0^-\}$, ${}_q\widehat{\mathcal{G}}_n = \{x_n^+, x_n^-, a_n, n \in \{\mathbb{Z} - 0\}\}$.

5.1. The case $l = 1$

From the commutation relations for $x_m^\pm$ and a_m (see subsection 7.1 of Appendix) it follows that in this realization of the quantum group $U_q(\widehat{sl}_2)$, the generators $x_m^\pm, a_m \in \mathcal{G}_m$, $x_0^\pm \in \mathcal{G}_0$. The solution

$$e^{-\beta\lambda_j\cdot\phi} = e^{-\beta\lambda_j\cdot\phi_0} \frac{\langle\Lambda_j| e^{-a_1 z_+} e^{Q\phi_-} e^{a_{-1} z_-} |\Lambda_j\rangle}{\langle\Lambda_0| e^{-a_1 z_+} e^{Q\phi_-} e^{a_1 z_-} |\Lambda_0\rangle^{m_j}},$$

where $|\Lambda_0\rangle = |1 \otimes 1\rangle$, $|\Lambda_1\rangle = |1 \otimes e^{\frac{\alpha}{2}}\rangle$ and the homogeneous grading quantum vertex operator is

$$\phi_- = exp\left(\sum_{k=1}^{\infty} \frac{a_{-k}}{[2k]} q^{\frac{7k}{2}} \zeta^k\right) exp\left(-\sum_{k=1}^{\infty} \frac{a_k}{[2k]} q^{-\frac{5k}{2}} \zeta^{-k}\right) \otimes e^{\frac{\alpha}{2}} (-q^3\zeta)^{\frac{(\partial\alpha + i)}{2}}.$$

Using the fact that [5] $[a_k, \phi_-] = q^{\frac{7k}{2}} \frac{[k]}{k} \zeta^k$, $\phi_-, k > 0$, $[a_k, \phi_-] = q^{-\frac{5k}{2}} \frac{[k]}{k} \zeta^{-k}$, $\phi_-, k > 0$, we commute $\exp(-a_1 z_+)$ with $\exp(Q\phi_-)$ to the right and $\exp(Q\phi_-)$ with $\exp(a_{-1} z_-)$ to the left. The commutation of $\exp(-a_1 z_+)$ withexp$(a_{-1} z_-)$ gives $\exp(-z_+ z_-[2])$). Thus it follows that

$$\langle\Lambda_j| e^{-a_1 z_+} e^{Q\phi_-} e^{a_1 z_-} |\Lambda_j\rangle = \langle\Lambda_j| exp\left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}} \phi_-\right) exp\left(-z_+ z_- [2]\right) |\Lambda_j\rangle,$$

(recall that $\exp(-a_1 z_+)$ and $\exp(a_{-1} z_-)$ act on $|\Lambda_j\rangle$ and $\langle\Lambda_j|$ as identities). Then we expand $exp\left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}} \phi_-\right)$ as a series and apply the action of powers of operators $exp\left(\sum_{k=1}^{\infty} \frac{a_{-k}}{[2k]} q^{\frac{7k}{2}} \zeta^k\right)$ and $exp\left(-\sum_{k=1}^{\infty} \frac{a_k}{[2k]} q^{-\frac{5k}{2}} \zeta^{-k}\right)$, to the left and to the right. Powers of operators ϕ_- act on the second part of tensor product as follows:

$$(\phi_-)^n |1 \otimes e^{\frac{\alpha}{2}}\rangle = (-q^3\zeta)^{\frac{i+n}{2}} |1 \otimes e^{\frac{\alpha(n+1)}{2}}\rangle,\ (\phi_-)^n |1 \otimes 1\rangle = (-q^3\zeta)^{i + \frac{n-1}{2}} |1 \otimes e^{\frac{\alpha n}{2}}\rangle.$$

Thus we have

$$\begin{aligned} e^{-\beta\lambda_j\cdot\phi} &= e^{-\beta\lambda_j\cdot\phi_0} \frac{\langle\Lambda_j| e^{-z_+ z_- [2]} \sum_{n=1}^{\infty} \frac{1}{n!} \left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}}\right)^n (-q^3\zeta)^{\frac{i+n}{2}} |1 \otimes e^{\frac{\alpha(n+1)}{2}}\rangle}{\langle\Lambda_0| e^{-z_+ z_- [2]} \sum_{n=1}^{\infty} \frac{1}{n!} \left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}}\right)^n (-q^3\zeta)^{i + \frac{n-1}{2}} |1 \otimes e^{\frac{\alpha n}{2}}\rangle^{m_j}} \\ &= e^{-\beta\lambda_j\cdot\phi_0} \frac{\langle\Lambda_j| e^{-z_+ z_- [2]} exp\left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}} (-q^3\zeta)^{\frac{i}{2}}\right) (-q^3\zeta)^{\frac{i}{2}} \otimes exp\left(e^{\frac{\alpha}{2}}\right) |\Lambda_j\rangle}{\langle\Lambda_0| e^{-z_+ z_- [2]} exp\left(Q e^{-q^{\frac{7}{2}} z_+ \zeta - q^{-\frac{5}{2}} z_- \zeta^{-1}} (-q^3\zeta)^{\frac{1}{2}}\right) (-q^3\zeta)^{i - \frac{1}{2}} \otimes exp\left(e^{\frac{\alpha}{2}}\right) |\Lambda_0\rangle^{m_j}}. \end{aligned} \quad (16)$$

In the limit $q \to 1$ we obtain ordinary soliton solutions.

5.2. The case $l = 2$

As in [13], if we put $\widehat{\phi}^{\pm}_{x,1} = 0$, then $E_\pm = a_{\pm 2} + a_{\pm 1}$, and one can integrate the equations for ${}_q\mu_\pm$ to obtain ${}_q\mu_\pm(z^\pm) = {}_q\mu_\pm(0)e^{(a_{\pm 2}+a_{\pm 1})z^\pm}$. Then the quantum soliton solution to the quantized (3) is

$$: e^{-\beta\widehat{\phi}(z^+,z^-)} :=: e^{-\beta\widehat{\phi}_0(z^+,z^-)} : \frac{{}_q\langle\Lambda_1|e^{(a_{+1}+a_{+2})z^+}\, {}_q\mu(0)e^{(a_{-1}+a_{-2})z^-}|\Lambda_1\rangle_q}{{}_q\langle\Lambda_0|e^{(a_{+1}+a_{+2})z^+}\, {}_q\mu(0)e^{(a_{-1}+a_{-2})z^-}|\Lambda_0\rangle_q},$$

where ${}_q\mu(0)$ should be chosen the same as in [13]. Then we have

$$: e^{-\beta\widehat{\phi}(z^+,z^-)} :=: e^{-\beta\widehat{\phi}_0(z^+,z^-)} : \frac{{}_q\langle\Lambda_1|e^{-\frac{\alpha}{2}} exp\big(i(-1)^{\partial_\alpha+1}Q\ W_2\cdot {}_q\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_1\rangle_q}{{}_q\langle\Lambda_0|e^{-\frac{\alpha}{2}} exp\big(i(-1)^{\partial_\alpha+1}Q\ W_2\cdot {}_q\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_0\rangle_q}$$

$$=: e^{-\beta\widehat{\phi}_0(z^+,z^-)} : \frac{1+iW_2Q}{1-iW_2Q}\zeta^{\frac{1}{2}},$$

where $W_2 = \exp\left(\sum_{k=1}^{2} \frac{7k}{2}\frac{[k]}{k}\zeta^k z^+ - \sum_{k=1}^{2}\frac{5k}{2}\frac{[k]}{k}\zeta^{-k}z^-\right)$. Similarly,

$${}_q\langle 1|F_1^+|1;1\rangle_q = e^{2(\widehat{\phi}_0^- - \widehat{\phi})}$$
$$\times\partial_+\left({}_q\langle 1|e^{(a_{+1}+a_{+2})z^+}\,{}_q\mu(0)e^{(a_{-1}+a_{-2})z^-}|1;1\rangle_q \times \frac{{}_q\langle\Lambda_1|e^{(a_{+1}+a_{+2})z^+}\,{}_q\mu(0)e^{(a_{-1}+a_{-2})z^-}|\Lambda_1\rangle_q}{{}_q\langle\Lambda_0|e^{(a_{+1}+a_{+2})z^+}\,{}_q\mu(0)e^{(a_{-1}+a_{-2})z^-}|\Lambda_0\rangle_q}\right).$$

Thus,

$${}_q\langle 1|F_1^+|1;1\rangle_q = e^{2(\widehat{\phi}_0^- - \widehat{\phi})}$$
$$\times\partial_+\Big({}_q\langle 1|e^{(a_{+1}+a_{+2})z^+}e^{-\frac{\alpha}{2}}exp\left(i(-1)^{\partial_\alpha+1}Q\ W_2\cdot{}_q\Phi(\zeta)\right)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}e^{(a_{-1}+a_{-2})z^-}|1;1\rangle_q$$
$$\times\frac{{}_q\langle\Lambda_1|e^{-\frac{\alpha}{2}} exp\big(i(-1)^{\partial_\alpha+1}Q\ W_2\cdot{}_q\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_1\rangle_q}{{}_q\langle\Lambda_0|e^{-\frac{\alpha}{2}} exp\big(i(-1)^{\partial_\alpha+1}QW_2\cdot{}_q\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_0\rangle_q}\Big).$$

Finally,

$${}_q\langle 1|F_1^+|1;1\rangle_q = e^{2(\widehat{\phi}_0^- - \widehat{\phi})}\partial_+\left((1+iW_2Q[2])\cdot\frac{1-iW_2Q}{1+iW_2Q}\zeta^{-\frac{1}{2}}\right).$$

5.3. Case l=3

The states

$$|\Lambda_0\rangle^{(m)} = |\prod_{k=1}^{m+1} a_{-(m-k)}\otimes 1\rangle,\ |\Lambda_1\rangle^{(m)} = |\prod_{k=1}^{m+1} a_{-(m-k)}\otimes e^{\frac{\alpha}{2}}\rangle,\ |\Lambda_0\rangle^{(1)} = |\Lambda_0\rangle,\ |\Lambda_1\rangle^{(1)} = |\Lambda_1\rangle,$$

are annihilated by the action of $\mathcal{G}_n, n \geq m$. Therefore for F_m^+ we have

$${}^{(1)}\langle\Lambda_1|e^{-\sum\limits_{k=1}^{3}a_k z_+}e^{Q\phi_-}e^{\sum\limits_{k=1}^{3}a_{-k}z_-}|\Lambda_1\rangle^{(m)}$$
$$= {}^{(1)}\langle\Lambda_1|exp\left(Qe^{-q^{\frac{7}{2}}z_+\zeta - q^{\frac{-5}{2}}z_-\zeta^{-1}}\phi_-\right)exp\left(-z_+z_-\sum_{k=1}^{3}\frac{[2k]}{k}[k]\right)e^{-\sum\limits_{k=1}^{3}a_k z_+}|\Lambda_1\rangle^{(m)}.$$

Action by the operators $e^{-\sum\limits_{k=1}^{3} a_k z_+}$ on $|\Lambda_1\rangle^{(m)}$, $m = 1,\, 2,\, 3$ we get for instance,

$$e^{-\sum\limits_{k=1}^{3} a_k z_+} |\Lambda_1\rangle^{(3)} = |\Lambda_1\rangle^{(3)} - z_+(C_2 + a_{-2})|\Lambda_1\rangle^{(1)} + z_+^2 C_1 C_2 |\Lambda_1\rangle^{(1)},$$

where $C_k = \frac{[2k]}{k}[k]$. Then we expand e^{ϕ_-} again and act on the states. Therefore we get an infinite series over $|\Lambda_1\rangle^{(3)}, |\Lambda_1\rangle^{(2)}, |\Lambda_1\rangle^{(1)}$ which contain $C_k, (k = 1, 2, 3)$, z_+ and tensor $\otimes$-part due to powers of $e^{\frac{\alpha}{2}} (-q^3\zeta)^{\frac{(\partial\alpha + i)}{2}}$.

6. Appendix

6.1. Affine Kac–Moody algebras

Here we recall facts about affine Kac–Moody algebras [7], [4]. Consider an untwisted affine Kac-Moody algebra $\widehat{\mathcal{G}}$ endowed with an integral grading $\widehat{\mathcal{G}} = \bigoplus_{n\in \mathbb{Z}} \widehat{\mathcal{G}}_n$, and denote $\widehat{\mathcal{G}}_\pm = \bigoplus_{n>0} \widehat{\mathcal{G}}_{\pm n}$. By an affine Lie algebra we mean a loop algebra corresponding to a finite dimensional simple Lie algebra $\mathcal{G}$ of rank r, extended by the center C and the derivation D. According to Tkac, integral gradings of $\widehat{\mathcal{G}}$ are labelled by a set of co-prime integers $\mathbf{s} = (s_0, s_1, \dots s_r)$, and the grading operators are given by

$$Q_{\mathbf{s}} \equiv H_{\mathbf{s}} + N_{\mathbf{s}}\, D - \frac{1}{2N_{\mathbf{s}}} \, \mathrm{Tr}\,(H_{\mathbf{s}})^2 \; C\, . \tag{17}$$

Here $H_{\mathbf{s}} \equiv \sum_{a=1}^r s_a \lambda_a^v \cdot H^0$, $N_{\mathbf{s}} \equiv \sum_{i=0}^r s_i m_i^\psi$, $\psi = \sum_{a=1}^r m_a^\psi \alpha_a\, , \quad m_0^\psi = 1$. H^0 is an element of Cartan subalgebra of $\mathcal{G}$; α_a, $a = 1, 2, \dots r$, are its simple roots; ψ is its maximal root; m_a^ψ the integers in expansion $\psi = \sum_{a=1}^r m_a^\psi \alpha_a$; and λ_a^v are the fundamental co–weights satisfying the relation $\alpha_a \cdot \lambda_b^v = \delta_{ab}$.

The principal grading operator Q_{ppal} is given by (17) where $N_{\mathbf{s}} = h$ is Coxeter number. Therefore $\widehat{\mathcal{G}}_0 = \{H_a^0\, , a = 1, 2, \dots r\, ; C; Q_{\mathrm{ppal}}\}$, $\widehat{\mathcal{G}}_m = \{E^0_{\alpha^{(m)}}, E^1_{-\alpha^{(h-m)}}\}$, $\widehat{\mathcal{G}}_{-m} = \{E^0_{-\alpha^{(m)}}, E^{-1}_{\alpha^{(h-m)}}\}$ where $0 < m < h$, and $\alpha^{(m)}$ are positive roots of height m. The element B is parameterized as $B = e^{\varphi\cdot \tilde{H}^0}\, e^{\nu\, C}\, e^{\eta Q_{\mathrm{ppal}}} = e^{\varphi\cdot H^0}\, e^{\tilde{\nu}\, C}\, e^{\eta Q_{\mathrm{ppal}}}$, where $\tilde{H}^0$ was defined in [4] as $\widetilde{H}_a^0 = H_a^0 - \frac{1}{N_{\mathbf{s}}}\, \mathrm{Tr}\left(H_{\mathbf{s}}\, H_a^0\right) C = H_a^0 - \frac{2}{\alpha_a^2}\frac{s_a}{N_{\mathbf{s}}}\, C$, and $\tilde{\nu} = \nu - \frac{2}{h}\widehat{\delta}\cdot\varphi$, with $\widehat{\delta} = \sum_{a=1}^r \frac{\lambda_a}{\alpha_a^2}$, and λ_a being the fundamental weights of $\mathcal{G}$. Let us denote by H^n, $E^n_\pm$, D, C the Chevalley basis generators of $\widehat{sl}_2$. The commutation relations are

$$[H^m\, ,\, H^n] = 2\, m\, C\, \delta_{m+n,0}, \quad [H^m\, ,\, E^n_\pm] = \pm 2\, E^{m+n}_\pm,$$
$$[E^m_+\, ,\, E^n_-] = H^{m+n} + m\, C\, \delta_{m+n,0}, \quad [D\, ,\, T^m] = m\, T^m\, , \quad T^m \equiv H^m, E^m_\pm.$$

The grading operator for the principal grading ($\mathbf{s} = (1,1)$) is $Q \equiv \frac{1}{2}H^0 + 2D$. Then the eigensubspaces are $\widehat{\mathcal{G}}_0 = \{H^0, C, Q\}$, $\widehat{\mathcal{G}}_{2n+1} = \{E^n_+, E^{n+1}_-\}$, $n \in \mathbb{Z}$, $\widehat{\mathcal{G}}_{2n} = \{H^n\}, n \in \{\mathbb{Z} - 0\}$.

6.2. Higher grading affine Toda system

In this and the next sections we recall [4] the affine Toda system consrtuction. Consider a two dimensional manifold $\mathcal{M}$ with local coordinates $z_\pm$. Up to a gauge transformation, $(1,0)$-component lying in (see subsection 6.1 of Appendix) $\bigoplus_{n=0}^{l} \widehat{\mathcal{G}}_{+n}$ and $(0,1)$-component in $\bigoplus_{n=0}^{l} \widehat{\mathcal{G}}_{-n}$ of a flat connection $\mathcal{A}$ in the trivial holomorphic principal fibre bundle $\mathcal{M}\times\widehat{G} \longrightarrow \mathcal{M}$ ($l > 0$ is fixed integer) satisfy the zero curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+\, ,\, A_-] = 0. \tag{18}$$

The components $A_\pm$ are the following (we keep notations of [4])

$$A_+ = -B\,F^+\,B^{-1}\,, \qquad A_- = -\partial_- B\,B^{-1} + F^-. \tag{19}$$

Here B is a mapping $\mathcal{M} \longrightarrow \widehat{G}_0$ ($\widehat{G}_0$ is a group with the Lie algebra $\widehat{\mathcal{G}}_0$) and $F^\pm$ ($1 \le m \le l-1$) are mappings to $\bigoplus_{n=1}^{l} \widehat{\mathcal{G}}_{\pm n}$

$$F^\pm = E_{\pm l} + \sum_{m=1}^{l-1} F_m^\pm,$$

where $E_{\pm l}$ are some fixed elements of $\widehat{\mathcal{G}}_{\pm l}$ and $F_m^\pm \in \widehat{\mathcal{G}}_{\pm m}$, ($1 \le m \le l-1$). Substituting (19) into (18) one arrives at the equations of motion

$$\begin{aligned}
\partial_+\left(\partial_- B\,B^{-1}\right) &= [E_{-l}\,,\,B\,E_l\,B^{-1}] + \sum_{n=1}^{l-1}[F_n^-\,,\,B\,F_n^+\,B^{-1}]\,, && (20)\\
\partial_- F_m^+ &= [E_l\,,\,B^{-1}\,F_{l-m}^-\,B] + \sum_{n=1}^{l-m-1}[F_{n+m}^+\,,\,B^{-1}\,F_n^-\,B]\,, && (21)\\
\partial_+ F_m^- &= -[E_{-l}\,,\,B\,F_{l-m}^+\,B^{-1}] - \sum_{n=1}^{l-m-1}[F_{n+m}^-\,,\,B\,F_n^+\,B^{-1}]\,. && (22)
\end{aligned}$$

Since $Q_{\mathbf{s}}$, $C \in \widehat{\mathcal{G}}_0$ then B can be parameterized as $B = b\,e^{\eta\,Q_{\mathbf{s}}}\,e^{\nu\,C}$ where b is a mapping to G_0, the subgroup of $\widehat{G}_0$ generated by all elements of $\widehat{\mathcal{G}}_0$ other than $Q_{\mathbf{s}}$ and C. Substituting B into the equations of motion (20–22) one has

$$\begin{aligned}
\partial_+\left(\partial_- b b^{-1}\right) &+ \partial_+\partial_-\,\nu\,C = e^{l\eta}[E_{-l}\,,\,b\,E_l\,b^{-1}] + \sum_{n=1}^{l-1} e^{n\eta}\,[F_n^-\,,\,b\,F_n^+ b^{-1}]\,, && (23)\\
\partial_- F_m^+ &= e^{(l-m)\eta}\,[E_l\,,\,b^{-1}\,F_{l-m}^-\,b] + \sum_{n=1}^{l-m-1} e^{n\eta}[F_{m+n}^+\,,\,b^{-1}\,F_n^-\,b]\,, && (24)\\
\partial_+ F_m^- &= -e^{(l-m)\eta}\,[E_{-l}\,,\,b\,F_{l-m}^+\,b^{-1}] - \sum_{n=1}^{l-m-1} e^{n\eta}[F_{m+n}^-\,,\,b\,F_n^+\,b^{-1}]\,, && (25)\\
\partial_+\partial_-\,\eta\,Q_{\mathbf{s}} &= 0\,. && (26)
\end{aligned}$$

Now consider the case $l = 1$. Let us parameterize the element B in the homogeneous grading of $\widehat{\mathcal{G}}$, [4]. From the equations (23–26) for an infinite dimensional Lie algebra $\widehat{\mathcal{G}}$ in the principal grading we obtain the affine Toda field theory systems of equations

$$\partial_+\partial_-\phi + \tfrac{4\mu}{\beta}\sum_{i=1}^{r}\left(m_i\tfrac{\alpha_i}{\alpha_i^2}\exp(\beta\alpha_i\cdot\phi) - \tfrac{\psi}{2}\exp(-\beta\cdot\phi)\right) = 0. \tag{27}$$

The formal general solution to the above equation was introduced in [12]:

$$e^{-\beta\lambda_i\cdot\phi} = e^{-\beta\lambda_i\cdot\phi_0}\frac{{}^{(1)}\langle\lambda_i|(\gamma_0^+)^{-1}\mu_+^{-1}(z_+)\mu_-(z_-)(\gamma_0^-)|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|(\gamma_0^+)^{-1}\mu_+^{-1}(z_+)\mu_-(z_-)(\gamma_0^-)|\lambda_0\rangle^{(1)}\right)^{m_j}} = e^{-\beta\lambda_i\cdot\phi_0}\frac{(1)\langle\lambda_i|B^{-1}|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|B^{-1}|\lambda_0\rangle^{(1)}\right)^{m_j}}, \tag{28}$$

The general solutions to the matter fields $F_i^\pm$ may be written in the following form. For $m = 1$ in (23–26) one has [4]

$$\langle i|F_1^+|i;i\rangle = f_i^+ = e^{\sum\limits_{l=0}^{r} k_{il}(\phi^- - \phi)_l}\, e^{\nu_0}\partial_+\left(\langle i|\mu_+^{-1}\mu_-|i;i\rangle\frac{\langle 0|\mu_+^{-1}\mu_-|0\rangle^{m_i}}{\langle i|\mu_+^{-1}\mu_-|i\rangle}\right).$$

Here $|i;i\rangle$ denotes an element of the Verma module which is result of the action of the lowering generator on the highest state vector. The fact that (28) is indeed a solution to (27) may be checked by using the representation theory of $\widehat{\mathcal{G}}$. A map $g: \mathcal{M} \longrightarrow G$ appearing in the gradient form of the flat connection $A_\pm = g^{-1}\partial_\pm g$, may be factorized (according to the Lie algebra decomposition $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$) by the modified Gauss decomposition $g = \mu_-\nu_+\gamma_{0-}$ or $g = \mu_+\nu_-\gamma_{0+}$ with maps $\gamma_{0\pm}: \mathcal{M} \longrightarrow G_0$, $\mu_\pm, \nu_\pm : \mathcal{M} \longrightarrow G_\pm$. The grading condition provides the holomorphic property of $\mu_\pm$, i.e., they satisfy the initial value problem

$$\partial_\pm \mu_\pm(z_\pm) = \mu_\pm(z_\pm)\tilde{\mathcal{E}}_\pm(z_\pm), \tag{29}$$

$$\tilde{\mathcal{E}}_\pm(z_\pm) = \sum_{m=1}^{M} \tilde{\mathcal{E}}^\pm_m(\Phi^\pm), \qquad \tilde{\mathcal{E}}^\pm_m(\Phi^\pm) = \sum_{\alpha\in\Delta^+_m} \Phi^{\pm m}_\alpha(z_\pm) X_{\pm\alpha}, \tag{30}$$

with arbitrary functions $\Phi^{\pm m}_\alpha(z_\pm)$ determining the general solution to the system. Note that the summations in (30) are performed over the set of positive roots Δ^+_m of $\mathcal{G} = \sum_{m\in\mathbb{Z}} \mathcal{G}_m$ in the subspace $\mathcal{G}_m$.

7. Soliton solution for the sine–Gordon in homogeneous grading

Another way to construct soliton solutions [13] to the sine–Gordon equation is to consider the formal general solution (27) in the homogeneous grading and to use vertex operators [7] which are related to the homogeneous Heisenberg subalgebra of $\widehat{sl_2}$. Take the general solution (28) to the affine Toda system (27). In the homogeneous grading the mappings $\gamma_\pm$ can be parameterized as $\gamma_\pm = e^{d\phi_d} e^{c\phi_c} e^{\phi^\pm_0 x^\pm_0}$, where d is the grading operator, c is the center of $\widehat{sl_2}$ and $x^\pm_k$ are generators of the subspaces $\widehat{\mathcal{G}}_k$ corresponding to the homogeneous grading. The mappings $\mu_\pm$ satisfy (29) where $\kappa_\pm(z^\pm) = a_{\pm1} + \phi^\pm x^\pm_1$. In order to obtain a soliton solution we put $\phi^\pm = 0$, $\phi^\pm_0 = 0$. Then the general solution reduces to

$$e^{-\beta\phi(z^+,z^-)} = \langle\Lambda_1|e^{a_{+1}z^+}\mu(0)e^{a_{-1}z^-}|\Lambda_1\rangle \left(\langle\Lambda_0|e^{a_{+1}z^+}\mu(0)e^{a_{-1}z^-}|\Lambda_0\rangle\right)^{-1}. \tag{31}$$

The following group element $\mu(0)$ in (31)

$$\mu(0) = e^{-\frac{\alpha}{2}N} \prod_{n=1}^{N} \left[exp\left((-1)^{\partial_\alpha+1} iQ_n \Phi(\zeta_n)\right) e^{\frac{\alpha}{2}} \zeta_n^{\frac{1}{2}\partial_\alpha} \right],$$

generates an N–soliton solution. Here the action of the operators $\frac{1}{2}\partial_\alpha$ and $e^{\frac{\alpha}{2}}$ on the highest vectors $|\Lambda_n\rangle = |1 \otimes e^{\frac{\alpha}{2}n}\rangle$, $n = 0, 1$ is the same as in the case of $U'_q(\widehat{sl_2})$ [7] when $q = 1$. The operator $\Phi(\zeta)$ is given by

$$\Phi(\zeta) = exp\left(\sum_{k=1}^{\infty} \frac{a_{-n}}{n}\zeta^n\right) exp\left(-\sum_{k=1}^{\infty} \frac{a_{+n}}{n}\zeta^{-n}\right),$$

and diagonalises the action of $a_{\pm k}$, $k \in \mathbb{N}$, i.e., $[a_{\pm k}, \Phi(\zeta)] = \zeta^{\pm k}\Phi(\zeta)$. The product of two vertex operators can be normal ordered as

$$\Phi(\zeta_1)\Phi(\zeta_2) = X(x) : \Phi(\zeta_1)\Phi(\zeta_2) :,$$

where $X(x) = exp(-\sum_{n=1}^{\infty} x^{2n}/n) = \exp\left(\log(1-x^2)\right)$. When $x = 1$, $X(x)$ vanishes which results in $\Phi(\zeta)\cdot\Phi(\zeta) = 0$. Therefore the exponential of $\Phi(\zeta)$ operator terminates after the first order.

In the limit $q \longrightarrow 1$ soliton–soliton, antisoliton–antisoliton and soliton–-antisoliton scattering reduce to the classical case, i.e.,

$$F^a(\zeta_1)F^b(\zeta_2) = \frac{1}{x}\frac{X(x)}{X(x^{-1})}F^b(\zeta_2)F^a(\zeta_1),$$

where $x^2 = \zeta_2/\zeta_1$, a, b denote soliton (antisoliton), and the factor $1/x$ comes from the commutation of $e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}$ and $e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}$ operators. Therefore the vertex operator generating a classical soliton solution is

$$F(\zeta) = Q\ \Phi(\zeta)\ e^{\frac{\alpha}{2}}\ \zeta_2^{\frac{1}{2}\partial_\alpha}.$$

Taking into account the properties of the operator $F(\zeta)$ we rewrite the solution (31) as

$$\begin{aligned} e^{-\beta\phi(z^+,z^-)} &= \frac{\langle\Lambda_1|\big(1+(-1)^{\partial_\alpha+1}iQ\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_1\rangle}{\langle\Lambda_0|\big(1+(-1)^{\partial_\alpha+1}iQ\Phi(\zeta)\big)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_0\rangle} \\ &= \left(1+iQe^{\zeta z^+-\zeta^{-1}z^-}\right)\left(1-iQe^{\zeta z^+-\zeta^{-1}z^-}\right)\zeta. \end{aligned}$$

The antisoliton solution can be associated with the vertex operator

$$\bar{F}(\zeta) = -Q\ \Phi(\zeta)\ e^{\frac{\alpha}{2}}\ \zeta^{\frac{1}{2}\partial_\alpha}.$$

7.1. Quantized universal enveloping algebra $U'q(\widehat{sl_2})$

In the spirit of [2], [5], the quantised enveloping algebra $U_q(sl_2)$ is an associative algebra generated by X^+, X^-, H with q-deformed commutation relations

$$X^+X^- - X^-X^+ = \left(q^H - q^{-H}\right)\left(q-q^{-1}\right)^{-1}, \qquad HX^\pm - X^\pm H = \pm 2X^\pm.$$

It possesses a Hopf algebra structure with the deformed adjoint action

$$(ad_{X^\pm})_q a = X^\pm a q^{H/2} - q^{\mp 1}q^{H/2}aX^\pm, \qquad (ad_H)_q a = Ha - aH,$$

for all $a \in U_q(sl_2)$. Let us recall the second Drinfeld realization of the quantized universal enveloping algebra $U'_q(\widehat{sl_2})$, (i.e., $U_q(\widehat{sl_2})$ without grading operator) [2], [6], which is a natural quantum analogue of the algebra $\widehat{sl_2}$ in the loop realizations. $U'_q(\widehat{sl_2})$ is an associative algebra generated by $\{x_k^\pm, k \in \mathbb{Z}; a_n, n \in \{\mathbb{Z}-0\}; \gamma^{\pm\frac{1}{2}}, K\}$, where $\gamma^{\pm\frac{1}{2}}$ belong to the center of the algebra, satisfying the commutation relations

$$\begin{gathered} [K, a_k] = 0, \quad [a_k, a_l] = \delta_{k,-l}\tfrac{[2k]}{k}\tfrac{\gamma^k-\gamma^{-k}}{q-q^{-1}}, \quad Kx_k^\pm K^{-1} = q^{\pm 2}x_k^\pm, \\ \left[a_n, x_k^\pm\right] = \pm\tfrac{[2n]}{n}\gamma^{\mp\frac{|n|}{2}}x_{n+k}^\pm, \quad \left[x_k^+, x_n^-\right] = \tfrac{\gamma^{(k-n)/2}\psi_{k+n}-\gamma^{(n-k)/2}\phi_{k+n}}{q-q^{-1}}, \\ x_{k+l}^\pm x_l^\pm - q^{\pm 2}x_l^\pm x_k^\pm = q^{\pm 2}x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm. \end{gathered}$$

The generators ϕ_k and ψ_{-k}, $k \in \mathbb{Z}_+$ are related to a_k and a_{-k} by means of the expressions $\sum_{k=0}^{\infty}\psi_m z^{-m} = Kexp\left((q-q^{-1})\sum_{k=1}^{\infty}a_k z^{-k}\right)$, $\sum_{k=0}^{\infty}\phi_{-m}z^m = K^{-1}exp\left(-(q-q^{-1})\sum_{k=1}^{\infty}a_{-k}z^k\right)$, i.e., $\psi_m = 0$, $m < 0$; $\phi_m = 0$, $m > 0$. Here $[k] = \frac{q^k-q^{-k}}{q-q^{-1}}$.

It is easy to define the grading operators corresponding to the principal and homogeneous grading of $U'_q(\widehat{sl_2})$ by analogy with the grading of $U'_q(\mathcal{G})$ where $\mathcal{G}$ is a simple Lie algebra. The principal grading can be realized with the help of the operator $D_p x = \frac{1}{2} q K^{-1} \left(\frac{d}{dq} (K x K^{-1}) \right) K + 2\lambda \frac{d}{d\lambda} x$, where $x \in U_q(\widehat{sl_2})$ and λ is an affinization parameter. The power of λ is denoted by the subscript of $U'_q(\widehat{sl_2})$ generators. Then the grading subspaces are ${}_q\widehat{\mathcal{G}}_0 = \{K, \gamma\}$, ${}_q\widehat{\mathcal{G}}_{2n+1} = \{x_n^+, x_{n+1}^-, n \in \mathbb{Z}\}$, ${}_q\widehat{\mathcal{G}}_{2n} = \{a_n, n \in \{\mathbb{Z} - 0\}\}$. The grading operator for the homogeneous grading is $D_h x = 2\lambda \frac{d}{d\lambda} x$, so that the grading subspaces are ${}_q\widehat{\mathcal{G}}_0 = \{K, \gamma, x_0^+, x_0^-\}$, ${}_q\widehat{\mathcal{G}}_n = \{x_n^+, x_n^-, a_n, n \in \{\mathbb{Z} - 0\}\}$.

The level one irreducible integrable highest weight representation of $U'_q(\widehat{sl_2})$ can be constructed in the following way [6]. Let $P = \mathbb{Z}\frac{\alpha}{2}$, $Q = \mathbb{Z}\alpha$ be the weight/root lattice of sl_2. Consider the group algebras $F[P]$, $F[Q]$ of P and Q. The multiplicative basis of $F[P]$ is formed by $e^{\frac{\alpha}{2}n}$, $n \in \mathbb{Z}$. The $F[Q]$-module is split into $F[P] = F[P]_0 \oplus F[P]_1$ where $F[P]_n = F[Q]e^{\frac{\alpha}{2}n}$. The sl_2-module structure on the space $W = F[a_{-1}, a_{-2}, ...] \otimes F[P]$ is given by the action of the $a_k, k \in \{\mathbb{Z} - 0\}$ and e^α, $\partial_\alpha = a_0$ generators in accordance with the rules

$$a_k(f \otimes e^\beta) = (a_k f \otimes e^\beta), \quad k < 0, \ a_k(f \otimes e^\beta) = ([a_k, f] \otimes e^\beta), \quad k > 0,$$
$$e^\alpha(f \otimes e^\beta) = (f \otimes e^{\alpha+\beta}), \quad \partial_\alpha(f \otimes e^\beta) = (\alpha, \beta)(f \otimes e^\beta),$$
$$K = 1 \otimes q^{\partial_\alpha}, \quad \gamma = q \otimes id.$$

Then W is a $U'_q(\widehat{sl_2})$-module. Its submodules are isomorphic to irreducible highest weight modules $V(\Lambda_n)$ with the highest vectors $|\Lambda_n\rangle = |1 \otimes e^{\frac{\alpha n}{2}}\rangle$, $n = 0, 1$.

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