# Singular limits of compressible fluids

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# Scaled Navier-Stokes system

## Continuity equation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

## Momentum equation

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \mathbf{f} \times \mathbf{u} + \boxed{\frac{1}{\varepsilon^{2m}}} \nabla_x p(\varrho)$$

$$= \boxed{\varepsilon^{\alpha}} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon^{2n}}} \varrho \nabla_x G$$

## Newtonian viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu > 0$$

## $f$-plane approximation

$$\mathbf{f} = [0, 0, 1], \ \ \nabla_x G = [0, 0, -1]$$

# Spatial domain and boundary conditions

## Infinite slab

$$\Omega = R^2 \times (0,1)$$

## Complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\rm tan}|_{\partial\Omega} = 0$$

## Far field conditions

$$\varrho \to \tilde{\varrho}_\varepsilon, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

## Static density distribution

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\varrho}_\varepsilon \nabla_x G, \ \tilde{\varrho}_\varepsilon \to 1 \text{ as } \varepsilon \to 0$$

# Singular limits

## Low Mach number

Mach number $\approx \varepsilon^m$:

$$\text{compressible} \ \rightarrow \text{incompressible}$$

## Low Rossby number

Rossby number $\approx \varepsilon$:

$$\text{3D flow} \ \rightarrow \text{2D flow}$$

## High Reynolds number

Reynolds number $\approx \varepsilon^{-\alpha}$:

$$\text{viscous (Navier-Stokes)} \ \rightarrow \text{inviscid (Euler)}$$

## Low stratification

$$\frac{m}{2} > n \geq 1$$

# Uniform bounds

## Energy inequality

$$\int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^{2m}} \left( H(\varrho) - H'(\tilde\varrho_\varepsilon)(\varrho - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon) \right) \right] (\tau, \cdot) \ \mathrm{d}x$$

$$+ \varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left[ \frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left( H(\varrho_{0,\varepsilon}) - H'(\tilde\varrho_\varepsilon)(\varrho_{0,\varepsilon} - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon) \right) \right] \ \mathrm{d}x$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z, \ \ p(\varrho) \approx a \varrho^\gamma, \ \ \gamma > \frac{3}{2}$$

## Ill-prepared initial data

$$\varrho_{0,\varepsilon} = \tilde\varrho_\varepsilon + \varepsilon^m \varrho^{(1)}_{0,\varepsilon}, \ \ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \ \text{in} \ L^2(\Omega), \ \ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty} \leq c,$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \text{in} \ L^2(\Omega; R^3)$$

# Limit system

## Limit density deviation

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot)-1\|_{L^\gamma_{\rm loc}(\Omega)} \leq \varepsilon^m c$$

## Limit velocity

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v} \left\{ \begin{array}{l} \text{weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)), \\ \\ \boxed{\text{strongly in } L^1_{\rm loc}((0,T)\times\Omega;R^3)}, \end{array} \right.$$

## Euler system

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x\Pi = 0 \text{ in } (0,T)\times R^2$$

$$\mathbf{v}_0 = \mathbf{H}\left[\int_0^1 \mathbf{u}_0 \ {\rm d}x_3\right]$$

# Relative entropy inequality

## Relative entropy

$$\mathcal{E}_\varepsilon \left[ \varrho, \mathbf{u} \middle| r, \mathbf{U} \right]$$

$$= \int_\Omega \left[ \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^{2m}} \Big( H(\varrho) - H'(r)(\varrho - r) - H(r) \Big) \right] \ \mathrm{d}x$$

## Relative entropy inequality

$$\mathcal{E}_\varepsilon \left( \varrho, \mathbf{u} \ \middle| \ r, \mathbf{U} \right) (\tau) + \varepsilon^\alpha \int_0^\tau \int_\Omega \Big( \mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big( \nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \ \mathrm{d}x \, \mathrm{d}t$$

$$\leq \mathcal{E}_\varepsilon \left( \varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \middle| \ r(0,\cdot), \mathbf{U}(0,\cdot) \right) + \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

## Test functions

$$r > 0, \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (r - \tilde{\varrho}_\varepsilon), \ \mathbf{U} \to 0 \text{ as } |x| \to \infty$$

# Remainder

$$\int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \varrho \left( \partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot \left( \mathbf{U} - \mathbf{u} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho (\mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \Big[ (r - \varrho) \partial_t H'(r) + \nabla_x \left( H'(r) - H'(\tilde{\varrho}_\varepsilon) \right) \cdot \left( r \mathbf{U} - \varrho \mathbf{u} \right) \Big] \ \mathrm{d}x \ \mathrm{d}t$$

$$- \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big( p(\varrho) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon^{2n}} \int_0^\tau \int_\Omega (\varrho - r) \nabla_x G \cdot \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

# Reformulation

## Decomposition

$$r_\varepsilon = \frac{\varrho_\varepsilon - 1}{\varepsilon^m} = q_\varepsilon + s_\varepsilon, \ \ \varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

$[q_\varepsilon, \mathbf{v}_\varepsilon]$ .................................... non-oscillatory component

$[s_\varepsilon, \mathbf{V}_\varepsilon]$ ........................................ oscillatory component

## "Acoustic analogy" - Poincaré waves

$$\varepsilon^m \partial_t \left[ \frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] + \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] = 0$$

$$\varepsilon^m \partial_t [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \varepsilon^{m-1} \mathbf{f} \times [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \left[ \frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] = \varepsilon \mathbf{f}_\varepsilon$$

# Test function ansatz

## Density deviation

$$r = \tilde{\varrho}_\varepsilon + \varepsilon^m (q_\varepsilon + s_\varepsilon)$$

## Velocity decomposition

$$\mathbf{U} = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

## Initial data

$$\varrho^{(1)}_{0,\varepsilon} = (q_\varepsilon + s_\varepsilon)(0,\cdot), \ \mathbf{u}_{0,\varepsilon} = (\mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon)(0,\cdot)$$

# Non-oscillatory - Euler system

## Diagnostic equation

$$\omega \mathbf{f} \times \mathbf{v}_\varepsilon + \nabla_x q_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

$$\omega \mathrm{curl} \mathbf{v} = -\Delta q_\varepsilon$$

## Perturbed Euler system

$$\partial_t \left( \Delta q_\varepsilon - \omega^2 q_\varepsilon \right) - \frac{1}{\omega} \nabla^t q_\varepsilon \cdot \nabla \left( \Delta q_\varepsilon - \omega^2 q_\varepsilon \right) = 0$$

## Initial data

$$\left( \Delta q_\varepsilon - \omega^2 q_\varepsilon \right) (0, \cdot) = \omega \mathrm{curl} \left[ \int_0^1 \mathbf{u}_{0,\varepsilon} \ \mathrm{d}x_3 \right] - \omega^2 \int_0^1 \varrho_{0,\varepsilon} \ \mathrm{d}x_3$$

# Oscillatory - vanishing part

## Poincaré waves

$$\varepsilon^m \partial_t s_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon^m \partial_t \mathbf{V}_\varepsilon + \omega \mathbf{f} \times \mathbf{V}_\varepsilon + \nabla_x s_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

## Antisymmetric acoustic propagator

$$\mathcal{B}(\omega) : \left[ \begin{array}{c} s \\ \mathbf{V} \end{array} \right] \mapsto \left[ \begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \omega \mathbf{f} \times \mathbf{V} + \nabla_x s \end{array} \right].$$

# Fourier representation

## Poincaré waves

$$\varepsilon^m \partial_t \begin{bmatrix} s_\varepsilon(\xi,k,\omega) \\ \mathbf{V}_\varepsilon(\xi,k,\omega) \end{bmatrix} = \mathrm{i}\mathcal{A}(\xi,k,\omega) \begin{bmatrix} s_\varepsilon(\xi,k,\omega) \\ \mathbf{V}_\varepsilon(\xi,k,\omega) \end{bmatrix}$$

## Hermitian matrix

$$\mathrm{i}\mathcal{B}(\omega) \approx \mathcal{A}(\xi,k,\omega) = \begin{bmatrix} 0 & \xi_1 & \xi_2 & k \\ \xi_1 & 0 & \omega\mathrm{i} & 0 \\ \xi_2 & -\omega\mathrm{i} & 0 & 0 \\ k & 0 & 0 & 0 \end{bmatrix}.$$

## Eigenvalues

$$\lambda_{1,2}(\xi,k,\omega) = \pm \left[ \frac{\omega^2 + |\xi|^2 + k^2 + \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

$$\lambda_{3,4}(\xi,k,\omega) = \pm \left[ \frac{\omega^2 + |\xi|^2 + k^2 - \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

# Fourier analysis

$$k \text{ fixed}, \ \psi \in C_c^\infty(0,\infty), \ 0 \leq \psi \leq 1$$

**Frequency cut-off**

$$Z(\tau, x_h, k, \omega) = \mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\left(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\right)\psi(|\xi|)\hat{h}(\xi)\right], \ \tau = t/\varepsilon^m$$

$$\|Z(\tau t, \cdot, k, \omega)\|_{L^\infty(R^2)}$$

$$\leq \left\|\mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\left(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\right)\psi(|\xi|)\right]\right\|_{L^\infty(R^2)} \|h\|_{L^1(R^2)}$$

**Fourier transform of radially symmetric function**

$$\mathcal{F}^{-1}_{\xi \to x_h}\left[\exp\left(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau\right)\psi(|\xi|)\right](x_h)$$

$$= \int_0^\infty \exp\left(\pm \mathrm{i}\lambda_j(r, k, \omega)\tau\right)\psi(r) r J_0(r|x_h|)\ \mathrm{d}r,$$

# van Corput's lemma

**Lemma**

*Let* $\Lambda = \Lambda(z)$ *be a smooth function away from the origin,*

$$\partial_z \Lambda(z) \ \textit{monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

*for all* $z \in [a, b]$, $0 < a < b < \infty$. *Let* $\Phi$ *be a smooth function on* $[a, b]$. *Then*

$$\left| \int_a^b \exp\left( \mathrm{i} \Lambda(z) \tau \right) \Phi(z) \ \mathrm{d}z \right| \leq c \frac{1}{\tau \Lambda_0} \left[ |\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ \mathrm{d}z \right],$$

*where* $c$ *is an absolute constant* $\boxed{\textit{independent}}$ *of the specific shape* $\Lambda$ *and* $\Phi$.

# Decay estimates

## $L^p - L^q$ estimates

$$\|Z(\tau,\cdot,k,\omega)\|_{L^p(R^2)} \le c(\psi,p,k)\max\left\{\frac{1}{\omega\tau^{1-\beta/2}};\frac{1}{\tau^{\beta/2}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$

$$\text{for } p\ge 2,\ \frac{1}{p}+\frac{1}{p'}=1,\ \beta>0,\ \lambda_j\ne 0.$$

## Scaling

$$\omega\approx\varepsilon^{m-1},\ \tau\approx t/\varepsilon^m$$

## Dispersive decay

$$\left\|Z\left(\frac{t}{\varepsilon^m},\cdot,k,\omega\right)\right\|_{L^p(R^2)} \le c\ \varepsilon^{\frac{1}{2}-\frac{1}{p}}\max\left\{\frac{1}{t^{1-1/2m}};\frac{1}{t^{1/2m}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$