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Abstract

We consider the inviscid incompressible limit of the compressible Navier-Stokes system on a large domain, the radius of which becomes infinite in the asymptotic limit. We show that the limit solutions satisfy the incompressible Euler system on the whole physical space R^3 as long as the radius of the domain is larger than the speed of acoustic waves inversely proportional to the Mach number. The rate of convergence is estimated in terms of the Mach and Reynolds numbers and the radius of the family of spatial domains.

Key words: Compressible Navier-Stokes system, large domain, inviscid limit, incompressible limit

Contents

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1 Introduction

Scale analysis provides a valuable insight in the behaviour of complex fluid systems in the regime, where some of the characteristic dimensionless parameters become small or infinitely large. We consider a scaled COMPRESSIBLE NAVIER-STOKES SYSTEM:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \nu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.3}$$

supplemented with the no-slip boundary condition

$$\mathbf{u}|_{\partial \Omega_M} = 0, \tag{1.4}$$

where $\Omega_M \subset R^3$ is a smooth, bounded, simply connected domain.

Motivated by Kelliher, Lopes, and Nussenzveig-Lopes [5], where the authors consider the inviscid limit of the *incompressible* Navier-Stokes system on a family of domains $\Omega_M = M\Omega$, $M \to \infty$, we consider a family of domains $\{\Omega_M\}_{M>0}$ enjoying the following properties:

- $\Omega_M \subset R^3$ are simply connected, bounded C^2 domains, uniformly for $M \to \infty$;
- there exists $\alpha > 0$ such that
$$\left\{ x \in R^3 \ \middle| \ |x| < \alpha M \right\} \subset \Omega_M; \tag{1.5}$$
- there exists $\beta > 0$ such that
$$|\partial \Omega_M|_2 \leq \beta M^2, \tag{1.6}$$
where $|\cdot|_2$ denotes the standard two-dimensional Hausdorff measure.

Our goal is to identify the triple singular limit, where

$$\varepsilon \to 0, \ \nu \to 0, \ \text{while } M \to \infty.$$

The present situation is more complex than that considered in [5] as the *compressible* Navier-Stokes system in the low Mach number limit generates rapidly oscillating acoustic waves, at least for the so-called ill-prepared initial data.

Similarly to Masmoudi [9], [10], Wang and Jiang [12], and others, our approach is based on the relative entropy (modulated energy) inequality put in the general framework introduced in [3]. We consider the *ill-prepared initial data* in the form

$$\varrho(0, \cdot) = \varrho_{0,\varepsilon} = 1 + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \ \mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.7}$$

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(R^3), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(R^3; R^3) \text{ as } \varepsilon \to 0. \tag{1.8}$$

Remark 1.1 *The far-field value of the density has been set to $\overline{\varrho} = 1$ for the sake of simplicity. The general case $\overline{\varrho} > 0$ can be handled with obvious modifications.*

In particular, the initial perturbation of the density is proportional to the Mach number. Under such circumstances, the relative entropy inequality specified in Section 3 yields similar bounds for any (weak) solution of (1.1 - 1.8) uniform for $t \in [0, T]$.

Formally, it is easy to identify the limit system for $\varepsilon, \nu \to 0$, $M \to \infty$. Indeed, $\varrho = \varrho_{\varepsilon,\nu,M}$, $\mathbf{u} = \mathbf{u}_{\varepsilon,\nu,M}$ being a solution of the problem (1.1 - 1.8), we expect that

$$\varrho \to 1, \ \mathbf{u} \to \mathbf{v},$$

where $\mathbf{v}$ is a solution of the incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{v} = 0 \text{ in } R^3. \tag{1.9}$$

The principal difficulties of a rigorous proof of such a scenario are:

- The *target* Euler system is defined on R^3 while the *primitive* system (1.1 - 1.8) on Ω_M, the solution $\mathbf{v}$ is not an admissible test function in the relative entropy inequality.
- The same problem occurs with the solutions of the associated acoustic system.

The afore-mentioned difficulties require introducing a corrector in the relative entropy inequality. A careful analysis of the extra terms in the relative entropy inequality due to the presence of the corrector as well as estimates of the actual *convergence rate* in terms of the singular parameters $\varepsilon, \mathbf{u}, M$ are the main novelties of the present paper.

The paper is organized as follows. In Section 2, we collect the necessary preliminaries and formulate our main result. Next, in Section 3, we introduce the relative entropy inequality and use it to derive uniform bounds on the family of solutions independent of the parameters ε, ν, and M.In Section 4, we analyze the behavior of acoustic waves. The cut-off operators are introduced in Section 5. The proof of the main result is completed in Section 6.

2 Main result

We consider the class of *finite energy weak solutions* of the compressible Navier-Stokes system (1.1-1.4) satisfying, besides the standard weak formulation of the equations (1.1 - 1.3), the *energy inequality*

$$\int_{\Omega_M} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} H(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x + \nu \int_0^\tau \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \tag{2.1}$$
$$\leq \int_{\Omega_M} \left[\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} H(\varrho_{0,\varepsilon}) \right] \ \mathrm{d}x \text{ for a.a. } \tau > 0,$$

where we have set

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z.$$

2.1 Solutions of the primitive system

Suppose that the pressure $p = p(\varrho)$ satisfies

$$p \in C[0,\infty) \cap C^3(0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty, \tag{2.2}$$

where

$$\gamma > \frac{3}{2}. \tag{2.3}$$

Then the (primitive) Navier-Stokes-Fourier system (1.1 - 1.4) admits a global in time finite energy weak solution for any finite energy initial data, see [2], Lions [8].

2.2 Solutions of the target system

We suppose that the limit velocity field $\mathbf{u}_0$ introduced in (1.8) is sufficiently smooth and confined to a compact set, specifically,

$$\mathbf{u}_0 \in C^m(R^3;R^3) \text{ for a certain } m > 4,\ \mathrm{supp}[\mathbf{u}_0] \text{ compact in } R^3.$$

As is well known, see e.g. Kato and Lai [4], the limit Euler system (1.9), endowed with the initial datum

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0], \tag{2.4}$$

where $\mathbf{H}$ denotes the standard Helmholtz projection onto the space of solenoidal functions, possesses a smooth solution

$$\mathbf{v} \in C^k([0,T_{\max}); W^{m-k,2}(R^3;R^3)), k = 1,..,m-1 \tag{2.5}$$

defined on a maximal time interval $[0,T_{\max})$, $T_{\max} > 0$.

Remark 2.1 *As a matter of fact, it is enough to assume* $\mathbf{u}_0 \in W^{m,2}(R^3;R^3)$*, with* $m > \frac{5}{2}$*. Compactness of* $\mathrm{supp}[\mathbf{u}_0]$ *is assumed solely for the purposes of the proposed singular limit.*

2.3 Acoustic system

Acoustic system is determined by the "compressible part" of the system (1.1), (1.2). Formally, we can rewrite (1.1), (1.2) in the form of *Lighthill's acoustic analogy*:

$$\varepsilon\partial_t \frac{\varrho - 1}{\varepsilon} + \mathrm{div}_x(\varrho\mathbf{u}) = 0,$$

$$\varepsilon\partial_t(\varrho\mathbf{u}) + p'(1)\nabla_x \frac{\varrho-1}{\varepsilon} = \varepsilon\left[\nu\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) - \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) - \nabla_x\left(p(\varrho) - p'(1)\frac{\varrho-1}{\varepsilon} - p(1)\right)\right],$$

where the quantity on the right-hand side of the second equation is termed Lighthill's tensor, see Lighthill [6], [7]. Accordingly, we consider the *acoustic system* in the form

$$\varepsilon\partial_t s + \Delta\Psi = 0,\ \varepsilon\partial_t\nabla_x\Psi + a\nabla_x s = 0,\ a = p'(1) > 0, \tag{2.6}$$

supplemented with the initial data

$$s(0,\cdot) = \varrho_0^{(1)},\ \nabla_x\Psi(0,\cdot) = \nabla_x\Psi_0 = \mathbf{u}_0 - \mathbf{H}[\mathbf{u}_0]. \tag{2.7}$$

2.4 Main results

We are ready to state our main result:

Theorem 2.1 *Let the pressure p satisfy the hypotheses (2.2), (2.3). Let $\{\Omega_M\}_{M>0}$ be a family of uniformly C^2-domains in R^3 such that (1.5), (1.6) hold for $M = M(\varepsilon)$,*

$$\varepsilon M(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \tag{2.8}$$

Let the initial data $[\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}]$ for the compressible Navier-Stokes system (1.1 - 1.4) be of the form

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = 1 + \varepsilon\varrho_{0,\varepsilon}^{(1)},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\ \|\varrho_{0,\varepsilon}^{(1)}\|_{L^2\cap L^\infty(R^3)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2(R^3;R^3)} \leq D. \tag{2.9}$$

In addition, suppose we are given functions $\mathbf{u}_0$, $\varrho_0^{(1)}$ such that

$$\mathbf{u}_0 \in C^m(R^3;R^3),\ \varrho_0^{(1)} \in C^m(R^3),\ \|\mathbf{u}_0\|_{C^m(R^3;R^3)} + \|\varrho_0^{(1)}\|_{C^m(R^3)} \leq D,\ m > 4, \tag{2.10}$$

$$\mathrm{supp}[\mathbf{u}_0],\ \mathrm{supp}[\varrho_0^{(1)}] \text{ compact in } R^3. \tag{2.11}$$

Let $T_{\max} > 0$ be the life-span of the smooth solution $\mathbf{v}$ of the Euler system (1.9), endowed with the initial datum $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$, and let $0 < T < T_{\max}$. Let $[s,\Psi]$ be the solution of the acoustic system (2.6), with the initial data (2.7).

Then

$$\left\|\sqrt{\varrho}\Big(\mathbf{u} - \nabla_x\Psi - \mathbf{v}\Big)(\tau,\cdot)\right\|_{L^2(\Omega_M;R^3)} + \left\|\left(\frac{\varrho-1}{\varepsilon}\right)(\tau,\cdot) - s(\tau,\cdot)\right\|_{L^2+L^\gamma(\Omega_M)} \tag{2.12}$$

$$\leq c(D,T,\alpha)\left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|_{L^2(\Omega_M;R^3)} + \left\|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right\|_{L^2(\Omega_M)} + \left(\nu + \varepsilon^\alpha + \frac{1}{\varepsilon M(\varepsilon)}\right)^{1/2}\right],$$

$$\tau \in [0,T],\ 0 < \alpha < 1,\ \varepsilon \to 0,$$

for any weak solution $[\varrho, \mathbf{u}]$ of the compressible Navier-Stokes system (1.1 - 1.4).

Corollary 2.1 *In addition to the hypotheses of Theorem 2.1, assume that*

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \ \mathit{in}\ L^2(R^3),\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \mathit{in}\ L^2(R^3;R^3) \ \mathit{as}\ \varepsilon \to 0.$$

Then

$$\mathrm{ess}\sup_{t\in(0,T)} \|\varrho - 1\|_{L^2+L^\gamma(K)} \to 0 \ \mathit{as}\ \varepsilon \to 0,$$

$$\mathrm{ess}\sup_{t\in(\delta,T)} \left\| \sqrt{\varrho}\Big(\mathbf{u} - \mathbf{v}\Big)(t,\cdot)\right\|_{L^2(K;R^3)} \to 0 \ \mathit{as}\ \varepsilon \to 0 \tag{2.13}$$

for any $0 < \delta < T$ *and any compact* $K \subset R^3$.

Remark 2.2 *The acoustic waves represented by the pair are* $[s, \nabla_x \Psi]$ *enjoy dispersive estimates specified in Section 4 below. On the other hand, however, the acoustic equation conserves energy; whence (2.13) holds only on* compact *subsets of the physical space.*

The rest of the paper is devoted to the proof of Theorem 2.1 and Corollary 2.1.

3 Relative entropy

Motivated by [3], we introduce the *relative entropy* functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) = \int_{\Omega_M} \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2}\Big(H(\varrho) - H'(r)(\varrho - r) - H(r) \Big) \right] \ \mathrm{d}x$$

along with the relative entropy inequality

$$\left[\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \right]_{t=0}^{t=\tau} \tag{3.1}$$

$$+\nu \int_0^\tau \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t \leq \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) \ \mathrm{d}t,$$

with the remainder $\mathcal{R}$

$$\mathcal{R}\left(\varrho, \mathbf{u}, r, \mathbf{U}\right) = \int_{\Omega_M} \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U}\Big) \cdot (\mathbf{U} - \mathbf{u})\Big) \ \mathrm{d}x + \nu \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x \tag{3.2}$$

$$+\frac{1}{\varepsilon^2} \int_{\Omega_M} \Big((r - \varrho)\partial_t P(r) + \nabla_x P(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \Big) \ \mathrm{d}x - \frac{1}{\varepsilon^2} \int_{\Omega_M} \Big(p(\varrho) - p(r) \Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x,$$

where $P = H'$.

As shown in [1], *any* finite energy weak solution $[\varrho, \mathbf{u}]$ to the compressible Navier-Stokes system (1.1 - 1.4) satisfies the relative entropy inequality for any pair of sufficiently smooth "test" functions

$$r > 0, \ \mathbf{U}|_{\partial\Omega_M} = 0.$$

3.1 Uniform bounds

Before stating the uniform bounds on the family of solutions $[\varrho, \mathbf{u}]$, it is convenient to introduce the following decomposition:

$$h = h_{\rm ess} + h_{\rm res},$$

where

$$h_{\rm ess} = \chi(\varrho)h, \ h_{\rm res} = 1 - \chi(\varrho),$$

with

$$\chi \in C_c^\infty(0,\infty), \ 0 \le \chi \le 1, \ \chi(\varrho) = 1 \text{ for all } \varrho \text{ in a certain neighborhood of } \varrho = 1.$$

Taking $r \equiv 1$, $\mathbf{U} = 0$ in the relative entropy inequality (3.1) we deduce the following uniform bounds:

$$\operatorname*{ess\,sup}_{t \in [0,T]} \| \sqrt{\varrho}\mathbf{u}(t,\cdot) \|_{L^2(\Omega_M;R^3)} \le c(D), \tag{3.3}$$

$$\operatorname*{ess\,sup}_{t \in [0,T]} \left\| \left[\frac{\varrho - 1}{\varepsilon}(t,\cdot) \right]_{\rm ess} \right\|_{L^2(\Omega_M)} \le c(D), \tag{3.4}$$

and

$$\operatorname*{ess\,sup}_{t \in [0,T]} \Big(\| 1_{\rm res}(t,\cdot) \|_{L^1(\Omega_M)} + \| \varrho^\gamma_{\rm res}(t,\cdot) \|_{L^1(\Omega_M)} \Big) \le \varepsilon^2 c(D). \tag{3.5}$$

4 Energy and dispersive estimates for the acoustic system

As already pointed out, the behavior of acoustic waves is governed by the standard wave equation

$$\varepsilon \partial_t s + \Delta \Psi = 0, \ \varepsilon \partial_t \nabla_x \Psi + a \nabla_x s = 0, \ a = p'(1) > 0,$$

supplemented with the initial data

$$s(0,\cdot) = \varrho_0^{(1)}, \ \nabla_x \Psi(0,\cdot) = \nabla_x \Psi_0 = \mathbf{u}_0 - \mathbf{H}[\mathbf{u}_0].$$

4.1 Finite speed of propagation

For s we obtain

$$\varepsilon^2 \partial^2_{t,t} s - a \Delta s = 0, \ s(0,\cdot) = \varrho_0^{(1)}, \ \partial_t s(0,\cdot) = -\frac{1}{\varepsilon} \Delta \Psi_0. \tag{4.1}$$

Since, by virtue of (2.11), both $\varrho_0^{(1)}$ and $\Delta \Psi_0$ have compact support we deduce that

$$s(t,x), \ \Delta \Psi(t,x) \equiv 0 \text{ for } x \ge c + \sqrt{a}\frac{t}{\varepsilon} \text{ for all } t \ge 0. \tag{4.2}$$

4.2 Energy estimates

The acoustic system conserves energy, specifically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3}\left(as^2+|\nabla_x\Psi|^2\right)(t,\cdot)\mathrm{d}x\ \mathrm{d}t=0. \tag{4.3}$$

Moreover, in view of (4.2) and the hypotheses (2.8), (2.11), we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_M}\left(as^2+|\nabla_x\Psi|^2\right)(t,\cdot)\ \mathrm{d}x\ \mathrm{d}t=0. \tag{4.4}$$

Higher order energy estimates give rise to

$$\|\nabla_x\Psi(\tau,\cdot)\|_{W^{k,2}(R^3;R^3)}+\|s(\tau,\cdot)\|_{W^{k,2}(R^3)}\leq c\|\nabla_x\Psi_0\|_{W^{k,2}(R^3;R^3)}+\|\varrho_0^{(1)}\|_{W^{k,2}(R^3)},\ k=0,1,\ldots,m, \tag{4.5}$$

for any $\tau>0$.

4.3 $L^p - L^q$ estimates

Finally, we recall the well-known $L^p - L^q$ estimates

$$\|\nabla_x\Psi(\tau,\cdot)\|_{L^p(R^3;R^3)}+\|s(\tau,\cdot)\|_{L^p(R^3)}\leq c\left(1+\frac{\tau}{\varepsilon}\right)^{-\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\|\nabla_x\Psi_0\|_{W^{k,q}(R^3;R^3)}+\|\varrho_0^{(1)}\|_{W^{k,q}(R^3)}\right) \tag{4.6}$$

$$k>3,\ 2\leq p\leq\infty,\ \frac{1}{p}+\frac{1}{q}=1,$$

see Strichartz [11].

5 Cut-off operators

In accordance with the hypotheses (1.5), (2.8), the acoustic waves emanating from a compact set (cf. (2.10), (2.11)) will never reach the boundary $\partial\Omega_M$ in a finite time lap $(0,T)$. Consequently, by virtue of (4.2),

$$\nabla_x\Psi|_{\partial\Omega_M}=\nabla_x\Psi_0|_{\partial\Omega_M}\text{ for all }t\in(0,T).$$

Similarly, as the Eulerian velocity is determined by the Biot-Savart law

$$\mathbf{v}=-\mathbf{curl}\Delta^{-1}[\mathbf{curl}[\mathbf{v}]],$$

$\mathbf{v}$ is a **curl** of a harmonic function outside a bounded ball.

We introduce the cut-off functions

$$\mathbf{w}_M = -\eta_M \mathbf{v} - \eta_M \nabla_x \Psi_0, \tag{5.1}$$

where η_M satisfy

$$\eta_M \in C_c^\infty(R^3),\ 0 \le \eta_M \le 1,\ \eta_M|_{\partial\Omega_M} = 1,\ \eta_M(x) = 0 \text{ whenever } \mathrm{dist}[x, \partial\Omega_M] > 1. \tag{5.2}$$

It follows from the previous discussion that both $\mathbf{v}$ and $\nabla_x \Psi_0$ behave like $\frac{1}{|x|^2}$ on $\partial\Omega_M$, therefore, by virtue of the hypothesis (1.6), we get

$$\|\partial_t \mathbf{w}_M(\tau,\cdot)\|_{L^p(\Omega_M;R^3)} + \|\mathbf{w}_M(\tau,\cdot)\|_{W^{2,p}(\Omega_M;R^3)} \le c M^{2\left(\frac{1}{p}-1\right)} \text{ for } 1 \le p \le \infty \tag{5.3}$$

for any $\tau \in (0,T)$.

6 Convergence

Our goal in this section is to show the estimate (2.12) stated in Theorem 2.1. We start with a proper choice of the test functions in the relative entropy inequality (3.1). Specifically, we take

$$r = 1 + \varepsilon s,\ \mathbf{U} = \mathbf{v} + \nabla_x \Psi + \mathbf{w}_M,$$

where $\mathbf{v}$ is the solution of the Euler system (1.9), and s, $\nabla_x \Psi$ represent the oscillatory component solving the acoustic wave equation (2.6). The corrector $\mathbf{w}_M$ introduced in Section 5 satisfies the boundary condition

$$\mathbf{w}_M|_{\partial\Omega_M} = -(\mathbf{v} + \nabla_x \Psi)|_{\partial\Omega_M};$$

whence r and $\mathbf{U}$ are admissible in (3.1).

In the remaining part of this section, we analyze term by term the relative entropy inequality (3.1).

6.1 Initial data

We have,

$$\left[\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)\right](0) = \int_{\Omega_M} \frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon} - \mathbf{H}[\mathbf{u}_0] - \nabla_x\Psi_0 - \mathbf{w}_M(0,\cdot)|^2 \, \mathrm{d}x \tag{6.1}$$

$$+ \int_{\Omega_M} \frac{1}{\varepsilon^2}\left(H(\overline{\varrho} + \varepsilon\varrho_{0,\varepsilon}^{(1)}) - \varepsilon H'(1 + \varepsilon\varrho_{0,\varepsilon}^{(1)})(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}) - H(1 + \varepsilon\varrho_{0,\varepsilon}^{(1)})\right) \, \mathrm{d}x$$

$$\le c(D)\left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|^2_{L^2(\Omega_M;R^3)} + \|\mathbf{w}_M(0,\cdot)\|^2_{L^2(\Omega_M;R^3)} + \|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\|^2_{L^2(\Omega_M)}\right]$$

$$\le c(D)\left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|^2_{L^2(\Omega_M;R^3)} + \|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\|^2_{L^2(\Omega_M)} + \frac{1}{M^2}\right],$$

where the last inequality follows from (5.3).

6.2 Estimates of the remainder in the relative entropy inequality

6.2.1 Dissipation

By virtue of Korn's inequality, we obtain

$$\nu \int_0^\tau \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \frac{\nu}{2} \int_0^\tau \int_{\Omega_M} (\mathbb{S}(\nabla_x \mathbf{U}) - \mathbb{S}(\nabla_x \mathbf{u})) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t + c\nu \int_0^\tau \int_{\Omega_M} |\mathbb{S}(\nabla_x \mathbf{U})|^2 \ \mathrm{d}x \ \mathrm{d}t.$$

Thus the relative entropy inequality (3.1) reduces to:

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau, \cdot) \tag{6.2}$$

$$+\frac{\nu}{2} \int_0^\tau \int_{\Omega_M} \mathbb{S}(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq c(D) \left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|^2_{L^2(\Omega_M; R^3)} + \|\varrho^{(1)}_{0,\varepsilon} - \varrho^{(1)}_0\|^2_{L^2(\Omega_M)} + \frac{1}{M^2} \right]$$

$$+c\nu \int_0^\tau \int_{\Omega_M} \left(|\nabla_x \mathbf{v}|^2 + |\nabla_x^2 \Psi|^2 + |\nabla_x \mathbf{w}_M|^2 \right) \ \mathrm{d}x \ \mathrm{d}t.$$

$$+ \int_0^\tau \int_{\Omega_M} \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big((r - \varrho)\partial_t P(r) + \nabla_x P(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \Big) \ \mathrm{d}x \ \mathrm{d}t - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big(p(\varrho) - p(r) \Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t.$$

$$\leq c(D, T) \left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\|^2_{L^2(\Omega_M; R^3)} + \|\varrho^{(1)}_{0,\varepsilon} - \varrho^{(1)}_0\|^2_{L^2(\Omega_M)} + \frac{1}{M^2} + \nu \right]$$

$$+ \int_0^\tau \int_{\Omega_M} \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big((r - \varrho)\partial_t P(r) + \nabla_x P(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \Big) \ \mathrm{d}x \ \mathrm{d}t - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big(p(\varrho) - p(r) \Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t.$$

6.2.2 Convective terms

We write

$$\int_0^\tau \int_{\Omega_M} \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^\tau \int_{\Omega_M} \varrho\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau \int_{\Omega_M} \varrho\nabla_x \mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t,$$

where the term

$$\int_0^\tau \int_{\Omega_M} \varrho\nabla_x \mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t \leq \int_0^\tau \int_{\Omega_M} \varrho|\mathbf{u}-\mathbf{U}|^2|\nabla_x \mathbf{U}|\ \mathrm{d}x\ \mathrm{d}t$$

can be controlled in terms of the data, specifically,

$$\int_0^\tau \int_{\Omega_M} \varrho|\mathbf{u}-\mathbf{U}|^2|\nabla_x \mathbf{U}|\ \mathrm{d}x\ \mathrm{d}t \leq \int_0^\tau \Big\|\nabla_x \mathbf{v} + \nabla_x \mathbf{w}_M + \nabla_x^2\Psi\Big\|_{L^\infty(\Omega_M;R^3)} \mathcal{E}\ \mathrm{d}t. \tag{6.3}$$

Next, we have

$$\int_0^\tau \int_{\Omega_M} \varrho\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= \int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\Big(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v}\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\partial_t\Big(\nabla_x\Psi + \mathbf{w}_M\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\otimes(\nabla_x\Psi + \mathbf{w}_M) : \nabla_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t$$

$$+\int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\otimes\mathbf{v} : \Big(\nabla_x^2\Psi + \nabla_x \mathbf{w}_M\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\frac{1}{2}\int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\nabla_x|\nabla_x\Psi + \mathbf{w}_M|^2\ \mathrm{d}x\ \mathrm{d}t.$$

Now observe that, in view of the uniform bounds (3.3 - 3.5), the last three integrals above can be controlled by

$$c(D)\int_0^\tau \Big[\|\nabla_x\Psi\|_{W^{1,p}(R^3;R^3)} + \|\mathbf{w}_M\|_{W^{1,p}(\Omega_M)}\Big]\ \mathrm{d}t \text{ for some } p\in(2,\infty) \text{ large enough} \tag{6.4}$$

$$\leq c(D,T)\left(\varepsilon^\alpha + \frac{1}{M^{2\alpha}}\right),\ \alpha = 1-\frac{1}{p} > \frac{1}{2}.$$

Indeed it follows from (3.3 - 3.5) that

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho\mathbf{u}(t,\cdot)\|_{L^2+L^q(\Omega_M;R^3)} \leq c(D) \text{ for a certain } q>1,$$

while

$$\text{ess}\sup_{t\in(0,T)} \|\mathbf{v}(t,\cdot)\|_{W^{1,2}\cap W^{1,\infty}(R^3;R^3)} \leq c(D).$$

The desired bounds on $\nabla_x\Psi$, $\mathbf{w}_M$ follow from (4.6), (5.3), respectively.

Furthermore, we have

$$\int_0^\tau \int_{\Omega_M} \varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\Big(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\Big)\ \mathrm{d}x\ \mathrm{d}t = -\int_0^\tau\int_{\Omega_M}\varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t \tag{6.5}$$

$$= \int_0^\tau\int_{\Omega_M}\varrho\mathbf{u}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t - \int_0^\tau\int_{\Omega_M}\varrho\mathbf{U}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t.$$

Thus using the weak formulation of the continuity equation we get

$$\int_0^\tau\int_{\Omega_M}\varrho\mathbf{u}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t = -\int_0^\tau\int_{\Omega_M}\varrho\partial_t\Pi\ \mathrm{d}x + \left[\int_{\Omega_M}\varrho\Pi\ \mathrm{d}x\right]_{t=0}^{t=\tau} \tag{6.6}$$

$$= -\varepsilon\int_0^\tau\int_{\Omega_M}\frac{\varrho-1}{\varepsilon}\partial_t\Pi\ \mathrm{d}x + \varepsilon\left[\int_{\Omega_M}\frac{\varrho-1}{\varepsilon}\Pi\ \mathrm{d}x\right]_{t=0}^{t=\tau},$$

where, in view of (3.4), (3.5) (2.5), the last two integrals are of order $c(D,T)\varepsilon$. As for the second integral on the right-hand side of (6.5), using $\mathbf{U}=0$ on $\partial\Omega$ and $\mathrm{div}_x\mathbf{v}=0$ we have

$$\left|\int_0^\tau\int_{\Omega_M}\varrho\mathbf{U}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t\right| \leq \varepsilon\left|\int_0^\tau\int_{\Omega_M}\frac{\varrho-1}{\varepsilon}\mathbf{U}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t\right| + \left|\int_0^\tau\int_{\Omega_M}\mathbf{U}\cdot\nabla_x\Pi\ \mathrm{d}x\ \mathrm{d}t\right|$$

$$\leq c(D,T)\varepsilon + \left|\int_0^\tau\int_{\Omega_M}\Pi\mathrm{div}_x\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t\right|$$

$$\leq c(D,T)\varepsilon + c(D)\int_0^\tau \|\Delta\Psi\|_{L^\infty(\Omega_M)} + \|\mathrm{div}_x\mathbf{w}_M\|_{L^\infty(\Omega_M)}\ \mathrm{d}t$$

$$\leq c(D,T)\left(\varepsilon\ln(1+\varepsilon^{-1}T)+\frac{1}{M^2}\right),$$

where the log-term results from integrating (4.6), (5.3). Finally,

$$\int_0^\tau\int_{\Omega_M}\varrho\Big(\mathbf{U}-\mathbf{u}\Big)\cdot\partial_t\Big(\nabla_x\Psi+\mathbf{w}_M\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$= -\int_0^\tau\int_{\Omega_M}\varrho\mathbf{u}\cdot\partial_t\Big(\nabla_x\Psi+\mathbf{w}_M\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_0^\tau\int_{\Omega_M}\varrho\mathbf{v}\cdot\partial_t\Big(\nabla_x\Psi+\mathbf{w}_M\Big)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\frac{1}{2}\int_0^\tau\int_{\Omega_M}\varrho\partial_t\left|\nabla_x\Psi+\mathbf{w}_M\right|^2\ \mathrm{d}x\ \mathrm{d}t,$$

where

$$\int_0^\tau \int_{\Omega_M} \varrho \mathbf{v} \cdot \partial_t \Big(\nabla_x \Psi + \mathbf{w}_M \Big) \ \mathrm{d}x \ \mathrm{d}t = \varepsilon \int_0^\tau \int_{\Omega_M} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \partial_t \Big(\nabla_x \Psi + \mathbf{w}_M \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_{\Omega_M} \mathbf{v} \cdot \partial_t \Big(\nabla_x \Psi + \mathbf{w}_M \Big) \ \mathrm{d}x \ \mathrm{d}t.$$

Now, we use the acoustic equation (2.6) to rewrite

$$\varepsilon \int_0^\tau \int_{\Omega_M} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \partial_t \nabla_x \Psi \ \mathrm{d}x \ \mathrm{d}t = -a \int_0^\tau \int_{\Omega_M} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \nabla_x s \ \mathrm{d}x \ \mathrm{d}t,$$

which is controlled by

$$c(D) \int_0^\tau \| \nabla_x s \|_{L^\infty(\Omega_M; R^3)} \leq c(D,T) \varepsilon \ln(1 + \varepsilon^{-1} T).$$

While, as $M >> \frac{1}{\varepsilon}$,

$$\int_0^\tau \int_{\Omega_M} \mathbf{v} \cdot \partial_t \nabla_x \Psi \ \mathrm{d}x \ \mathrm{d}t = \int_{\partial \Omega_M} \mathbf{v} \cdot \mathbf{n} \ \partial_t \Psi \ \mathrm{d}S_x = \int_{\partial \Omega_M} \mathbf{v} \cdot \mathbf{n} \ \partial_t \Psi_0 \ \mathrm{d}S_x = 0$$

Thus we may write the relative entropy inequality in the form

$$\mathcal{E} \left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right) (\tau, \cdot) \tag{6.7}$$

$$\leq c(D) \Big[\| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \|^2_{L^2(\Omega_M; R^3)} + \| \varrho^{(1)}_{0,\varepsilon} - \varrho^{(1)}_0 \|^2_{L^2(\Omega_M; R^3)} \Big] + \int_0^\tau \Big\| \nabla_x \mathbf{v} + \nabla_x \mathbf{w}_M + \nabla_x^2 \Psi \Big\|_{L^\infty(\Omega_M; R^3)} \mathcal{E} \ \mathrm{d}t$$

$$+ c(D,T) \left(\varepsilon^\alpha + \frac{1}{M} + \nu \right) + \frac{1}{2} \left[\int_{\Omega_M} |\nabla_x \Psi|^2 \ \mathrm{d}x \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\Omega_M} \varrho \mathbf{u} \cdot \partial_t \nabla_x \Psi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big((r - \varrho) \partial_t P(r) + \nabla_x P(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \Big) \ \mathrm{d}x \ \mathrm{d}t - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big(p(\varrho) - p(r) \Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t.$$

6.2.3 Terms depending on the pressure

In order to bound the remaining integrals in (6.7) we first realize that

$$\int_{\Omega_M} \nabla_x P(r) \cdot r \mathbf{U} \ \mathrm{d}x = - \int_{\Omega_M} p(r) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x. \tag{6.8}$$

Next, we get

$$\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \nabla_x P(r) \cdot (\varrho \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t = \frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_M} P'(r) \nabla_x s \cdot (\varrho \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_{\Omega_M} \frac{P'(1+\varepsilon s) - P'(1)}{\varepsilon} \nabla_x s \cdot (\varrho \mathbf{u}) \ {\rm d}x \ {\rm d}t + \frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_M} p'(1) \nabla_x s \cdot (\varrho \mathbf{u}) \ {\rm d}x \ {\rm d}t,$$

where the first integral on the right-hand side is controlled by the dispersive estimates. Furthermore, using the acoustic equation, we deduce

$$\frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_M} p'(1) \nabla_x s \cdot (\varrho \mathbf{u}) \ {\rm d}x \ {\rm d}t = - \int_0^\tau \int_{\Omega_M} \varrho \mathbf{u} \cdot \partial_t \nabla_x \Psi \ {\rm d}x \ {\rm d}t.$$

Now we write

$$\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} \Big[(r - \varrho) \partial_t P(r) - p(\varrho) {\rm div}_x \mathbf{U} \Big] \ {\rm d}x \ {\rm d}t \tag{6.9}$$

$$= \frac{1}{\varepsilon} \int_0^\tau \int_{\Omega_M} (r - \varrho) P'(r) \partial_t s \ {\rm d}x \ {\rm d}t + \int_0^\tau \int_{\Omega_M} \frac{p(\varrho) - p(1)}{\varepsilon^2} \Delta \Psi \ {\rm d}x \ {\rm d}t - \int_0^\tau \frac{1}{\varepsilon^2} \int_{\Omega_M} p(\varrho) {\rm div}_x \mathbf{w}_M \ {\rm d}x \ {\rm d}t$$

$$= - \frac{1}{\varepsilon^2} \int_{\Omega_M} p(\varrho) {\rm div}_x \mathbf{w}_M \ {\rm d}x$$

$$+ \int_0^\tau \int_{\Omega_M} \frac{1 - \varrho}{\varepsilon} P'(r) \partial_t s \ {\rm d}x \ {\rm d}t + \int_0^\tau \int_{\Omega_M} s P'(r) \partial_t s \ {\rm d}x \ {\rm d}t$$

$$\frac{1}{\varepsilon^2} \int_0^\tau \int_{\Omega_M} p'(1)(\varrho - 1) \Delta \Psi \ {\rm d}x \ {\rm d}t - \int_0^\tau \int_{\Omega_M} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2} \Delta \Psi \ {\rm d}x \ {\rm d}t,$$

where

$$\left| \int_0^\tau \int_{\Omega_M} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2} \Delta \Psi \ {\rm d}x \ {\rm d}t \right| \leq \int_0^\tau \| \Delta \Psi \|_{L^\infty(\Omega_M)} \mathcal{E} \ {\rm d}t. \tag{6.10}$$

Furthermore, we have

$$\frac{1}{\varepsilon^2} \int_{\Omega_M} p(\varrho) {\rm div}_x \mathbf{w}_M \ {\rm d}x = \frac{1}{\varepsilon^2} \int_{\Omega_M} \Big(p(\varrho) - p(1) \Big) {\rm div}_x \mathbf{w}_M \ {\rm d}x + \frac{1}{\varepsilon^2} \int_{\Omega_M} p(1) {\rm div}_x \mathbf{w}_M \ {\rm d}x,$$

where, in accordance with(3.4), (3.5),

$$\left| \frac{1}{\varepsilon^2} \int_{\Omega_M} \Big(p(\varrho) - p(1) \Big) {\rm div}_x \mathbf{w}_M \ {\rm d}x \right| \tag{6.11}$$

$$\leq \frac{1}{\varepsilon} \int_{\Omega_M} \left| \left[\frac{p(\varrho) - p(1)}{\varepsilon} \right]_{\rm ess} \right| |{\rm div}_x \mathbf{w}_M| \ {\rm d}x + \frac{1}{\varepsilon^2} \int_{\Omega_M} \left| \left[\frac{p(\varrho) - p(1)}{\varepsilon} \right]_{\rm res} \right| |{\rm div}_x \mathbf{w}_M| \ {\rm d}x$$

$$\leq c(D) \left(\frac{1}{\varepsilon M} + \frac{1}{M^2} \right),$$

while, by Green's theorem,

$$\int_{\Omega_M} p(1) {\rm div}_x \mathbf{w}_M \ {\rm d}x = -p(1) \int_{\partial \Omega_M} \left(\mathbf{v} \cdot \mathbf{n} + \nabla_x \Psi_0 \cdot \mathbf{n} \right) \ {\rm d}S_x$$

$$-p(1)\int_{\Omega_M}\left(\mathrm{div}_x\mathbf{v}+\mathrm{div}_x\mathbf{u}_0\right)\ \mathrm{d}x=0.$$

Finally, neglecting small terms, we get

$$\left|\int_0^\tau\int_{\Omega_M}\frac{1-\varrho}{\varepsilon}P'(r)\partial_t s\ \mathrm{d}x\ \mathrm{d}t\right|\leq\int_0^\tau\int_{\Omega_M}\frac{1-\varrho}{\varepsilon}p'(1)\partial_t s\ \mathrm{d}x\ \mathrm{d}t+c(D)\int_0^\tau\|\Delta\Psi\|_{L^\infty(\Omega_M)}\mathcal{E}\ \mathrm{d}t,$$

where the first integral will cancel out with its counterpart in (6.9), and

$$\left|\int_0^\tau\int_{\Omega_M}sP'(r)\partial_t s\ \mathrm{d}x\ \mathrm{d}t\right|\leq p'(1)\left[\frac{1}{2}\int_{\Omega_M}s^2\ \mathrm{d}x\right]_{t=0}^{t=\tau}+c(D)\int_0^\tau\|\Delta\Psi\|_{L^\infty(\Omega_M)}\mathcal{E}\ \mathrm{d}t.$$

Using the energy equality (4.4) for the acoustic system, we may apply Gronwall's lemma to (6.7) to conclude that

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle|r,\mathbf{U}\right)(\tau,\cdot) \tag{6.12}$$

$$\leq c(D,T)\left[\|\mathbf{u}_{0,\varepsilon}-\mathbf{u}_0\|^2_{L^2(\Omega_M;R^3)}+\|\varrho^{(1)}_{0,\varepsilon}-\varrho^{(1)}_0\|_{L^2(\Omega_M)}\right]$$

$$+c(D,T,\alpha)\left(\nu+\varepsilon^\alpha+\frac{1}{M}+\frac{1}{\varepsilon M}+\frac{1}{M^2}\right)$$

$$\leq c(D,T)\left[\|\mathbf{u}_{0,\varepsilon}-\mathbf{u}_0\|^2_{L^2(\Omega_M;R^3)}+\|\varrho^{(1)}_{0,\varepsilon}-\varrho^{(1)}_0\|_{L^2(\Omega_M)}\right]+c(D,T,\alpha)\left(\nu+\varepsilon^\alpha+\frac{1}{\varepsilon M}\right),$$

which is nothing other than (2.12).

Thus we have proved Theorem 2.1. Finally, we conclude by observing that Corollary (2.1) follows from the inequality (2.12), combined with the estimates established in Section 4.

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