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The effective energy in the Allen–Cahn model with deformation

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Abstract The sharp interface limit of a diffuse interface theory of phase transitions is considered in static situations. The diffuse interface model is of the Allen–Cahn type with deformation, with a parameter ε measuring the width of the interface. Equilibrium states of a given elongation and a given interface width are considered and the asymptotics for $\varepsilon \to 0$ of the equilibrium energy is determined. The interface energy is defined as the excess energy over the corresponding two phase state with a sharp interface without the interface energy. It is shown that to within the term of order $o(\varepsilon)$ the interface energy is equal to $\sigma\varepsilon$ where the coefficient σ is given by a new formula that involves the mechanical contribution to the total energy. Also the corresponding equilibrium states are determined and shown to converge to a sharp interface state for $\varepsilon \to 0$.

Keywords Phase transitions, diffuse and sharp phase interface, interfacial energy

MSC 2000 classification 74A50, 49Q20

Contents

1 Introduction

By the Allen–Cahn model with deformation we mean a model of a phase transformation in a solid which in addition to the displacement of the body introduces an extra scalar field ϕ called the phase field. The phase field is an internal state variable which takes different values in the two phases of the body and which changes continuously but steeply across the phase interface, which is therefore "diffuse." Throughout the paper, we consider the 1 dimensional static case in which the body is an interval $(0,1)$ of material points x and the displacement u is given by a scalar valued function $u = u(x)$ of the scalar variable x. The displacement u gives rise to the strain $e(x) = u'(x)$ where the prime denotes the differentiation with respect to x.

The evolution equation for this model is a generalization of the Allen–Cahn equation [1].

States of the body are determined by the displacement u over the body and the phase field ϕ over the body. The total energy of the state (u, ϕ) is given by

$$\mathsf{F}_\varepsilon(u,\phi) = \int_0^1 \varepsilon^2 \phi'^2(x)/2 + f(u'(x), \phi(x))\, dx; \tag{1.1}$$

here ε is a small parameter that will eventually tend to 0 and f is a given function of the indicated variables called the coarse grain energy. It will be seen that the quadratic term in the integrand gives rise to the interfacial energy which is approximatively proportional to ε. Setting formally $\varepsilon = 0$ gives the theory in which the interfacial energy is neglected.

For a given strain e, the value of ϕ corresponding to the pure phase of the strain e is determined by the pointwise minimization of $f(e, \phi)$ with respect to ϕ. For the two-phase system it is appropriate to assume that for a given e there are exactly two local minima $\phi_i(e)$, $i = 1$ or 2, of which one is an absolute minimum. The absolute minimum indicates the stable phase corresponding to the strain e just mentioned while the nonabsolute local minimum the unstable phase complementary to the stable phase. A point $x \in (0,1)$ is in the phase i where $i = 1$ or 2 if $\phi(x)$ is close to the value $\phi_i(e(x))$. One can introduce the energy f_i of the phase i by

$$f_i(e) = f(e, \phi_i(e))$$

where $i = 1$ or 2 and e runs over the set of all reals. Typically each of the functions f_i describes a potential well. The minimum energy

$$w(e) = \min\{f_1(e), f_2(e)\}$$

is typically a double well potential as shown in Figure 1.1.

w

E_1 E_2 e

Fig. 1.1.

Given the elongation λ, the equilibrium state is obtained by seeking the infimum

$$\mathsf{E}_\varepsilon(\lambda) = \inf\{\mathsf{F}_\varepsilon(u,\phi) : \text{the state } (u,\phi) \text{ satisfies } u(0) = 0,\ u(1) = \lambda\}. \tag{1.2}$$

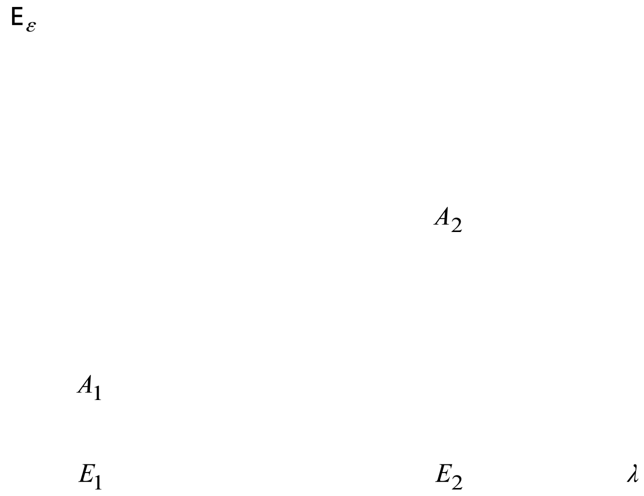

Fig. 1.2.

Conditions will be given below under which the infimum is a minimum. The state (u, ϕ) which realizes the infimum is the equilibrium state under the given elongation.

In the present paper, we examine the properties of the function $\mathsf{E}_\varepsilon(\lambda)$ and the minimizers in (1.2) for $\varepsilon \to 0$. It turns out that for small values of $\varepsilon > 0$, the typical shape of the function $\mathsf{E}_\varepsilon(\cdot)$ is shown by the bold line in Figure 1.2. This is justified by considering first the energy of the theory with $\varepsilon = 0$ and then the theory with $\varepsilon > 0$ as follows.

In Figure 1.2, the line segment $[A_1, A_2]$ is the part of the common tangent to the graph of w. The graph of the convex envelope w^{co} of w consists of the graph of w outside the segment (E_1, E_2) while in that segment the graph of w is replaced by the segment $[A_1, A_2]$. Setting $\varepsilon = 0$ and using definition (1.2) one obtains the well known common tangent construction of the gibbsian thermostatics saying

$$\mathsf{E}_0(\lambda) = w^{\mathrm{co}}(\lambda)$$

for all elongations λ. The underlying interpretation is that for λ outside the interval (E_1, E_2) the minimum is achieved by the single phase homogeneous state of strain λ which represents the pure phase. For $\lambda \in (E_1, E_2)$ the infimum is realized by a two phase state which mixes particular amounts of the pure phase states A_1 and A_2 in such a way as to satisfy the constraint that the elongation be λ. The mentioned amounts are linear (affine) functions of λ and the resulting minimum energy is linear (affine) in (E_1, E_2).

If $\varepsilon > 0$ is small then the infimum energy is realized on homogeneous single phase states for λ outside the interval (E_1, E_2) while the bold line inside the interval (E_1, E_2) corresponds to a mixture of two phase states satisfying the elongation constraint and bearing the interfacial energy which shifts the graph above the common tangent segment, as will be shown in the present paper. Furthermore, the homogeneous single phase states minimize the energy slightly beyond the point A_1 and slightly before the point A_2 as the two phase state has a bigger energy due to the contribution of the interface.

We define the interfacial energy corresponding to the parameter ε as the excess energy over the energy $w^{\mathrm{co}}(\lambda)$ of the theory with $\varepsilon = 0$. The main result of the paper says that under natural assumptions the interfacial energy of a two phase equilibrium state is asymptotically linear in ε for $\varepsilon \to 0$. More precisely it will be proved that for any $\varepsilon > 0$ and $\lambda \in \mathbf{R}$ we have

$$\mathsf{E}_\varepsilon(\lambda) = \begin{cases} w(\lambda) & \text{if } \lambda \notin (E_1, E_2), \\ w^{\mathrm{co}}(\lambda) + \varepsilon\sigma + \mathrm{o}(\varepsilon, \lambda) & \text{if } \lambda \in (E_1, E_2) \end{cases} \tag{1.3}$$

where $\sigma > 0$ is a constant given by an explicit formula in terms of f (see (2.8), below) and

$$\mathrm{o}(\varepsilon, \lambda)/\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \text{ for any } \lambda \in (E_1, E_2).$$

Moreover, the equilibrium states of a given elongation will be described explicitly for any $\varepsilon > 0$ and it will be proved that for $\varepsilon \to 0$ and $\lambda \in (E_1, E_2)$ they approach the mixture of the two pure phase states separated by a sharp interface corresponding to $\varepsilon = 0$. These results show that for $\varepsilon > 0$ the present theory approximates that with the sharp interface which bears the energy of the magnitude $\varepsilon\sigma$.

The scaling with respect to ε as in (1.1) and the above interpretation is different from the one adopted in the literature so far. In the absence of deformation the standard scaling amounts to dealing with the sequence of functionals

$$\mathsf{H}_\varepsilon(\phi) = \int_0^1 \varepsilon\phi'^2(x)/2 + h(\phi(x))/\varepsilon \, dx$$

where h is a nonnegative double well potential vanishing only at ± 1. The relationship between the theory with $\varepsilon > 0$ and its sharp interface limit is well understood both in the dynamical situations (see [2, 4, 10] and the references therein) and statical situations [8–9], in any dimension. The main static result asserts that the gamma limit of H_ε is proportional to the area of the interface, via a coefficient σ given by

$$\sigma = \int_{-1}^{1} \sqrt{2h(\eta)} \, d\eta. \tag{1.4}$$

The interpretation is that in the limit the state is a mixture of the states $\phi = \pm 1$ with definite relative weights. Introducing the energy

$$\mathsf{F}_\varepsilon(\phi) = \varepsilon \mathsf{H}_\varepsilon(\phi) = \int_0^1 \varepsilon^2\phi'^2(x)/2 + h(\phi(x)) \, dx$$

we obtain the scaling similar to (1.1) and the mentioned gamma limit can be rephrased as the assertion that

$$\mathsf{F}_\varepsilon(\phi) = \varepsilon\sigma \times \text{the area of the interface} + \mathrm{o}(\varepsilon, \phi)$$

with

$$\mathrm{o}(\varepsilon, \phi)/\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \text{ for any } \phi.$$

This form of the result is analogous to $(1.3)_2$.

In the theory with deformation Leo, Lowengrub & Jou [7], Fried & Gurtin [5], Garcke [6] consider the following sequence of functionals

$$\mathsf{H}_\varepsilon(u,\phi) = \int_0^1 \varepsilon\phi'^2(x)/2 + h(\phi(x))/\varepsilon + f(e(x),\phi(x))\,dx \tag{1.5}$$

$\varepsilon > 0$, where h is a double well potential with minima at ± 1, f is the coarse grain energy, and e is the strain tensor. (Bodies of arbitrary dimension $n \geq 1$ are considered in the cited papers.) It turns out that the limit of H_ε is proportional to the area of the (sharp) interface with the coefficient of proportionality given by (1.4). We observe that the limit is identical to the case neglecting the deformation, with the coefficient σ independent of f. In contrast, the coefficient σ as in $(1.3)_2$ [Equation (2.8), below] depends on f in an essential way. The separation of variables as in (1.5) seems to be hard to motivate in view of the results of the present paper.

2 Assumptions

Any pair $(u,\phi) \in H := W^{1,2}((0,1),\mathbf{R}^2)$ is referred to as state; then u is interpreted as the displacement and ϕ as a phase field. For $\varepsilon \geq 0$, we define the energy of the state (u,ϕ) by

$$\mathsf{F}_\varepsilon(u,\phi) = \int_0^1 \varepsilon^2\phi'^2(x)/2 + f(u'(x),\phi(x))\,dx \tag{2.1}$$

where $f : \mathbf{R}^2 \to \mathbf{R}$ is a given twice continuously differentiable function subject to Hypotheses H_1–H_4 listed below. These hypotheses imply that the right hand side of (2.1) defines an absolutely convergent integral. For a given $\lambda \in \mathbf{R}$ we define the collection of states

$$D(\lambda) = \{(u,\phi) \in W^{1,2}((0,1),\mathbf{R}^2) : u(0) = 0, u(1) = \lambda\}$$

of elongation λ and the ***effective energy corresponding to the elongation*** λ**,** the main object of the present paper, by

$$\mathsf{E}_\varepsilon(\lambda) = \inf\{\mathsf{F}_\varepsilon(u,\phi) : (u,\phi) \in D(\lambda)\} \tag{2.2}$$

for every $\varepsilon > 0$.

We denote the generic variable of f by $(e,\eta) \in \mathbf{R}^2$ with e the small strain and define the response function for the stress by

$$\hat{s} = \mathrm{D}_e f$$

where $\mathrm{D}_e f$ denotes the derivative of f with respect to its first argument e. We furthermore put

$$\hat{p} = \mathrm{D}_\eta f$$

where $\mathrm{D}_\eta f$ denotes the derivative of f with respect to its second argument η.

We make the following four hypotheses:

H_1 For each $\phi \in \mathbf{R}$ the function $f(\cdot,\phi)$ is strictly convex in the sense that $\hat{s}(\cdot,\phi)$ is strictly increasing; moreover, the range of $\hat{s}(\cdot,\phi)$ is $\mathbf{R}$.

H_2 There exist constants $c_j, j = 1,\dots,4$, with $c_1 > 0$, such that

$$c_1(e^2+\eta^2) + c_2 \leq f(e,\eta) \leq c_3(e^2+\eta^2) + c_4 \tag{2.3}$$

$$|\hat{s}(e,\eta)| + |\hat{p}(e,\eta)| \le c_3(e^2+\eta^2)+c_4 \qquad (2.4)$$

for all $(e,\eta) \in \mathbf{R}^2$.

H_3 There exist numbers S, G and points $(E_i, \Phi_i) \in \mathbf{R}^2$, $i = 1, 2$, with $E_1 < E_2$, $\Phi_1 < \Phi_2$, such that

$$f(e,\eta) \ge G + Se \qquad (2.5)$$

for all $(e,\eta) \in \mathbf{R}^2$ with the equality holding if and only if $(e,\eta) = (E_i, \Phi_i)$ for some $i = 1, 2$.

H_4 The function $w : \mathbf{R} \to \mathbf{R}$, defined by

$$w(e) = \min\{f(e,\eta) : \phi \in \mathbf{R}\}, \quad e \in \mathbf{R},$$

[with the minimum existing by $(2.3)_1$], is continuously differentiable at E_1 and E_2, and the restrictions of w to the intervals $(-\infty, E_1]$ and $[E_2, \infty)$ are convex.

Here H_1 is the basic convexity assumption. It will be seen that H_1 in conjunction with the form of the integral in (2.1) and the coercivity assumption $(2.3)_1$ guarantee that the infimum in (2.2) is achieved. The corresponding minimizer is the equilibrium state of the given elongation. The role of the coercivity in H_2 has already been commented; the quadratic growth conditions in H_2 guarantee that the minimizer in (2.2) satisfies the Euler Lagrange equations which is our main tool in analysing the asymptotic behavior of the model for $\varepsilon \to 0$. Condition H_3 says that the plane $P = \{(e,\eta,z) : (e,\eta) \in \mathbf{R}^2 : z = G + Se\}$ is the tangent hyperplane to the graph of f which touches that graph at exactly two points, viz., at (E_i, Φ_i) for $i = 1, 2$, and moreover, the graph of f is 'above' P. The points (E_i, Φ_i) are obtained by the common tangent construction. They represent the phases that can coexist in a single state in the theory with $\varepsilon = 0$. It will be seen that for $\lambda \in (E_1, E_2)$ the equilibrium state of elongation λ corresponding to $\varepsilon \to 0$ aprroaches a simple two phase state with phases (E_i, Φ_i) present in amounts that lead to the total elongation λ. Finally, H_4 is a simple assumption that guarantees that the convex envelope w^{co} of w is given by

$$w^{\mathrm{co}}(e) = \begin{cases} w(e) & \text{if } e \in \mathbf{R} \sim (E_1, E_2), \\ G + Se & \text{if } e \in [E_1, E_2], \end{cases} \qquad (2.6)$$

$e \in \mathbf{R}$, see Remark 4.1 (below). The function w^{co} is the infimum energy for $\varepsilon = 0$ by the common tangent construction of gibbsian thermostatics.

Hypothesis H_1 allows us to define the Gibbs function $g : \mathbf{R}^2 \to \mathbf{R}$ by

$$g(s,\phi) = f(\tilde{e}(s,\phi),\phi) - s\tilde{e}(s,\phi)$$

for every $(s,\phi) \in \mathbf{R}^2$ where $\tilde{e}(s,\phi)$ is the unique point with $\hat{s}(\tilde{e}(s,\phi),\phi) = s$. We note that

$$\hat{p}(\tilde{e}(s,\phi),\phi) = \mathrm{D}_\phi g(s,\phi). \qquad (2.7)$$

By Remark 4.1 (below), the function $Q : \mathbf{R} \to \mathbf{R}$, defined by

$$Q(\eta) = g(S,\eta) - G,$$

$\eta \in \mathbf{R}$, is nonnegative (and vanishes at E_1 and E_2). We can thus define the ***interface constant*** σ by

$$\sigma := \int_{\Phi_1}^{\Phi_2} \sqrt{2Q(\eta)}\, d\eta. \qquad (2.8)$$

3 The main results

The following three theorems, the main results of this paper, show that for $\varepsilon \to 0$ the diffuse interface model can be approximated by the sharp interface model with the interface energy $\varepsilon\sigma$. The proofs are given in Section 4.

Theorem 3.1. *If* $\lambda \in \mathbf{R}$ *and* $\varepsilon > 0$ *then*

$$\mathsf{E}_\varepsilon(\lambda) = \begin{cases} w(\lambda) & \text{if} \quad \lambda \notin (E_1, E_2), \\ w^{\mathrm{co}}(\lambda) + \varepsilon\sigma + \mathrm{o}(\varepsilon, \lambda) & \text{if} \quad \lambda \in (E_1, E_2) \end{cases} \tag{3.1}$$

where

$$\mathrm{o}(\varepsilon, \lambda)/\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0 \text{ for any } \lambda \in (E_1, E_2).$$

Moreover,

$$w^{\mathrm{co}}(\lambda) < \mathsf{E}_\varepsilon(\lambda) \le \min\left\{w(\lambda), w^{\mathrm{co}}(\lambda) + \varepsilon\sigma\right\} \tag{3.2}$$

for $\lambda \in (E_1, E_2)$.

Thus if λ is outside the ***Maxwell interval*** (E_1, E_2), the effective energy $\mathsf{E}(\lambda)$ coincides with $w(\lambda)$; this value of $\mathsf{E}(\lambda)$ corresponds to a homogeneous state $(u, \phi) \in D(\lambda)$ in which the strain u' is λ and ϕ is constant delivering the minimum value of $f(\lambda, \phi)$. On the other hand, if $\lambda \in (E_1, E_2)$ then the effective energy $\mathsf{E}(\lambda)$ is above the maxwellian value $w^{\mathrm{co}}(\lambda)$, but is bounded as in (3.2) and asymptotically for $\varepsilon \to 0$ is given by $(3.1)_2$. This corresponds to a creation of a two phase state in which the strain is $\simeq E_1$ on a region of length r and $\simeq E_2$ on a region of length $1 - r$ where

$$rE_1 + (1 - r)E_2 = \lambda, \tag{3.3}$$

separated by a single interface of energy $\varepsilon\sigma$.

The following result gives an explicit description of the states minimizing the right hand side of (2.2):

Theorem 3.2. *For each* $\lambda \in \mathbf{R}$ *and* $\varepsilon > 0$ *there exists an* (u, ϕ) *such that*

$$\mathsf{F}_\varepsilon(u, \phi) = \mathsf{E}_\varepsilon(\lambda) \quad \text{and} \quad (u, \phi) \in D(\lambda)$$

and one of the following two possibilities occurs:

(i) ϕ *is constant; then* $(u(x), \phi(x)) = (\lambda x, \hat{\phi})$ *for all* $x \in (0, 1)$ *where* $\hat{\phi}$ *is such that* $w(\lambda) = f(\lambda, \hat{\phi})$ *and*

$$\mathsf{E}_\varepsilon(\lambda) = w(\lambda);$$

(ii) *there exists a minimizer with* ϕ *is strictly increasing; then there exists* $s \in \mathbf{R}$ *such that with* $\alpha := \phi(0), \beta := \phi(1)$ *the function* $q(\cdot) := g(s, \cdot) - g(s, \alpha)$ *is positive on* (α, β),

$$x/\varepsilon = \int_\alpha^{\phi(x)} \frac{d\eta}{\sqrt{2q(\eta)}}, \qquad u(x)/\varepsilon = \int_\alpha^{\phi(x)} \frac{\tilde{e}(s, \eta) d\eta}{\sqrt{2q(\eta)}} \tag{3.4}$$

for every $x \in [0, 1]$, *and*

$$\mathsf{E}_\varepsilon(\lambda) = g(s, \alpha) + s\lambda + \varepsilon \int_\alpha^\beta \sqrt{2q(\eta)}\, d\eta. \tag{3.5}$$

If $\lambda \in \mathbf{R} \sim (E_1, E_2)$ *then Possibility* (i) *occurs; if* $\lambda \in (E_1, E_2)$ *and*

$$\varepsilon < \big(w(\lambda) - w^{\mathrm{co}}(\lambda)\big)/\sigma \tag{3.6}$$

then Possibility (ii) *occurs.*

Thus every minimizer (u, ϕ) is such that either the strain u' and ϕ are constant with ϕ delivering a minimum of $f(\lambda, \phi)$ or else it can be chosen such that ϕ is strictly increasing and (3.4) hold with some $s \in \mathbf{R}$. We note that if (u, ϕ) is a minimizer and $(\bar{u}, \bar{\phi})$ is defined by

$$\bar{u}(x) = u(1-x) - \lambda(1-x), \quad \bar{\phi}(x) = \phi(1-x),$$

$x \in (0,1)$, then $(\bar{u}, \bar{\phi}) \in D(\lambda)$ and $\mathsf{F}_\varepsilon(u, \phi) = \mathsf{F}_\varepsilon(\bar{u}, \bar{\phi})$ and hence also $(\bar{u}, \bar{\phi})$ is a minimizer; if ϕ is increasing then $\bar{\phi}$ is decreasing and vice versa. The minimizers (u, ϕ) with u' and ϕ constant occur for λ outside the Maxwell interval (E_1, E_2); on the other hand if $\lambda \in (E_1, E_2)$ and ε is sufficiently small in the sense of (3.6), then necessarily the minimizers are of the type described (ii); these have the interface of energy $\varepsilon\sigma$ as described above.

Finally, the following result shows that for $\lambda \in (E_1, E_2)$ and $\varepsilon \to 0$ the minimizers approach a two phase state with the values of strain and phase field equal to (E_1, Φ_1) and (E_2, Φ_2) on intervals $[0, r)$ and $(r, 1]$ where r is given by (3.3).

Theorem 3.3. *Let* $\lambda \in (E_1, E_2)$ *be fixed and for each* $\varepsilon > 0$ *satisfying* (3.6) *let* $(u_\varepsilon, \phi_\varepsilon)$ *be a minimizer in the sense that*

$$\mathsf{E}_\varepsilon(\lambda) = \mathsf{F}_\varepsilon(u_\varepsilon, \phi_\varepsilon) \quad \text{and} \quad (u_\varepsilon, \phi_\varepsilon) \in D(\lambda) \tag{3.7}$$

such that ϕ_ε *is an increasing function; then*

$$(u_\varepsilon'(x), \phi_\varepsilon(x)) \to \begin{cases} (E_1, \Phi_1) & \text{if} \quad 0 \le x < r, \\ (E_2, \Phi_2) & \text{if} \quad r < x \le 1, \end{cases}$$

as $\varepsilon \to 0$ *where r is determined by* (3.3).

4 Proofs

Remark 4.1. *We have*

$$w(E_i) = G + SE_i, \quad g(S, \Phi_i) = G, \quad i = 1, 2, \tag{4.1}$$

the convex envelope w^{co} *of w is given by* (2.6),

$$f(e, \eta) - G - Se \ge Q(\eta) \ge 0 \tag{4.2}$$

for any $(e, \eta) \in \mathbf{R}^2$, *and Q vanishes only at* Φ_1 *and* Φ_2.

Proof From (2.5) we deduce that $f(E_i, \phi) \ge G + SE_i$ for any ϕ with the equality holding if and only if $\phi = \Phi_i$. Hence $f(E_i, \phi) \ge f(E_i, \Phi_i) = G + SE_i$ for all ϕ from which $w(E_i) = f(E_i, \Phi_i) = G + SE_i$ which proves $(4.1)_1$. From (2.5) we deduce that if $i \in \{1, 2\}$ then

$$f(e, \Phi_i) \ge w(e) \ge G + Se$$

for all $e \in \mathbf{R}$ with the equalities holding throughout if $e = E_i$. It follows by differentiation that $\mathrm{D}_e w(e_i) = \mathrm{D}_e f(e_i, \Phi_i) = S$ and hence also $\tilde{e}(S, \Phi_i) = E_i$. The definition of g then gives $(4.1)_2$. To prove (2.6), let $c : \mathbf{R} \to \mathbf{R}$ be given by

$$c(e) = \begin{cases} w(e) & \text{if} \quad e < E_1, \\ G + Se & \text{if} \quad e \in [E_1, E_2], \\ w(e) & \text{if} \quad e > E_2, \end{cases} \tag{4.3}$$

$e \in \mathbf{R}$, and note that c is convex since the three regimes of (4.3) define a convex function on the intervals specified in (4.3) and the limits of these functions and of their derivatives at E_i from the two sides of E_i coincide. Hence also the right hand side of (2.6) is a convex function; this function clearly does not exceed w and is the largest convex function that does not exceed w; thus (2.6) holds. Further, if $\phi \in \mathbf{R}$ then

$$g(S, \phi) = f(\tilde{e}(S, \phi), \phi) - S\tilde{e}(S, \phi) \geq G$$

by (2.5) with the equality holding only if $(\tilde{e}(S, \phi), \phi) = (E_i, \Phi_i)$ for some $i = 1, 2$, i.e., only if $\phi = \Phi_i$ for some $i = 1, 2$; thus Q is nonnegative and vanishes only if $\phi = \Phi_i$ for some $i = 1, 2$. Finally, to prove $(4.2)_1$, we note that the function $f(\cdot, \phi)$ is convex at $\tilde{e}(S, \phi)$ and since $\hat{s}(\tilde{e}(S, \phi), \phi) = S$, the convexity inequality reads

$$f(e, \eta) \geq f(\tilde{e}(S, \phi), \phi) + S(e - \tilde{e}(S, \phi)) = g(S, \phi) + Se;$$

a rearrangement gives $(4.2)_1$. □

***Proof of* $(3.1)_1$** If $\lambda \in \mathbf{R}$ and $(u, \phi) \in D(\lambda)$ then

$$\mathsf{F}_\varepsilon(u, \phi) \geq \int_0^1 f(u'(x), \phi(x))\, dx \geq \int_0^1 w(u'(x))\, dx \geq \int_0^1 w^{\mathrm{co}}(u'(x))\, dx \geq w^{\mathrm{co}}(\lambda)$$

by Jensen's inequality and the conditions $u(0) = 0, u(1) = \lambda$; taking the infimum over all $(u, \phi) \in D(\lambda)$ we obtain $\mathsf{E}_\varepsilon(\lambda) \geq w^{\mathrm{co}}(\lambda)$ for every $\lambda \in \mathbf{R}$. On the other hand let $(u, \phi) \in D(\lambda)$ be defined by $u(x) = \lambda x, \phi(x) = \hat{\phi}$ for every $x \in (0, 1)$ where $\hat{\phi}$ is such that $f(\lambda, \hat{\phi}) = w(\lambda)$ then $\mathsf{F}_\varepsilon(u, \phi) = f(\lambda, \hat{\phi}) = w(\lambda)$ and hence $\mathsf{E}_\varepsilon(\lambda) \leq w(\lambda)$. To summarize, we have

$$w^{\mathrm{co}}(\lambda) \leq \mathsf{E}_\varepsilon(\lambda) \leq w(\lambda) \tag{4.4}$$

for every $\lambda \in \mathbf{R}$. If $\lambda \in \mathbf{R} \sim (E_1, E_2)$ then $w^{\mathrm{co}}(\lambda) = w(\lambda)$ by (2.6) and thus we have $(3.1)_1$. □

***Proof of* (3.2)** Let $\lambda \in (E_1, E_2)$. By Remark 4.1 the function Q is nonnegative and vanishes only at Φ_1, Φ_2. Let $\theta_0 \in (\Phi_1, \Phi_2)$ be arbitrary. Using $Q(\Phi_i) = Q'(\Phi_i) = 0$ for $i = 1, 2$ one finds that

$$\int_{\Phi_1}^{\theta_0} \frac{d\eta}{\sqrt{2Q(\eta)}} = \int_{\theta_0}^{\Phi_2} \frac{d\eta}{\sqrt{2Q(\eta)}} = \infty \tag{4.5}$$

which implies that there exists an increasing function $\theta : \mathbf{R} \to (\Phi_1, \Phi_2)$ such that

$$t = \int_{\theta_0}^{\theta(t)} \frac{d\eta}{\sqrt{2Q(\eta)}} \tag{4.6}$$

for every $t \in \mathbf{R}$. We have

$$\theta(t) \to \Phi_1 \quad \text{for } t \to -\infty, \quad \theta(t) \to \Phi_2 \quad \text{for } t \to \infty \tag{4.7}$$

by (4.5), and differentiating (4.6) using $Q(\theta(t)) > 0$ one finds that θ is continuously differentiable and satisfies

$$\theta'^2(t) = 2Q(\theta(t)) \tag{4.8}$$

for every $t \in \mathbf{R}$. Integrating using (4.7) we obtain

$$\int_{\mathbf{R}} \theta'^2(t)\,dt = \int_{\mathbf{R}} \sqrt{2Q(\theta(t))}\theta'(t)\,dt = \int_{\Phi_1}^{\Phi_2} \sqrt{2Q(\eta)}\,d\eta = \sigma.$$

Next note that there exists at least one $r \in \mathbf{R}$ such that

$$\int_0^1 \tilde{e}(S, \theta((x-r)/\varepsilon))\,dx = \lambda \tag{4.9}$$

since the integral is a continuous function of r and it converges to E_1 for $r \to -\infty$ and to E_2 for $r \to \infty$ by (4.7). Let $(u, \phi) \in H$ be defined by

$$\phi(x) = \theta((x-r)/\varepsilon), \quad u(x) = \int_0^x \tilde{e}(S, \phi(y))\,dy,$$

$x \in (0,1)$; we have $(u, \phi) \in D(\lambda)$ by (4.9). Furthermore,

$$\varepsilon^2 \phi'^2(x)/2 = Q(\phi(x))$$

as a consequence of (4.8); from $u'(x) = \tilde{e}(S, \phi(x))$ we obtain

$$f(u'(x), \phi(x)) = g(S, \phi(x)) + Su'(x) = \varepsilon^2 \phi'^2(x)/2 + G + Su'(x).$$

Hence

$$\mathsf{F}_\varepsilon(u, \phi) = \int_0^1 \varepsilon^2 \phi_\varepsilon'^2(x) + Su'(x)\,dx + G = \int_0^1 \varepsilon^2 \phi_\varepsilon'^2(x)\,dx + w^{\mathrm{co}}(\lambda)$$

where we have used that $S\lambda + G = w^{\mathrm{co}}(\lambda)$ by (2.6). Estimating

$$\int_0^1 \varepsilon^2 \phi_\varepsilon'^2(x)\,dx = \int_0^{1/\varepsilon} \varepsilon \theta'^2(t-r)\,dt \le \int_{\mathbf{R}} \varepsilon \theta'^2(t-r)\,dt = \varepsilon\sigma$$

we obtain

$$\mathsf{E}_\varepsilon(\lambda) \le \mathsf{F}_\varepsilon(u, \phi) \le w^{\mathrm{co}}(\lambda) + \varepsilon\sigma.$$

Combining with (4.4) we obtain (3.2) with the nonstrict inequality sign in the left inequality. Let us show that the equality $w^{\mathrm{co}}(\lambda) = \mathsf{E}_\varepsilon(\lambda)$ cannot hold. Indeed, assuming this equality and referring to Theorem 3.2 (to be proved below independently of the present argument), we find $(u, \phi) \in D(\lambda)$ such that

$$\int_0^1 \phi'^2(x)/2 + f(u'(x), \phi(x))\,dx = w^{\mathrm{co}}(\lambda) \equiv \int_0^1 G + Su'(x)\,dx,$$

i.e.,

$$\int_0^1 \phi'^2(x)/2 + f(u'(x), \phi(x)) - G - Su'(x)\,dx = 0.$$

Since the term $f(u'(x), \phi(x)) - G - Su'(x)$ is nonnegative for a.e. $x \in (0,1)$ by (2.5), we deduce that

$$\phi'^2(x) = 0, \quad f(u'(x), \phi(x)) - G - Su'(x) = 0$$

for a.e. $x \in (0,1)$, which in turn implies that ϕ is constant, and $(u'(x), \phi) \in \{(E_i, \Phi_i) : i = 1, 2\}$ for a.e. $x \in (0,1)$ by (2.5). Thus $\phi = \Phi_i$ and $u'(x) = E_i$ for some $i \in \{1, 2\}$. But then $\int_0^1 u'(x)\, dx = E_i \neq \lambda$ in contradiction with $(u, \phi) \in D(\lambda)$.

□

***Proof of Theorem* 3.2** To prove the existence of the minimizer, let $\varepsilon > 0$ and λ be fixed and let $(u_k, \phi_k) \in D(\lambda)$ be a minimizing sequence in the sense that $\mathsf{F}_\varepsilon(u_k, \phi_k) \to \mathsf{E}_\varepsilon(\lambda)$ for $k \to \infty$. From the coercivity [see $(2.3)_1$] we deduce that the $L^2(0,1)$ norms of the sequences ϕ_k, ϕ_k' and u_k' are bounded; since $u_k(0) = 0$, we deduce that also the L^2 norm of u_k is bounded. Thus passing to a subsequence (not relabelled) we can assume that

$$(u_k, \phi_k) \rightharpoonup (u, \phi) \quad \text{in} \quad W^{1,2}((0,1), \mathbf{R}^2)$$

for some $(u, \phi) \in D(\lambda)$. As a consequence,

$$(u_k, \phi_k) \to (u, \phi) \quad \text{in} \quad L^2((0,1), \mathbf{R}^2).$$

The integrand occurring in (2.1) is convex in (u', ϕ') by Assumption H_1 and as the hypotheses of the lowersemicontinuity theorem [3; Theorem 3.23] are satisfied, we have

$$\mathsf{E}_\varepsilon(\lambda) = \lim_{k \to \infty} \mathsf{F}_\varepsilon(u_k, \phi_k) \geq \mathsf{F}_\varepsilon(u, \phi) \geq \mathsf{E}_\varepsilon(\lambda)$$

and thus (u, ϕ) is a minimizer.

Let $(u, \phi) \in D(\lambda)$ be any minimizer. Hypothesis H_2 implies via [3; Theorem 3.37] that (u, ϕ) satisfy the weak form of the Euler Lagrange equations

$$\int_0^1 \hat{s}(u'(x), \phi(x)) v'(x)\, dx = 0, \tag{4.10}$$

$$\int_0^1 \varepsilon^2 \phi'(x) \psi'(x) + \hat{p}(u'(x), \phi(x)) \psi(x)\, dx = 0 \tag{4.11}$$

for every $v \in C_0^1(0,1)$ and every $\psi \in C^1([0,1])$ where (4.11) follows for $\psi \in C_0^1(0,1)$ directly from [3; Theorem 3.37] and generally for $\psi \in C^1([0,1])$ by a straightforward extension of the proof of [3; Theorem 3.37]. Equation (4.10) implies that there exists a $s \in \mathbf{R}$ such that

$$s = \hat{s}(u'(x), \phi(x)), \quad u'(x) = \tilde{e}(s, \phi(x)) \tag{4.12}$$

for a.e. $x \in (0,1)$. Integrating by parts, we rewrite the second condition as

$$\int_0^1 \Big(\varepsilon^2 \phi'(x) - \int_0^x \hat{p}(u'(y), \phi(y))\ dy\Big) \psi'(x) dx + \psi(1) \int_0^1 \hat{p}(u'(y), \phi(y))\, dy = 0.$$

The arbitrariness of $\psi \in C^1([0,1])$ then gives

$$\varepsilon^2 \phi'(x) = \int_0^x \hat{p}(u'(y), \phi(y))\, dy \tag{4.13}$$

for a.e. $x \in (0,1)$ and

$$\int_0^1 \hat{p}(u'(y), \phi(y))\, dy = 0.$$

From $(4.12)_2$ and the continuity of ϕ we deduce that $u \in C^1([0,1])$ and with this knowledge (4.13) implies that $\phi \in C^2([0,1])$,

$$\varepsilon^2 \phi''(x) = \hat{p}(u'(x), \phi(x)) \tag{4.14}$$

for every $x \in [0,1]$ and

$$\phi'(0) = \phi'(1) = 0. \tag{4.15}$$

The condition $\phi \in C^2([0,1])$ and $(4.12)_2$ finally imply $u \in C^2([0,1])$. Hence the Euler Lagrange equations hold in the classical sense and this in turn implies the first integral

$$\varepsilon^2 \phi'^2(x)/2 = f(u'(x), \phi(x)) - su'(x) - G_0 \equiv g(s, \phi(x)) - G_0 \tag{4.16}$$

for $x \in [0,1]$ where G_0 is a constant. By (4.15), G_0 satisfies $g(s,\alpha) - G_0 = g(s,\beta) - G_0 = 0$; this gives

$$g(s,\alpha) = g(s,\beta)$$

and (4.16) reduces to

$$\varepsilon^2 \phi'^2(x)/2 = g(s, \phi(x)) - g(s, \alpha). \tag{4.17}$$

If ϕ is constant then the condition of minimization gives Assertion (i).

Assume that (u, ϕ) is a minimizer with a nonconstant ϕ. Thus there exists a point $y \in (0,1)$ such that $\phi'(y) \neq 0$; assume for definitenes that $\phi'(y) > 0$. Introduce the sets N_+, N_-, and N_0 by

$$N_\pm := \{x \in [0,1] : \pm\phi'(x) > 0\}, \quad N_0 := \{x \in [0,1] : \phi'(x) = 0\},$$

note that the sets $N_\pm$ are open, N_0 closed, and

$$\phi'(x) = \pm\sqrt{2q(\phi(x))}/\varepsilon \tag{4.18}$$

if $x \in N_\pm$. Prove that $N_+ = (0,1)$, $N_- = \emptyset$ and $N_0 = \{0,1\}$. Let $J = (a,b) \subset (0,1)$ be the component of N_+ which contains y. The assertions above will be proved if we show that $J = (0,1)$. To prove the last statement, put $\gamma = \phi(a)$, $\delta = \phi(b)$; (4.17) gives that q is positive on (γ, δ) and $q(\gamma) = q(\delta) = 0$. The integration of (4.18) gives

$$x - a = \varepsilon \int_\gamma^{\phi(x)} \frac{d\eta}{\sqrt{2q(\eta)}}, \tag{4.19}$$

and hence $b - a = \Delta$ where

$$\Delta = \varepsilon \int_\gamma^\delta \frac{d\eta}{\sqrt{2q(\eta)}}. \tag{4.20}$$

From the finiteness of the integral (4.20) we deduce that $q'(\delta) \neq 0$ since otherwise the Taylor expansion of q at δ shows that the integral would diverge at least as $\int_\gamma^\delta 1/(\delta - t)\, dt$. Moreover, since $q(\psi) > 0$ for all $\psi < \delta$ sufficiently close to δ, we have $q'(\delta) < 0$. Recalling that ϕ is twice continuously differentiable, from (4.18) with the plus sign we deduce that $\phi''(b) = q'(\delta)/2$ is negative. New we distinguish the following two possibilities: $b = 1$ and $b < 1$. The first possibility is trivial from the analysis that follows and let us now consider the second possibility. Assume therefore $b < 1$. Then the solution ϕ exists for all $x > b$ sufficiently close to b. The conditions $\phi'(b) = 0$ and $\phi''(b) < 0$ then imply that ϕ has a strict local maximum at $x = b$ and therefore $\phi(x) < \delta$ and $\phi'(x) < 0$ for all $x > b$ sufficiently close to b. Thus we see that we have (4.18) with the minus sign for the indicated values of x. Moreover $q > 0$ everywhere on (γ, δ) and thus (4.17) tells us that ϕ' is different from 0 and therefore does not change its sign as long as $\phi(x)$ exists and $\phi(x) > \gamma$. Thus we have (4.18) with the minus sign. Thus the subinterval (a, b) of $(0, 1)$ with the positive sign of ϕ' is followed by a subinterval (b, c) with the negative sign of ϕ'. Equation (4.18) then shows that $\phi(b - \xi) = \phi(b + \xi)$ for all $\xi \in (0, b - a)$. The above properties of $N_\pm$ and N_0 and the conditions $\phi'(0) = \phi'(1) = 0$ show that the interval $(0, 1)$ is divided into m intervals of equal length such that the sign of ϕ' changes alternatively on these intervals, and moreover, ϕ is a restriction of an even periodic class 2 function on $\mathbf{R}$ of period $2/m$ with $\phi'(0) = 0$. Let us prove that $m = 1$. Assuming $m > 1$, we will construct $(\bar{u}, \bar{\phi}) \in D(\lambda)$ such that

$$\mathsf{F}_\varepsilon(u, \phi) > \mathsf{F}_\varepsilon(\bar{u}, \bar{\phi}). \tag{4.21}$$

Indeed, from $(4.12)_2$ we see that also u' is a restriction of a periodic class 1 function of period $2/m$ and hence

$$\int_0^{1/m} \phi'^2(x)\, dx = \int_0^1 \phi'^2(x)\, dx/m, \qquad \int_0^{1/m} u'(x)\, dx = \lambda/m. \tag{4.22}$$

Let now $(\bar{u}, \bar{\phi})$ be defined by

$$\bar{u}(x) = mu(x/m), \quad \bar{\phi}(x) = \phi(x/m)$$

$x \in (0, 1)$. Then (4.22) give $(\bar{u}, \bar{\phi}) \in D(\lambda)$ and

$$\int_0^1 f(\bar{u}'(x), \bar{\phi}(x))\, dx = \int_0^1 f(u'(x), \phi(x))\, dx,$$

while

$$\int_0^1 \bar{\phi}'^2(x)\, dx = \int_0^1 \phi'^2(x)\, dx/m^2 < \int_0^1 \phi'^2(x)\, dx$$

and hence we have (4.21). This proves that (u, ϕ) is not a minimizer unless $m = 1$. Thus we see that if (u, ϕ) is a minimizer with ϕ nonconstant then either $\phi' > 0$ everywhere on $(0, 1)$ of $\phi' < 0$ everywhere on $(0, 1)$. We can assume $\phi' > 0$; then (4.19) reduces to $(3.4)_1$. The integration of $(4.12)_2$ and the conditions $u(0) = 0, u(1) = \lambda$ give

$$\lambda = \int_0^1 \tilde{e}(s, \phi(x))\, dx \tag{4.23}$$

which reduces to $(3.4)_2$ by making the change of variables $x \to \phi$. To prove (3.5), we note that $(4.12)_1$ gives

$$f(u', \phi) = g(s, \phi) + s\tilde{e}(s, \phi) = q(\phi) + g(s, \phi(0)) + s\tilde{e}(s, \phi)$$

and hence

$$\mathsf{F}_\varepsilon(u, \phi) = \int_0^1 2q(\phi(x))\, dx + g(s, \phi(0)) + s\lambda$$

by (4.17) and (4.23). The formula (3.5) follows from

$$\int_0^1 2q(\phi(x))\,dx = \varepsilon\int_0^1 \sqrt{2q(\phi(x))}\phi'(x)\,dx = \varepsilon\int_\alpha^\beta \sqrt{2q(\eta)}\,d\eta$$

This completes the proof of (ii).

If $\lambda \in \mathbf{R} \sim (E_1, E_2)$ then the pair (u, ϕ) described in Item (i) is a minimizer by $(3.1)_1$ and one easily finds that any other state $(u, \phi) \in D(\lambda)$ has $\mathsf{F}_\varepsilon(u, \phi) > w(\lambda)$. If $\lambda \in (E_1, E_2)$ and $\varepsilon > 0$ satisfies (3.6) then the state (u, ϕ) with u', ϕ constant as in Item (i) has $\mathsf{F}_\varepsilon(u, \phi) = w(\lambda)$ and assuming $\mathsf{E}(\lambda) = w(\lambda)$ contradicts (3.2); thus the case described in Item (ii) necessarily occurs. □

Remark 4.2. *Let π, ρ be, respectively, the smallest and the largest point in the interval (Φ_1, Φ_2) for which Q attains a maximum on $[\Phi_1, \Phi_2]$ (the possibility $\pi = \rho$ not excluded) and let η_k and (e_k, η_k) be two sequences.*

(i) *If $Q(\eta_k) \to 0$ and $\eta_k \le \pi$ for all large k then $\eta_k \to \Phi_1$; if $Q(\eta_k) \to 0$ and $\eta_k \ge \rho$ for all large k then $\eta_k \to \Phi_2$;*

(ii) *if $f(e_k, \eta_k) - G - Se_k \to 0$ and $\eta_k \to \Phi_1$ then $e_k \to E_1$; if $f(e_k, \eta_k) - G - Se_k \to 0$ and $\eta_k \to \Phi_2$ then $e_k \to E_2$.*

Proof (i): By $(2.3)_1$ we have

$$g(s, \eta) = f(\tilde{e}(s, \eta)) - s\tilde{e}(s, \eta) \ge c_1(\tilde{e}^2(s, \eta) + \eta^2) + c_2 - s\tilde{e}(s, \eta) \ge c_1\eta^2 + \bar{c}_2$$

for any $(s, \eta) \in \mathbf{R}^2$ where $\bar{c}_2$ is the minimum $e \mapsto c_1 e^2 - se \ge \bar{c}_2$ on $\mathbf{R}$ and hence

$$Q(\eta) \ge c_1\eta^2 + \bar{c}_3 \tag{4.24}$$

for all $\eta \in \mathbf{R}$ and some $c_1, \bar{c}_3$ with $c_1 > 0$. Let η_k be a sequence with $Q(\eta_k) \to 0$ and $\eta_k \le \pi$ for all large k; prove that $\eta_k \to \Phi_1$ by contradiction. Hence assume that η_k contains a subsequence, not relabelled, such that $|\eta_k - \Phi_1| \ge \delta$ for all k and some $\delta > 0$. From $Q(\eta_k) \to 0$ and (4.24) we deduce that η_k is a bounded sequence and thus it contains a subsequence, not relabelled, such that $\eta_k \to \bar{\eta}$ for some $\bar{\eta} \in \mathbf{R}$. Then the limits in $Q(\eta_k) \to 0$ and in $\eta_k \le \pi$ give $Q(\bar{\eta}) = 0$ and $\bar{\eta} \le \pi$. Since Q is positive on $(-\infty, \Phi_1) \cup (\Phi_1, \pi]$ and vanishes only at Φ_1, this implies that $\bar{\eta} = \Phi_1$, in contradiction with the starting assumption. The second assertion of (i) is proved similarly. (ii): From $(2.3)_1$ we obtain that

$$f(e, \eta) - G - Se \ge \bar{c}_1(e^2 + \eta^2) + \bar{c}_2 \tag{4.25}$$

for all $(e, \eta) \in \mathbf{R}^2$ and some $\bar{c}_1 > 0, \bar{c}_2 \in \mathbf{R}$. We prove that if $f(e_k, \eta_k) - G - Se_k \to 0$ and $\eta_k \to \Phi_1$ then $e_k \to E_1$ by a similar contradiction as in Case (i). Namely, if for some subsequence we have $|e_k - E_1| \ge \delta$ for all k and some $\delta > 0$, then the sequence e_k is bounded by (4.25) and thus it contains a subsequence such that $e_k \to \bar{e}$ for some $\bar{e}$; the limit in $f(e_k, \eta_k) - G - Se_k \to 0$ then gives $f(\bar{e}, \Phi_1) - G - S\bar{e} = 0$ and then $\bar{e} = E_1$ by (2.5). The second assertion of (ii) is proved similarly. □

***Proof of Theorem* 3.3** Prove first that

$$\phi_\varepsilon(x) \to \begin{cases} \Phi_1 & \text{if} \quad 0 \le x < r, \\ \Phi_2 & \text{if} \quad r < x \le 1. \end{cases} \tag{4.26}$$

Suppose, that there exists a sequence $\varepsilon_k \to 0$ and a point y with $0 \le y < r$ such that $|\phi_k(y) - \Phi_1| \ge \delta$ for all k and some $\delta > 0$, where we write $\phi_k := \phi_{\varepsilon_k}$ and $u_k := u_{\varepsilon_k}$. We shall successively extract subsequences of ϕ_k without relabelling the symbols until we obtain a subsequence with $\phi_k(y) \to \Phi_1$; this contradiction will prove $(4.26)_1$; a similar argument proves $(4.26)_2$.

Thus let ϕ_k be a sequence as above. From (3.2) we deduce that $\mathsf{F}_{\varepsilon_k}(u_k, \phi_k) \to w^{\text{co}}(\lambda) \equiv \int_0^1 G + Su_k'(x)\,dx$ we deduce that

$$\int_0^1 f(u_k'(x), \phi_k(x)) - G - Su_k'(x)\,dx \to 0;$$

the integrand is nonnegative by (2.5) and thus by extracting a subsequence, we have

$$f(u_k'(x), \phi_k(x)) - G - Su_k'(x) \to 0 \tag{4.27}$$

for a.e. $x \in (0,1)$ which by $(4.2)_1$ also implies

$$Q(\phi_k(x)) \to 0 \tag{4.28}$$

for a.e. $x \in (0,1)$. Let π, ρ be as in Remark 4.2, let

$$s_k = \sup\{x \in [0,1] : \phi_k(z) \le \pi \ \text{ for every } \ z \in (0,x]\},$$

$$t_k = \inf\{x \in [0,1] : \phi_k(z) \ge \rho \ \text{ for every } \ z \in [x,1)\},$$

and note that since ϕ_k is increasing, $0 \le s_k \le t_k \le 1$. Passing to a subsequence if necessary, we assume that $s_k \to s, t_k \to 0$ for some s, t with $0 \le s \le t \le 1$. We have $Q(\phi_k(x)) \to 0$ for a.e. $x \in [0,1]$ and $\phi_k(x) < \pi$ for every $x \in [0,s)$ and all sufficiently large k and $\phi_k(x) > \rho$ for every $x \in (t,1]$ and all sufficiently large k; hence

$$\phi_k(x) \to \begin{cases} \Phi_1 & \text{for a.e. } x \in [0,s), \\ \Phi_2 & \text{for a.e. } x \in (t,1] \end{cases} \tag{4.29}$$

by Remark 4.2(i). Since the function Q is strictly positive on $[\pi, \rho]$, we have $Q(\phi_k(x)) \ge c > 0$ every $x \in (s,t)$ and every k sufficiently large; if $s < t$, this is inconsistent with (4.28); we thus conclude that $s = t$. Moreover, since ϕ_k are increasing functions, one deduces that the a.e. convergences in (4.29) can be replaced by the convergences everywhere. Thus letting $m = s = t$ we conclude that

$$\phi_k(x) \to \begin{cases} \Phi_1 & \text{if } \ 0 \le x < m, \\ \Phi_2 & \text{if } \ m < x \le 1. \end{cases} \tag{4.30}$$

Combining (4.30) with (4.27) we obtain

$$u_k'(x) \to \begin{cases} E_1 & \text{if } \ 0 \le x < m, \\ E_2 & \text{if } \ m < x \le 1 \end{cases} \tag{4.31}$$

by Remark 4.2(ii), also using the continuity of u_k'. The limit in $\int_0^1 u_k'(x)\,dx = \lambda$ then gives $mE_1 + (1-m)E_2 = \lambda$ and a comparison with (3.3) shows that $m = r$. Then (4.30) implies $\phi_k(y) \to \Phi_1$, a contradition; this proves (4.26).

We finally prove that

$$u'_\varepsilon(x) \to \begin{cases} E_1 & \text{if} \quad 0 \le x < r, \\ E_2 & \text{if} \quad r < x \le 1. \end{cases} \tag{4.32}$$

Suppose that there exists a sequence $\varepsilon_k \to 0$ and a point y with $0 \le y < r$ such that $|u'_k(y) - E_1| \ge \delta$ for all k and some $\delta > 0$. Then for a subsequence we have (4.30) and (4.31) with $m = r$, in particular, $u'_k(y) \to E_1$; a contradiction proving $(4.32)_1$. A similar argument proves $(4.32)_2$. □

Proof of $\mathbf{(3.1)_2}$ Let ε satisfy (3.6) and let $(u_\varepsilon, \phi_\varepsilon)$ be a minimizer in the sense of (3.7) with ϕ_ε an increasing function. Inserting $e = u'_\varepsilon(x), \eta = \phi_\varepsilon(x)$ in $(4.2)_1$ and integrating using the condition $\int_0^1 u'_\varepsilon(x)\,dx = \lambda$ we obtain

$$\int_0^1 f(u'_\varepsilon(x), \phi_\varepsilon(x))\,dx - w^{\text{co}}(\lambda) \ge \int_0^1 Q(\phi_\varepsilon(x))\,dx.$$

Then

$$\mathsf{F}_\varepsilon(u_\varepsilon, \phi_\varepsilon) - w^{\text{co}}(\lambda) \ge \int_0^1 \varepsilon^2 \phi_\varepsilon'^2(x)/2 + Q(\phi_\varepsilon(x))\,dx \ge \varepsilon \int_{\phi_\varepsilon(0)}^{\phi_\varepsilon(1)} \sqrt{2Q(\eta)}\,d\eta$$

where we have used

$$\varepsilon^2 \phi_\varepsilon'^2(x)/2 + Q(\phi_\varepsilon(x)) \ge \varepsilon \sqrt{2Q(\phi_\varepsilon(x))}\phi'_\varepsilon(x)$$

since $a^2 + b^2 \ge 2|a||b|$. Combining with (3.2) we thus obtain

$$\varepsilon\sigma \ge \mathsf{F}_\varepsilon(u_\varepsilon, \phi_\varepsilon) - w^{\text{co}}(\lambda) \ge \varepsilon \int_{\phi_\varepsilon(0)}^{\phi_\varepsilon(1)} \sqrt{2Q(\eta)}\,d\eta$$

and the proof of $(3.1)_2$ is completed by pointing out that

$$\int_{\phi_\varepsilon(0)}^{\phi_\varepsilon(1)} \sqrt{2Q(\eta)}\,d\eta \to \sigma$$

as $\varepsilon \to 0$ since $\phi_\varepsilon(0) \to \Phi_1$ and $\phi_\varepsilon(1) \to \Phi_2$ by Theorem 3.3. □

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