HC-CONVERGENCE THEORY OF L-NETS AND L-IDEALS AND SOME OF ITS APPLICATIONS

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Abstract. In this paper we introduce and study the concepts of HC-closed set and HC-limit (HC-cluster) points of L-nets and L-ideals using the notion of almost N-compact remoted neighbourhoods in L-topological spaces. Then we introduce and study the concept of HL-continuous mappings. Several characterizations based on HC-closed sets and the HC-convergence theory of L-nets and L-ideals are presented for HL-continuous mappings.

Keywords: L-topology, remoted neighbourhood, almost N-compactness, HC-closed set, HL-continuity, L-net, L-ideal, HC-convergence theory

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1. Introduction

Wang in [12], [13] established the Moore-Smith convergence theory in both L-topological spaces (in the sense of [7]) and L-topological molecular lattices [13] by using remoted neighbourhoods. Yang in [15] established the convergence theory of L-ideals in L-topological molecular lattices by using remoted neighbourhoods. In [1], [3], [5], some extended convergence theories are developed. In [2], [3], the concept of the N-convergence theory in L-topological spaces by means of the near N-compactness and remoted neighbourhoods is introduced. In this paper, we further develop the convergence theory in L-topological spaces by (i) introducing the concepts of the HC-convergence of L-nets and L-ideals, (ii) presenting the notions of the HC-closure and HC-interior operators in L-topological spaces, and (iii) giving a new definition of H-continuity in L-topological spaces for the so called HL-continuous mapping. Then we show several applications of HL-continuity by means of HC-convergence theory. In Section 3 we define an HC-closed (HC-open) set and discuss its basic properties. In Section 4 we introduce and study HC-convergence theory of L-nets and L-ideals, and discuss their various properties and mutual relationships. In

Section 5 we give and study the concept of an HL-continuous mapping. Several characterizations of HL-continuous mappings by HC-convergence theory of L-nets and L-ideals are given. In Section 6 we study the relationships between HL-continuous mappings and other L-valued Zadeh mappings such as L-continuous, CL-continuous and almost CL-continuous mappings.

2. Preliminaries and definitions

Throughout the paper L denotes a completely distributive complete lattice with different least and greatest elements 0 and 1 and with an order reversing involution $a \to a'$. By $M(L)$ we denote the set of all nonzero irreducible elements of L. Let X be a nonempty crisp set. L^X denotes the set of all L-fuzzy sets on X and $M(L^X) = \{x_\alpha \in L^X\colon\ x \in X, \alpha \in M(L)\}$ is the set of all nonzero irreducible elements (the so-called L-fuzzy points or molecules) of L^X; 0_X and 1_X denote respectively the least and the greatest elements of L^X.

Let (L^X, τ) be an L-topological space [7], briefly L-ts. For each $\mu \in L^X$, $\mathrm{cl}(\mu)$, $\mathrm{int}(\mu)$ and μ' will denote the closure, the interior and the pseudo-complement of μ, respectively.

An L-fuzzy set $\mu \in L^X$ is called regular closed (regular open) set iff $\mathrm{cl}(\mathrm{int}(\mu)) = \mu$ $(\mathrm{int}(\mathrm{cl}(\mu)) = \mu)$. The class of all regular closed and regular open sets in (L^X, τ) will be denoted by $\mathrm{RC}(L^X, \tau)$ and $\mathrm{RO}(L^X, \tau)$, respecively. An L-ts (L^X, τ) is called fully stratified [8] if for each $\alpha \in L$, the L-fuzzy set which assumes the value α at each point $x \in X$ belongs to τ. A mapping $F\colon\ L^X \to L^Y$ is said to be an L-valued Zadeh mapping induced by a mapping $f\colon\ X \to Y$, iff $F(\mu)(y) = \bigvee\{\mu(x)\colon\ f(x) = y\}$ for every $\mu \in L^X$ and every $y \in Y$ [13]. For $\Psi \subseteq L^X$ we define $\Psi' = \{\mu'\colon\ \mu \in \Psi\}$. An L-valued Zadeh mapping $F\colon\ (L^X, \tau) \to (L^Y, \Delta)$ is called L-continuous iff $F^{-1}(\eta) \in \tau'$ for each $\eta \in \Delta'$. In an obvious way L-topological spaces and L-continuous maps form a category denoted by L-TOP. For other undefined notions and symbols in this paper we refer to [7].

Definition 2.1 [12], [13]. Let (L^X, τ) be an L-ts and let $x_\alpha \in M(L^X)$. Then $\lambda \in \tau'$ is called a remoted neighbourhood (R-nbd, for short) of x_α if $x_\alpha \notin \lambda$. The set of all R-nbds of x_α is denoted by R_{x_α}.

Definition 2.2 [16]. Let (L^X, τ) be an L-ts and let $\mu \in L^X$. Now $\Psi \subset \tau'$ is called

(i) an α-remoted neighbourhood family of μ, briefly α-RF of μ, if for each molecule $x_\alpha \in \mu$, there is $\eta \in \Psi$ such that $\eta \in R_{x_\alpha}$;
(ii) an $\overline{\alpha}$-remoted neighbourhood family of μ, briefly $\overline{\alpha}$-RF of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is an γ-RF of μ where $\beta^*(\alpha) = \beta(\alpha) \cap M(L)$, and $\beta(\alpha)$ denotes the union of all minimal sets relative to α.

Definition 2.3 [6]. Let (L^X, τ) be an L-ts and let $\mu \in L^X$. Now $\Psi \subset \tau'$ is called

(i) an almost α-remoted neighbourhood family of μ, briefly almost α-RF of μ, if for each molecule $x_\alpha \in \mu$, there is $\eta \in \Psi$ such that $\mathrm{int}(\eta) \in R_{x_\alpha}$;

(ii) an almost $\overline{\alpha}$-remoted neighbourhood family of μ, briefly almost $\overline{\alpha}$-RF of μ, if there exists $\gamma \in \beta^*(\alpha)$ such that Ψ is an almost γ-RF of μ.

We denote the set of all nonempty finite subfamilies of Ψ by $2^{(\Psi)}$.

Definition 2.4 [6]. Let (L^X, τ) be an L-ts. $\mu \in L^X$ is almost N-compact in (L^X, τ), if for any $\alpha \in M(L)$ and every α-RF Ψ of μ there exists $\Psi_\circ \in 2^{(\Psi)}$ such that $\Psi_\circ$ is an almost $\overline{\alpha}$-RF of μ. An L-ts (L^X, τ) is called an almost N-compact space if 1_X is an almost N-compact set in (L^X, τ).

We need the following result.

Theorem 2.5 [6]. *Let (L^X, τ) be an L-ts and let $\mu \in L^X$. Then:*

(i) *If μ is an almost N-compact set, then for each $\varrho \in \tau'$ (or $\varrho \in \mathrm{RC}(L^X, \tau)$), $\mu \wedge \varrho$ is almost N-compact.*

(ii) *Every closed L-fuzzy set of an almost N-compact set is almost N-compact.*

(iii) *Every almost N-compact set in a fully stratified* LT_2*-space* [8] *is a closed L-fuzzy set.*

3. HC-closed L-fuzzy sets

In this section, we first introduce and study the concepts of the HC-closure (NC-closure) and the HC-interior (NC-interior) operators in L-topological spaces. Secondly, we discuss the relationships between the HC-closure (HC-interior), NC-closure (NC-interior), N-closure (N-interior) [3] and closure (interior) [13] operators. Finally, we give the definition of the HC$\cdot L$-topological space and NC$\cdot L$-topological space.

Definition 3.1. Let (L^X, τ) be an L-ts and let $\mu \in L^X$. A molecule $x_\alpha \in M(L^X)$ is called an HC-adherent (NC-adherent) point of μ, written as $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$ ($x_\alpha \in \mathrm{NC}\cdot\mathrm{cl}(\mu)$) iff $\mu \notin \lambda$ for each $\lambda \in \mathrm{HC}\,R_{x_\alpha}$ ($\lambda \in \mathrm{NC}\,R_{x_\alpha}$), where $\mathrm{HC}\,R_{x_\alpha}$ ($\mathrm{NC}\,R_{x_\alpha}$) is the family of all almost N-compact (N-compact) remoted neighbourhoods of x_α. Further $\mathrm{HC}\cdot\mathrm{cl}(\mu)$ ($\mathrm{NC}\cdot\mathrm{cl}(\mu)$) is called the HC-closure (NC-closure) of μ. If $\mathrm{HC}\cdot\mathrm{cl}(\mu) \leqslant \mu$ ($\mathrm{NC}\cdot\mathrm{cl}(\mu) \leqslant \mu$), then μ is called an HC-closed (NC-closed) L-fuzzy set. The complement of an HC-closed (NC-closed) L-fuzzy set is called an HC-open (NC-open) L-fuzzy set. Let $\mathrm{HC}\cdot\mathrm{int}(\mu) = \bigvee\{\varrho \in L^X\colon\ \varrho$ is an HC-open L-fuzzy set contained in $\mu\}$. We say that $\mathrm{HC}\cdot\mathrm{int}(\mu)$ is the HC-interior of μ. Similarly, we can define $\mathrm{NC}\cdot\mathrm{int}(\mu)$.

R e m a r k 3.2. It is clear that $\mathrm{NC}\,R_{x_\alpha} \subseteq \mathrm{HC}\,R_{x_\alpha}$, because every N-compact set [15] is almost N-compact [6]. So the properties and characterizations of an NC-closed set and its related notions are similar to those of an HC-closed set and hence omitted.

Proposition 3.3. *Let (L^X,τ) be an L-ts and let $\mu \in L^X$. Then the following hold:*

(i) $\mu \leqslant \mathrm{cl}(\mu) \leqslant \mathrm{HC}\cdot\mathrm{cl}(\mu) \leqslant N\cdot\mathrm{cl}(\mu) \leqslant \mathrm{NC}\cdot\mathrm{cl}(\mu)$ $(\mathrm{NC}\cdot\mathrm{int}(\mu) \leqslant N\cdot\mathrm{int}(\mu) \leqslant \mathrm{HC}\cdot\mathrm{int}(\mu) \leqslant \mathrm{int}(\mu) \leqslant \mu)$ *for every* $\mu \in L^X$.

(ii) *If* $\mu \leqslant \varrho$ *then* $\mathrm{HC}\cdot\mathrm{cl}(\mu) \leqslant \mathrm{HC}\cdot\mathrm{cl}(\varrho)$ $(\mathrm{HC}\cdot\mathrm{int}(\mu) \leqslant \mathrm{HC}\cdot\mathrm{int}(\varrho))$.

(iii) μ *is* HC-*open iff* $\mu = \mathrm{HC}\cdot\mathrm{int}(\mu)$.

(iv) $\mathrm{HC}\cdot\mathrm{cl}(\mathrm{HC}\cdot\mathrm{cl}(\mu)) = \mathrm{HC}\cdot\mathrm{cl}(\mu)$ $(\mathrm{HC}\cdot\mathrm{int}(\mathrm{HC}\cdot\mathrm{int}(\mu)) = \mathrm{HC}\cdot\mathrm{int}(\mu))$.

(v) $(\mathrm{HC}\cdot\mathrm{cl}(\mu))' = \mathrm{HC}\cdot\mathrm{int}(\mu')$ *and* $(\mathrm{HC}\cdot\mathrm{int}(\mu))' = \mathrm{HC}\cdot\mathrm{cl}(\mu')$.

(vi) $\mathrm{HC}\cdot\mathrm{cl}(\mu) = \bigwedge\{\eta \in L^X\colon\ \eta$ *is an* HC-*closed set containing* $\mu\}$.

P r o o f. (i), (ii) and (v) follow directly from the definitions.

(iii) Let $\mu \in L^X$ be HC-open, then $\mathrm{HC}\cdot\mathrm{int}(\mu) = \bigvee\{\varrho \in L^X\colon\ \varrho$ is HC-open set contained in $\mu\} = \mu$. Conversely; let $\mu = \mathrm{HC}\cdot\mathrm{int}(\mu)$. Since $\mathrm{HC}\cdot\mathrm{int}(\mu)$ is the join of all HC-open sets contained in μ, so $\mathrm{HC}\cdot\mathrm{int}(\mu)$ is HC-open and hence μ is HC-open.

(iv) Let $x_\alpha \in M(L^X)$ with $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mathrm{HC}\cdot\mathrm{cl}(\mu))$. Then $\mathrm{HC}\cdot\mathrm{cl}(\mu) \nleqslant \eta$ for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$. Hence there exists $y_\nu \in M(L^X)$ such that $y_\nu \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$ and $y_\nu \notin \eta$. So $\mu \nleqslant \eta$, that is $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. Thus $\mathrm{HC}\cdot\mathrm{cl}(\mathrm{HC}\cdot\mathrm{cl}(\mu)) \leqslant \mathrm{HC}\cdot\mathrm{cl}(\mu)$. On the other hand, $\mathrm{HC}\cdot\mathrm{cl}(\mu) \leqslant \mathrm{HC}\cdot\mathrm{cl}(\mathrm{HC}\cdot\mathrm{cl}(\mu))$ follows from (i) and (ii). Thus $\mathrm{HC}\cdot\mathrm{cl}(\mu) = \mathrm{HC}\cdot\mathrm{cl}(\mathrm{HC}\cdot\mathrm{cl}(\mu))$. The proof of the other case is similar.

(vi) By (i) and (iv), we have that $\mathrm{HC}\cdot\mathrm{cl}(\mu)$ is an HC-closed set containing μ and so $\mathrm{HC}\cdot\mathrm{cl}(\mu) \geqslant \bigwedge\{\eta \in L^X\colon\ \eta$ is an HC-closed set containing $\mu\}$. Conversely, let $x_\alpha \in M(L^X)$ be such that $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. Then $\mu \nleqslant \varrho$ for each $\varrho \in \mathrm{HC}\,R_{x_\alpha}$. Hence, if $\eta \in L^X$ is an HC-closed set containing μ, then $\eta \nleqslant \varrho$ and then $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\eta) = \eta$. This implies that $\mathrm{HC}\cdot\mathrm{cl}(\mu) \leqslant \bigwedge\{\eta \in L^X\colon\ \eta$ is an HC-closed set containing $\mu\}$. Thus, we have $\mathrm{HC}\cdot\mathrm{cl}(\mu) = \bigwedge\{\eta \in L^X\colon\ \eta$ is an HC-closed set containing $\mu\}$. □

Theorem 3.4. *Let (L^X,τ) be an L-ts. The following statements hold:*

(i) 1_X *and* 0_X *are both* HC-*closed* (HC-*open*).

(ii) *Every almost N-compact closed set is* HC-*closed.*

(iii) *The union (intersection) of finite* HC-*closed* (HC-*open*) *sets is* HC-*closed* (HC-*open*).

(iv) *The intersection (union) of arbitrary* HC-*closed* (HC-*open*) *sets is* HC-*closed* (HC-*open*).

(v) $\mu \in L^X$ *is* HC-*closed iff there exists* $\eta \in \mathrm{HC}\,R_{x_\alpha}$ *such that* $\mu \leqslant \eta$ *for each* $x_\alpha \in M(L^X)$ *with* $x_\alpha \notin \mu$.

Proof. (i) Obvious.

(ii) Let $\mu \in L^X$ be an almost N-compact closed set in (L^X, τ). Let $x_\alpha \in M(L^X)$ with $x_\alpha \notin \mu$. Since μ is almost N-compact closed, so $\mu \in \mathrm{HC}\, R_{x_\alpha}$. Also, since $\mu \leqslant \mu$, so by Definition 3.1 we have $x_\alpha \notin \mathrm{HC} \cdot \mathrm{cl}(\mu)$. Thus $\mathrm{HC} \cdot \mathrm{cl}(\mu) \leqslant \mu$ and hence μ is an HC-closed set.

(iii) Let $\mu, \eta \in L^X$ be two HC-closed sets in (L^X, τ). Let $x_\alpha \in M(L^X)$ and $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mu \vee \eta)$. Then for each $\varrho \in \mathrm{HC}\, R_{x_\alpha}$ we have $\mu \vee \eta \nleqslant \varrho$ and so $\mu \nleqslant \varrho$ or $\eta \nleqslant \varrho$. Hence $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mu)$ or $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\eta)$ and so $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mu) \vee \mathrm{HC} \cdot \mathrm{cl}(\eta) = \mu \vee \eta$. Thus $\mu \vee \eta$ is HC-closed. The proof of the other case is similar.

(iv) Let $\{\mu_j \in L^X \colon j \in J\}$ be a family of HC-closed sets. Let $x_\alpha \in M(L^X)$ be such that $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}\Big(\bigwedge\limits_{j \in J} \mu_j\Big)$. Then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ we have $\bigwedge\limits_{j \in J} \mu_j \nleqslant \eta$, equivalently, $\mu_j \nleqslant \eta$ for every $j \in J$. Hence $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mu_j) \leqslant \mu_j$ for every $j \in J$. Then $x_\alpha \in \bigwedge\limits_{j \in J} \mu_j$. Thus $\bigwedge\limits_{J \in J} \mu_j$ is an HC-closed set in (L^X, τ). The proof of the other case is similar.

(v) Suppose that $\mu \in L^X$ is HC-closed, $x_\alpha \in M(L^X)$ and $x_\alpha \notin \mu$. By Definition 3.1 there exists $\eta \in \mathrm{HC}\, R_{x_\alpha}$ such that $\mu \leqslant \eta$. Conversely, suppose that $\mu \in L^X$ is not HC-closed, then there exists $x_\alpha \in M(L^X)$ such that $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mu)$ and $x_\alpha \notin \mu$. Hence, $\mu \nleqslant \eta$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$, a contradiction with the hypothesis and so μ is HC-closed. □

Theorem 3.5. *Let (L^X, τ) be an L-ts. Then the families $\tau_{\mathrm{HC}} = \{\mu \in L^X \colon \mathrm{HC} \cdot \mathrm{cl}(\mu') = \mu'\}$ and $\tau_{\mathrm{NC}} = \{\mu \in L^X \colon \mathrm{NC} \cdot \mathrm{cl}(\mu') = \mu'\}$ are L-topologies on L^X. We call $(L^X, \tau_{\mathrm{HC}})$ and $(L^X, \tau_{\mathrm{NC}})$ the $\mathrm{HC} \cdot L$-topological space and $\mathrm{NC} \cdot L$-topological space induced by (L^X, τ).*

Proof. It is an immediate consequence of Definition 3.1 and Proposition 3.3 and Theorem 3.4. □

Theorem 3.6. *Let (L^X, τ) be an L-ts. Then:*

(i) $\tau_{\mathrm{NC}} \subseteq \tau_N[3] \subseteq \tau_{\mathrm{HC}} \subseteq \tau$.

(ii) *If (L^X, τ) is N-compact (nearly N-compact, almost N-compact), then $\tau = \tau_{\mathrm{NC}}$ $(\tau = \tau_N,\ \tau = \tau_{\mathrm{HC}})$.*

(iii) *If (L^X, τ) is an LR_2-space* [13], *then $\tau_{\mathrm{NC}} = \tau_N = \tau_{\mathrm{HC}}$.*

(iv) *If (L^X, τ) is an induced L-ts* [9], *then $\tau_N = \tau_{\mathrm{NC}}$.*

(v) *L-ts $(L^X, \tau_{\mathrm{NC}})$ is an N-compact space.*

(vi) *L-ts $(L^X, \tau_{\mathrm{HC}})$ is an almost N-compact space.*

Proof. Follows immediately from Definition 3.5. □

4. HC-convergence theory of L-nets and L-ideals

In this section we establish the HC-convergence theories of both the L-nets and the L-ideals. We discuss the relationship between the HC-convergence of L-ideals and that of L-nets.

Definition 4.1 [13], [14]**.** Let (L^X, τ) be an L-ts. An L-net in (L^X, τ) is a mapping $S\colon D \to M(L^X)$ denoted by $S = \{S(n); n \in D\}$, where D is a directed set. S is said to be in $\mu \in L^X$ if for every $n \in D$, $S(n) \in \mu$.

Definition 4.2. Let S be an L-net in an L-ts (L^X, τ) and let $x_\alpha \in M(L^X)$.

(i) x_α is said to be an HC-limit point of S, or net S HC-converges to x_α, in symbol $S \xrightarrow{\mathrm{HC}} x_\alpha$ if $(\forall \lambda \in \mathrm{HC}\, R_{x_\alpha})$ $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \lambda)$.

(ii) x_α is said to be an HC-cluster point of S, or net S HC-acumulates to x_α, in symbol $S \overset{\mathrm{HC}}{\propto} x_\alpha$ if $(\forall \lambda \in \mathrm{HC}\, R_{x_\alpha})$ $(\forall n \in D)$ $(\exists m \in D, m \geqslant n)$ $(S(m) \notin \lambda)$.

The union of all HC-limit points and HC-cluster points of S will be denoted by $\mathrm{HC} \cdot \lim(S)$ and $\mathrm{HC} \cdot \mathrm{adh}(S)$, respectively.

Theorem 4.3. *Suppose that S is an L-net in (L^X, τ), $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then the following results are true:*

(i) $x_\alpha \in \mathrm{HC} \cdot \lim(S)$ *iff* $S \xrightarrow{\mathrm{HC}} x_\alpha$ ($x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(S)$ *iff* $S \overset{\mathrm{HC}}{\propto} x_\alpha$).

(ii) $\lim(S)$ [14] $\leqslant \mathrm{HC} \cdot \lim(S)$ ($\mathrm{adh}(S)$ [14] $\leqslant \mathrm{HC} \cdot \mathrm{adh}(S)$).

(iii) $\mathrm{HC} \cdot \lim(S) \leqslant \mathrm{HC} \cdot \mathrm{adh}(S)$.

(iv) $\mathrm{HC} \cdot \lim(S)$ *and* $\mathrm{HC} \cdot \mathrm{adh}(S)$ *are* HC-*closed sets in* L^X.

Proof. (i) Let $S \xrightarrow{\mathrm{HC}} x_\alpha$, so by definition $x_\alpha \in \mathrm{HC} \cdot \lim(S)$. Conversely, let $x_\alpha \in \mathrm{HC} \cdot \lim(S)$ and $\lambda \in \mathrm{HC}\, R_{x_\alpha}$. Since $x_\alpha \notin \lambda$, so $\mathrm{HC} \cdot \lim(S) \nleqslant \lambda$. Therefore there exists $y_\beta \in M(L^X)$ such that $y_\beta \in \mathrm{HC} \cdot \lim(S)$ but $y_\beta \notin \lambda$ and so $\lambda \in \mathrm{HC}\, R_{y_\beta}$. Hence $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \lambda)$. Thus $S \xrightarrow{\mathrm{HC}} x_\alpha$. The proof of the other case is similar.

(ii) Let $x_\alpha \in \lim(S)$ and $\eta \in \mathrm{HC}\, R_{x_\alpha}$. Since $\mathrm{HC}\, R_{x_\alpha} \subseteq R_{x_\alpha}$, we have $\eta \in R_{x_\alpha}$. And since $x_\alpha \in \lim(S)$, we have $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \eta)$. Hence $x_\alpha \in \mathrm{HC} \cdot \lim(S)$. So $\lim(S) \leqslant \mathrm{HC} \cdot \lim(S)$. The proof of the other case is similar.

(iii) Obvious.

(iv) Let $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(S))$ and $\lambda \in \mathrm{HC}\, R_{x_\alpha}$. Then $\mathrm{HC} \cdot \lim(S) \nleqslant \lambda$. So there exists $y_\beta \in M(L^X)$ such that $y_\beta \in \mathrm{HC} \cdot \lim(S)$ and $y_\beta \notin \lambda$. Then $(\forall \varrho \in \mathrm{HC}\, R_{y_\beta})$ $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \varrho)$ and so $S(m) \notin \lambda$. Hence $x_\alpha \in \mathrm{HC} \cdot \lim(S)$. Thus $\mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(S)) \leqslant \mathrm{HC} \cdot \lim(S)$ and so $\mathrm{HC} \cdot \lim(S)$ is an HC-closed set. Similarly, one can easily verify that $\mathrm{HC} \cdot \mathrm{adh}(S)$ is an HC-closed set. □

Theorem 4.4. *Let (L^X,τ) be an L-ts, $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$ iff there is an L-net in μ which* HC*-converges to x_α.*

Proof. Let $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. Then $(\forall\lambda \in \mathrm{HC}\,R_{x_\alpha})$ $(\mu \nleqslant \lambda)$ and so there exists $\alpha(\mu,\lambda) \in L\backslash\{0\}$ such that $x_{\alpha(\mu,\lambda)} \in \mu$ and $x_{\alpha(\mu,\lambda)} \notin \lambda$. Since the pair $(\mathrm{HC}\,R_{x_\alpha}, \geqslant)$ is a directed set so we can define an L-net $S\colon\ \mathrm{HC}\,R_{x_\alpha} \to M(L^X)$ given by $S(\lambda) = x_{\alpha(\mu,\lambda)}$, $\forall\lambda \in \mathrm{HC}\,R_{x_\alpha}$. Then S is an L-net in μ. Now let $\varrho \in \mathrm{HC}\,R_{x_\alpha}$ be such that $\varrho \geqslant \lambda$, so there exists $S(\varrho) = x_{\alpha(\mu,\varrho)} \notin \varrho$. Then $x_{\alpha(\mu,\varrho)} \notin \lambda$. So $S\xrightarrow{\mathrm{HC}}x_\alpha$. Conversely; let S be an L-net in μ with $S\xrightarrow{\mathrm{HC}}x_\alpha$. Then $(\forall\lambda \in \mathrm{HC}\,R_{x_\alpha})$ $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \lambda)$. Since S is an L-net in μ, we have $\mu \geqslant S(m) > \lambda$. Hence $(\forall\lambda \in \mathrm{HC}\,R_{x_\alpha})$ $(\mu \nleqslant \lambda)$. So $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. □

Theorem 4.5. *Let both $S = \{S(n); n \in D\}$ and $T = \{T(n); n \in D\}$ be L-nets in L-ts (L^X,τ) with the same domain and for each $n \in D$, let $T(n) \geqslant S(n)$ hold. Then the following statements hold:*

(i) $\mathrm{HC}\cdot\lim(S) \leqslant \mathrm{HC}\cdot\lim(T)$.

(ii) $\mathrm{HC}\cdot\mathrm{adh}(S) \leqslant \mathrm{HC}\cdot\mathrm{adh}(T)$.

Proof. (i). Let $x_\alpha \in M(L^X)$ with $x_\alpha \in \mathrm{HC}\cdot\lim(S)$, then $(\forall\eta \in \mathrm{HC}\,R_{x_\alpha})$ $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \eta)$. Since $T(n) \geqslant S(n)$, $\forall n \in D$, so $T(m) \notin \eta$. Hence $(\forall\eta \in \mathrm{HC}\,R_{x_\alpha})$ $(\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(T(m) \notin \eta)$. So $x_\alpha \in \mathrm{HC}\cdot\lim(T)$. Hence $\mathrm{HC}\cdot\lim(S) \leqslant \mathrm{HC}\cdot\lim(T)$.

(ii) The proof is similar to that of (i) and is omitted. □

Theorem 4.6. *Let S be an L-net in an L-ts (L^X,τ) and let $x_\alpha \in M(L^X)$, then:*

(i) *$S\overset{\mathrm{HC}}{\propto}x_\alpha$ iff there exists an L-subnet T [14] of S such that $T\xrightarrow{\mathrm{HC}}x_\alpha$.*

(ii) *If $S\xrightarrow{\mathrm{HC}}x_\alpha$, then $T\xrightarrow{\mathrm{HC}}x_\alpha$ for each L-subnet T of S.*

Proof. (i) Sufficiency follows from the definition of an L-subnet and so we only prove necessity. Let $g\colon\ (\mathrm{HC}\,R_{x_\alpha}, D) \to D$, so $g(\eta,n) \in D$. Let $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(S)$, then $(\forall\eta \in \mathrm{HC}\,R_{x_\alpha})$ $(\forall n \in D)$ $(\exists g(\eta,n) \in D)$ $(g(\eta,n) \geqslant n)$ $(S(g(\eta,n)) \notin \eta)$. Let $E = \{(g(\eta,n),\eta)\colon\ \eta \in \mathrm{HC}\,R_{x_\alpha}, n \in D\}$ and define the relation $\leqslant$ on E as following: $(g(\eta_1,n_1),\eta_1) \leqslant (g(\eta_2,n_2),\eta_2)$ iff $n_1 \leqslant n_2$ and $\eta_1 \leqslant \eta_2$. It is easy to show that E is a directed set. So we can define an L-net $T\colon\ E \to M(L^X)$ as follows: $T(g(\eta,n),\eta) = S(g(\eta,n))$ and T is an L-subnet of S. Now we prove that $T\xrightarrow{\mathrm{HC}}x_\alpha$. Let $\eta \in \mathrm{HC}\,R_{x_\alpha}, n \in D$, so $(g(\eta,n),\eta) \in E$. Then $(\forall(g(\lambda,m),\lambda) \in E)$ $(g(\lambda,m),\lambda) \geqslant (g(\eta,n),\eta))$, hence $T(g(\lambda,m),\lambda) = S(g(\lambda,m)) \notin \lambda$. Since $\lambda \geqslant \eta$, so $T(g(\lambda,m),\lambda)) \notin \eta$. Hence $T\xrightarrow{\mathrm{HC}}x_\alpha$.

(ii) follows from the definition of an L-subnet. □

Definition 4.7 [15]. A nonempty family $\mathcal{L} \subset L^X$ is called an L-ideal if the following conditions are fulfilled, for each μ_1, $\mu_2 \in L^X$:

(i) If $\mu_1 \leqslant \mu_2$ and $\mu_2 \in \mathcal{L}$ then $\mu_1 \in \mathcal{L}$.
(ii) If $\mu_1, \mu_2 \in \mathcal{L}$, then $\mu_1 \vee \mu_2 \in \mathcal{L}$.
(iii) $1_X \notin \mathcal{L}$.

Definition 4.8. Let (L^X, τ) be an L-ts and let $x_\alpha \in M(L^X)$. An L-ideal $\mathcal{L}$ is said

(i) to HC-converge to x_α, in symbol $\mathcal{L} \xrightarrow{\mathrm{HC}} x_\alpha$ (or x_α is an HC-limit point of $\mathcal{L}$) if $\mathrm{HC}\, R_{x_\alpha} \subseteq \mathcal{L}$.
(ii) to HC-accumulates to x_α, in symbol $\mathcal{L} \overset{\mathrm{HC}}{\propto} x_\alpha$ (or x_α is an HC-cluster point of $\mathcal{L}$) if for each $\mu \in \mathcal{L}$ and $\eta \in \mathrm{HC}\, R_{x_\alpha}$, $\mu \vee \eta \neq 1_X$.

The union of all HC-limit points and HC-cluster points of $\mathcal{L}$ are denoted by $\mathrm{HC} \cdot \lim(L)$ and $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$, respectively.

Theorem 4.9. *Let $\mathcal{L}$ be an L-ideal in L-ts (L^X, τ) and let $x_\alpha \in M(L^X)$. Then the following statements hold:*
(i) $\mathrm{HC} \cdot \lim(\mathcal{L}) \leqslant \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$.
(ii) $\mathcal{L} \xrightarrow{\mathrm{HC}} x_\alpha$ *iff* $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L})$ ($\mathcal{L} \overset{\mathrm{HC}}{\propto} x_\alpha$ *iff* $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$).
(iii) $\lim(\mathcal{L})$ [15] $\leqslant \mathrm{HC} \cdot \lim(\mathcal{L})$ ($\mathrm{adh}(\mathcal{L})$ [15] $\leqslant \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$).

Proof.
(i) Let $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L})$. Then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ we have $\eta \in \mathcal{L}$. Hence for each $\mu \in \mathcal{L}$, we have $\eta \vee \mu \in \mathcal{L}$ and so $\eta \vee \mu \neq 1_X$. Hence $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$.
(ii) Let $\mathcal{L} \xrightarrow{\mathrm{HC}} x_\alpha$, then by Definition 4.8(i), $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L})$. Conversely, let $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L})$ and let $\eta \in \mathrm{HC}\, R_{x_\alpha}$. Since $x_\alpha \notin \eta = \mathrm{HC} \cdot \mathrm{cl}(\eta)$, so we have $\mathrm{HC} \cdot \lim(\mathcal{L}) \nleqslant \eta$. Therefore there exists $y_\gamma \in M(L^X)$ satisfying $y_\gamma \in \mathrm{HC} \cdot \lim(\mathcal{L})$ but $y_\gamma \notin \eta$, hence $\eta \in \mathrm{HC}\, R_{y_\gamma}$. So we have $\mathrm{HC}\, R_{x_\alpha} \subseteq \mathrm{HC}\, R_{y_\gamma} \subseteq \mathcal{L}$, hence $\mathrm{HC}\, R_{x_\alpha} \subseteq \mathcal{L}$. So $\mathcal{L} \xrightarrow{\mathrm{HC}} x_\alpha$. Similarly, one can easily verify that $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$.
(iii) Obvious. □

Definition 4.10 [15]**.** A nonempty family $\mathcal{B} \subset L^X$ is called an L-ideal base if it satisfies the following conditions, for each $\mu_1, \mu_2 \in L^X$:
(i) If $\mu_1, \mu_2 \in \mathcal{B}$, then there exists $\mu_3 \in \mathcal{B}$ such that $\mu_3 \geqslant \mu_1 \vee \mu_2 \in \mathcal{B}$.
(ii) $1_X \notin \mathcal{B}$.

Then $\mathcal{L} = \{\varrho \in L^X \colon\ \varrho \leqslant \mu \text{ for some } \mu \in \mathcal{B}\}$ is an L-ideal and it is said to be the L-ideal generated by $\mathcal{B}$.

Theorem 4.11. *Let $\mathcal{L}$ be an L-ideal in an L-ts (L^X, τ) and let $x_\alpha \in M(L^X)$. If $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$ then there is in L^X an L-ideal $\mathcal{J} \supseteq \mathcal{L}$ with $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{J})$.*

Proof. Let $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$, then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $\mu \in \mathcal{L}$, $\eta \vee \mu \neq 1_X$, hence there exists $x_\alpha \in M(L^X)$, $x_\alpha \notin \eta \vee \mu$. Choose $\mathcal{B} = \{\eta \vee \mu \colon \mu \in \mathcal{L}, \eta \in \mathrm{HC}\, R_{x_\alpha}\}$. Then $\mathcal{B}$ is an L-ideal base in L^X. Then $\mathcal{J} = \{\varrho \in L^X \colon \varrho \leqslant \lambda$ for some $\lambda = \eta \vee \mu\}$ is an L-ideal in L^X and we call $\mathcal{J}$ the L-ideal generated by $\mathcal{B}$. It is easy to show that $\mathcal{J} \supset \mathcal{L}$. Now let $\eta \in \mathrm{HC}\, R_{x_\alpha}$. Since $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$, so $\eta \vee \mu \neq 1_X$ for each $\mu \in \mathcal{L}$, hence $\eta \vee \mu \in \mathcal{B}$. Moreover, since $\eta \vee \mu \geqslant \eta \vee \mu$, so $\eta \vee \mu \in \mathcal{J}$ and since $\eta \leqslant \eta \vee \mu$, so $\eta \in \mathcal{J}$. Hence $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{J})$. □

Definition 4.12 [15]**.** An L-ideal $\mathcal{L}$ in L^X is called maximal if for every L-ideal $\mathcal{L}^*$, $\mathcal{L} \subseteq \mathcal{L}^*$ implies $\mathcal{L} = \mathcal{L}^*$.

Theorem 4.13. *If $\mathcal{L}$ is a maximal L-ideal in an L-ts (L^X, τ), then* $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}) = \mathrm{HC} \cdot \lim(\mathcal{L})$.

Proof. It follows from Theorems 4.9 (i) and 4.11. □

Theorem 4.14. *Let both $\mathcal{L}_1$ and $\mathcal{L}_2$ be L-ideals in L-ts (L^X, τ) with $\mathcal{L}_1 \subset \mathcal{L}_2$. Then the following statements hold:*

(i) $\mathrm{HC} \cdot \lim(\mathcal{L}_1) \leqslant \mathrm{HC} \cdot \lim(\mathcal{L}_2)$.

(ii) $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_1) \geqslant \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_2)$.

Proof.

(i) Let $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L}_1)$, then $\eta \in \mathcal{L}_1$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$, so $\eta \in \mathcal{L}_2$. Hence $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L}_2)$. Thus $\mathrm{HC} \cdot \lim(\mathcal{L}_1) \leqslant \mathrm{HC} \cdot \lim(\mathcal{L}_2)$.

(ii) Let $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_2)$, then $\eta \vee \mu \neq 1_X$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $\mu \in \mathcal{L}_2$. Since $\mathcal{L}_1 \subset \mathcal{L}_2$, so for each $\mu \in \mathcal{L}_1$ we have $\eta \vee \mu \neq 1_X$. Hence $x_\alpha \in \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_1)$. Thus $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_1) \geqslant \mathrm{HC} \cdot \mathrm{adh}(\mathcal{L}_2)$. □

Theorem 4.15. *Let $\mathcal{L}$ be an L-ideal in an L-ts (L^X, τ). Then both* $\mathrm{HC} \cdot \lim(\mathcal{L})$ *and* $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$ *are* HC*-closed set in L^X.*

Proof. Let $x_\alpha \in \mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(\mathcal{L}))$ and $\eta \in \mathrm{HC}\, R_{x_\alpha}$. Then $\mathrm{HC} \cdot \lim(\mathcal{L}) \nleqslant \eta$, so there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \in \mathrm{HC} \cdot \lim(\mathcal{L})$ and $y_\gamma \notin \eta$. Since $y_\gamma \in \mathrm{HC} \cdot \lim(\mathcal{L})$, so for each $\varrho \in \mathrm{HC}\, R_{y_\gamma}$ we have $\varrho \in \mathcal{L}$. Since $y_\gamma \notin \eta$, we have $\eta \in \mathrm{HC}\, R_{y_\gamma}$ and so $\eta \in \mathcal{L}$. Hence $x_\alpha \in \mathrm{HC} \cdot \lim(\mathcal{L})$. Thus $\mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(\mathcal{L})) \leqslant \mathrm{HC} \cdot \lim(\mathcal{L})$. On the other hand, since $\mathrm{HC} \cdot \lim(\mathcal{L}) \leqslant \mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(\mathcal{L}))$, so $\mathrm{HC} \cdot \mathrm{cl}(\mathrm{HC} \cdot \lim(\mathcal{L})) = \mathrm{HC} \cdot \lim(\mathcal{L})$. This means that $\mathrm{HC} \cdot \lim(\mathcal{L})$ is an HC-closed set. Similarly, one can easily verify that $\mathrm{HC} \cdot \mathrm{adh}(\mathcal{L})$ is an HC-closed set. □

Theorem 4.16. *Let (L^X,τ) be an L-ts, $\mu \in L^X$ and $x_\alpha \in M(L^X)$. Then $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$ iff there exists an L-ideal $\mathcal{L}$ in L^X such that $\mathcal{L}\xrightarrow{\mathrm{HC}}x_\alpha$ and $\mu \notin \mathcal{L}$.*

P r o o f. Let $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. Then for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$ we have $\mu \nleqslant \eta$. Let $\mathcal{L} = \{\varrho \in L^X\colon\ \varrho \leqslant \eta$ for some $\eta \in \mathrm{HC}\,R_{x_\alpha}\}$. It is easy to show that $\mathcal{L}$ is an L-ideal. It is clear that $\mu \notin \mathcal{L}$. Now we show that $\mathcal{L}\xrightarrow{\mathrm{HC}}x_\alpha$. Let $\lambda \in \mathrm{HC}\,R_{x_\alpha}$. We have $\lambda \in \mathcal{L}$, by the definition of $\mathcal{L}$. So $\mathrm{HC}\,R_{x_\alpha} \subseteq \mathcal{L}$. Thus $\mathcal{L}\xrightarrow{\mathrm{HC}}x_\alpha$. Conversely; let $\mathcal{L}$ be an L-ideal, $\mu \notin \mathcal{L}$ and $\mathcal{L}\xrightarrow{\mathrm{HC}}x_\alpha$. Then $\eta \in \mathcal{L}$ for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$. Since $\eta \in \mathcal{L}$ and $\mu \notin \mathcal{L}$, we conclude $\mu \nleqslant \eta$. Hence $x_\alpha \in \mathrm{HC}\cdot\mathrm{cl}(\mu)$. □

Theorem 4.17. *Let $F\colon (L^X,\tau) \to (L^Y,\Delta)$ be an L-valued Zadeh mapping and let $\mathcal{L}_1,\mathcal{L}_2$ be L-ideals in L^X, L^Y, respectively. Then $F^*(\mathcal{L}_1) = \{\eta \in L^Y\colon\ (\exists \mu \in \mathcal{L}_1)$ $(\forall x_\alpha \in M(L^X))$ $(x_\alpha \notin \mu)$ $(F(x_\alpha) \notin \eta)\}$ is an L-ideal in L^Y. Also, if F is onto, then $F^{-1}(\mathcal{L}_2) = \{F^{-1}(\eta)\colon\ \eta \in \mathcal{L}_2\}$ is an L-ideal in L^X.*

P r o o f. Straightforward. □

Definition 4.18 [14], [15]. Let $\mathcal{L}$ be an L-ideal in an L-ts (L^X,τ) and let $D(\mathcal{L}) = \{(x_\alpha,\mu)\colon\ x_\alpha \in M(L^X), \mu \in \mathcal{L}$ and $x_\alpha \notin \mu\}$. In $D(\mathcal{L})$ we define the ordering relation as follows: $(x_\alpha,\mu_1) \leqslant (y_\gamma,\mu_2)$ iff $\mu_1 \leqslant \mu_2$. Then $(D(\mathcal{L}),\leqslant)$ is a directed set. Now we define a mapping $S(\mathcal{L})\colon D(\mathcal{L}) \to M(L^X)$ as follows: $S(\mathcal{L})(x_\alpha,\mu) = x_\alpha$. So $S(\mathcal{L}) = \{S(\mathcal{L})(x_\alpha,\mu) = x_\alpha; (x_\alpha,\mu) \in D(\mathcal{L})\}$ is the L-net generated by $\mathcal{L}$.

On the other hand, let S be an L-net in (L^X,τ), then $\mathcal{L}(S) = \{\mu \in L^X\colon\ (\exists n \in D)$ $(\forall m \in D, m \geqslant n)$ $(S(m) \notin \mu)\}$ is the L-ideal generated by S.

Theorem 4.19. *Let $\mathcal{L}$ be an L-ideal in an L-ts (L^X,τ). Then the following equalities hold:*

(i) $\mathrm{HC}\cdot\lim(\mathcal{L}) = \mathrm{HC}\cdot\lim(S(\mathcal{L}))$.

(ii) $\mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}) = \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$.

P r o o f. (i) Let $x_\alpha \in \mathrm{HC}\cdot\lim(\mathcal{L})$, then $\eta \in \mathcal{L}$ for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$ (or $\mathrm{HC}\,R_{x_\alpha} \subseteq \mathcal{L}$). Since $\eta \in \mathcal{L}$ and $x_\alpha \notin \eta$, so $(x_\alpha,\eta) \in D(\mathcal{L})$ where $D(\mathcal{L}) = \{(x_\alpha,\eta)\colon\ x_\alpha \in M(L^X), \eta \in \mathcal{L}$ and $x_\alpha \notin \eta\}$. Since $\mathcal{L}\xrightarrow{\mathrm{HC}}x_\alpha$, hence for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$ there exists $\mu \in \mathcal{L}$ such that $\eta \leqslant \mu$. Since $\eta \leqslant \mu$ is equivalent to $(x_\alpha,\eta) \leqslant (y_\gamma,\mu)$, we have $S(\mathcal{L})((y_\gamma,\mu)) = y_\gamma \notin \eta$. So for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$ there exists $(x_\alpha,\eta) \in D(\mathcal{L})$ such that $S(\mathcal{L})((y_\gamma,\mu)) \notin \eta$ for each $(y_\gamma,\mu) \in D(\mathcal{L})$ and $(y_\gamma,\mu) \geqslant (x_\alpha,\eta)$. So $S(\mathcal{L})\xrightarrow{\mathrm{HC}}x_\alpha$. Hence $x_\alpha \in \mathrm{HC}\cdot\lim(S(\mathcal{L}))$. Thus $\mathrm{HC}\cdot\lim(\mathcal{L}) \leqslant \mathrm{HC}\cdot\lim(S(\mathcal{L}))$. Conversely, let $x_\alpha \in \mathrm{HC}\cdot\lim(S(\mathcal{L}))$, then for each $\eta \in \mathrm{HC}\,R_{x_\alpha}$ there exists $(z_\varepsilon,\lambda) \in D(\mathcal{L})$ such that $S(\mathcal{L})((y_\gamma,\mu)) \notin \eta$ for each $(y_\gamma,\mu) \in D(\mathcal{L})$ and $(y_\gamma,\mu) \geqslant (z_\varepsilon,\lambda)$. Since $(y_\gamma,\mu) \geqslant (z_\varepsilon,\lambda)$, we have $y_\gamma \notin \lambda$ (because $\mu \geqslant \lambda$) and from $S(\mathcal{L})((y_\gamma,\mu)) = y_\gamma \notin \eta$ we obtain $\eta \leqslant \lambda$. Since $\lambda \in \mathcal{L}$, we have $\eta \in \mathcal{L}$. Hence $x_\alpha \in \mathrm{HC}\cdot\lim(\mathcal{L})$. So $\mathrm{HC}\cdot\lim(S(\mathcal{L})) \leqslant \mathrm{HC}\cdot\lim(\mathcal{L})$. Hence the equality hold. Thus $\mathrm{HC}\cdot\lim(\mathcal{L}) = \mathrm{HC}\cdot\lim(S(\mathcal{L}))$.

(ii) Let $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L})$, then $\eta \vee \mu \neq 1_X$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $\mu \in \mathcal{L}$. Since $\eta \in \mathrm{HC}\, R_{x_\alpha}$, we have $\eta \vee \mu \neq 1_X$ for each $(y_\gamma, \mu) \in D(\mathcal{L})$. Therefore there exists a molecule $z_\varepsilon \in M(L^X)$ such that $z_\varepsilon \notin \eta, z_\varepsilon \notin \mu$. So $(z_\varepsilon, \mu) \in D(\mathcal{L})$ and $(z_\varepsilon, \mu) \geqslant (y_\gamma, \mu)$, so $S(\mathcal{L})(z_\varepsilon, \mu) = z_\varepsilon \notin \eta$. So for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $(y_\gamma, \mu) \in D(\mathcal{L})$ there exists $(z_\varepsilon, \mu) \in D(\mathcal{L})$ such that $(z_\varepsilon, \mu) \geqslant (y_\gamma, \mu)$ and $S(\mathcal{L})(z_\varepsilon, \mu) = z_\varepsilon \notin \eta$. So $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$. Hence $\mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}) \leqslant \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$. Conversely, let $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$. Let $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and $\mu \in \mathcal{L}$. Since $\mu \in \mathcal{L}$, so $\mu \neq 1_X$ and there exists $y_\gamma \in M(L^X)$ such that $y_\gamma \notin \mu$. So $(y_\gamma, \mu) \in D(\mathcal{L})$. Now since $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$, there exists $(z_\varepsilon, \lambda) \in D(\mathcal{L})$ such that $(z_\varepsilon, \lambda) \geqslant (y_\gamma, \mu)$ and $S(\mathcal{L})((z_\varepsilon, \lambda)) = z_\varepsilon \notin \eta$. Since $z_\varepsilon \notin \lambda, z_\varepsilon \notin \eta$, so $z_\varepsilon \notin \eta \vee \lambda$ and $\lambda \geqslant \mu$, so $z_\varepsilon \notin \eta \vee \mu$. Hence $\eta \vee \mu \neq 1_X$. So we have $\eta \vee \mu \neq 1_X$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $\mu \in \mathcal{L}$. Hence $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L})$. So $\mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L})) \leqslant \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L})$. Hence the equality is satisfied. Thus $\mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}) = \mathrm{HC}\cdot\mathrm{adh}(S(\mathcal{L}))$. □

Theorem 4.20. *Suppose that S is an L-net in an L-ts (L^X, τ), then:*
(i) $\mathrm{HC}\cdot\lim(S) = \mathrm{HC}\cdot\lim(\mathcal{L}(S))$.
(ii) $\mathrm{HC}\cdot\mathrm{adh}(S) \leqslant \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}(S))$.

Proof.
(i) Let $x_\alpha \in \mathrm{HC}\cdot\lim(S)$. Then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ there exists $m \in D$ such that $S(n) \notin \eta$ for each $n \in D$, $n \geqslant m$. Since $S(n) \notin \eta$, so by the definition of $\mathcal{L}(S)$ we have $\eta \in \mathcal{L}(S)$ for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$. So $\mathrm{HC}\, R_{x_\alpha} \subseteq \mathcal{L}(S)$. Hence $x_\alpha \in \mathrm{HC}\cdot\lim(\mathcal{L}(S))$. So $\mathrm{HC}\cdot\lim(S) \leqslant \mathrm{HC}\cdot\lim(\mathcal{L}(S))$. Conversely, let $x_\alpha \in \mathrm{HC}\cdot\lim(\mathcal{L}(S))$. Then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ there exists $\lambda \in \mathcal{L}(S)$ such that $\eta \leqslant \lambda$. Since $\lambda \in \mathcal{L}(S)$, so by the definition of $\mathcal{L}(S)$ for each $\lambda \in \mathcal{L}(S)$ there exists $m \in D$ such that $S(n) \notin \lambda$ for each $n \in D$, $n \geqslant m$. Since $\eta \leqslant \lambda$, so $S(n) \notin \eta$. Hence $x_\alpha \in \mathrm{HC}\cdot\lim(S)$. So $\mathrm{HC}\cdot\lim(\mathcal{L}(S)) \leqslant \mathrm{HC}\cdot\lim(S)$.
(ii) Let $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(S)$. Then for each $\eta \in \mathrm{HC}\, R_{x_\alpha}$ and each $m \in D$ there exists $n_1 \in D$ such that $n_1 \geqslant m$ and $S(n_2) \notin \eta$. By the definition of $\mathcal{L}(S)$, for each $\lambda \in \mathcal{L}(S)$ and each $m \in D$ there exists $n_2 \in D$ such that $n_2 \geqslant m$ and $S(n_2) \notin \lambda$. Since D is a directed set, there exists $n_3 \in D$ such that $n_3 \geqslant n_1$, $n_3 \geqslant n_2$ and $n_3 \geqslant m$. Thus $(\forall \eta \in \mathrm{HC}\, R_{x_\alpha})\ (\forall \lambda \in \mathcal{L}(S))\ (S(n_3) \notin \eta \vee \lambda)$. Hence $\eta \vee \lambda \neq 1_X$ and so $x_\alpha \in \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}(S))$. Hence $\mathrm{HC}\cdot\mathrm{adh}(S) \leqslant \mathrm{HC}\cdot\mathrm{adh}(\mathcal{L}(S))$. □

5. HL-continuous mapping

The concept of H-continuous mappings in general topology was introduced by Long and Hamlett in [10]. Recently, Dang and Behera extended the concept to I-topology [4] using the almost compactness introduced by Mukherjee and Sinha [11]. But the almost compactness has some shortcomings, for example, it is not a "good extension". In this section, we introduce a new definition of H-continuous mappings to be called HL-continuous on the basis of the notions of almost N-compactness due to [6] and R-nbds due to [12].

Definition 5.1. An L-valued Zadeh mapping $F\colon (L^X,\tau)\to(L^Y,\Delta)$ is said to be:

(i) H-continuous if $F^{-1}(\eta)\in\tau'$ for each almost N-compact closed set η in L^Y.

(ii) H-continuous at a molecule $x_\alpha\in M(L^X)$ if $F^{-1}(\lambda)\in R_{x_\alpha}$ for each $\lambda\in \operatorname{HC} R_{F(x_\alpha)}$.

Theorem 5.2. *Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be an L-valued Zadeh mapping. Then the following assertions are equivalent:*

(i) *F is* HL*-continuous.*

(ii) *F is* HL*-continuous at x_α, for each molecule $x_\alpha\in M(L^X)$.*

(iii) *If $\eta\in\Delta$ and η' is almost N-compact, then $F^{-1}(\eta)\in\tau$.*

These statements are implied by

(iv) *If $\eta\in L^Y$ is almost N-compact, then $F^{-1}(\eta)\in\tau'$.*

Moreover, if (L^Y,Δ) is a fully stratified LT_2*-space, all the statements are equivalent.*

Proof. (i) $\Longrightarrow$ (ii): Suppose that $F\colon (L^X,\tau)\to(L^Y,\Delta)$ is HL-continuous, $x_\alpha\in M(L^X)$ and $\lambda\in\operatorname{HC} R_{F(x_\alpha)}$, then $F^{-1}(\lambda)\in\tau'$. Since $F(x_\alpha)\notin\lambda$ is equivalent to $x_\alpha\notin F^{-1}(\lambda)$, so $F^{-1}(\lambda)\in R_{x_\alpha}$. Hence F is HL-continuous at x_α.

(ii) $\Longrightarrow$ (i): Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be HL-continuous at x_α for each $x_\alpha\in M(L^X)$. If F is not HL-continuous, then there is an almost N-compact closed set $\eta\in L^Y$ with $\operatorname{cl}(F^{-1}(\eta))\nleq F^{-1}(\eta)$. Then there exists $x_\alpha\in M(L^X)$ such that $x_\alpha\in\operatorname{cl}(F^{-1}(\eta))$ and $x_\alpha\notin F^{-1}(\eta)$. Since $x_\alpha\notin F^{-1}(\eta)$ implies that $F(x_\alpha)\notin\eta$, so $\eta\in\operatorname{HC} R_{F(x_\alpha)}$. But $F^{-1}(\eta)\notin R_{x_\alpha}$, a contradiction. Therefore, F must be HL-continuous.

(i) $\Longrightarrow$ (iii): Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be HL-continuous and $\eta\in\Delta$ with η' is almost N-compact. Then by the HL-continuity of F we have $F^{-1}(\eta')\in\tau'$, which is equivalent to $(F^{-1}(\eta))'\in\tau'$. So $F^{-1}(\eta)\in\tau$.

(iii) $\Longrightarrow$ (i): Let $\eta\in L^Y$ be an almost N-compact closed set, so $\eta'\in\tau$ and by (iii) we have $F^{-1}(\eta')\in\tau$. Then $F^{-1}(\eta)\in\tau'$. Hence F is HL-continuous.

(iv) $\Longrightarrow$ (i): Let $\eta\in L^Y$ be an almost N-compact closed set. By (iv), $F^{-1}(\eta)\in\tau'$. Hence F is HL-continuous.

Now suppose that (L^Y,Δ) is a fully stratified LT_2-space.

(i) $\Longrightarrow$ (iv): Let $\eta\in L^Y$ be an almost N-compact set. Since (L^Y,Δ) is a fully stratified LT_2-space, so $\eta\in\Delta'$. Thus by (i), $F^{-1}(\eta)\in\tau'$. $\square$

Theorem 5.3. *Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be a surjective L-valued Zadeh mapping. Then the following conditions are equivalent:*

(i) *F is* HL-*continuous.*

(ii) *For each $\mu\in L^X$, $F(\mathrm{cl}(\mu))\leqslant \mathrm{HC}\cdot\mathrm{cl}(F(\mu))$.*

(iii) *For each $\eta\in L^Y$, $\mathrm{cl}(F^{-1}(\eta))\leqslant F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta))$.*

(iv) *For each $\eta\in L^Y$, $F^{-1}(\mathrm{HC}\cdot\mathrm{int}(\eta))\leqslant \mathrm{int}(F^{-1}(\eta))$.*

(v) *$F^{-1}(\varrho)$ is open in L^X for each* HC-*open set ϱ in L^Y.*

(vi) *$F^{-1}(\lambda)$ is closed in L^X for each* HC-*closed set λ in L^Y.*

Proof. (i) $\Longrightarrow$ (ii): Let $\mu\in L^X$ and $x_\alpha\in\mathrm{cl}(\mu)$, then $F(x_\alpha)\in F(\mathrm{cl}(\mu))$. Further let $\lambda\in\mathrm{HC}\,R_{F(x_\alpha)}$, so $F^{-1}(\lambda)\in R_{x_\alpha}$ by (i). Since $x_\alpha\in\mathrm{cl}(\mu)$ and $F^{-1}(\lambda)\in R_{x_\alpha}$, so $\mu\nleqslant F^{-1}(\lambda)$. Since F is onto, so $F(\mu)>FF^{-1}(\lambda)=\lambda$. Thus $F(\mu)\nleqslant\lambda$ and $\lambda\in\mathrm{HC}\,R_{F(x_\alpha)}$. So $F(x_\alpha)\in\mathrm{HC}\cdot\mathrm{cl}(F(\mu))$. Thus $F(\mathrm{cl}(\mu))\leqslant\mathrm{HC}\cdot\mathrm{cl}(F(\mu))$.

(ii) $\Longrightarrow$ (iii): Let $\eta\in L^Y$. Then $F^{-1}(\eta)\in L^X$. By (ii) we have $F(\mathrm{cl}(F^{-1}(\eta)))\leqslant \mathrm{HC}\cdot\mathrm{cl}(FF^{-1}(\eta))\leqslant\mathrm{HC}\cdot\mathrm{cl}(\eta)$. Then $F(\mathrm{cl}(F^{-1}(\eta)))\leqslant\mathrm{HC}\cdot\mathrm{cl}(\eta)$ and so $F^{-1}F(\mathrm{cl}(F^{-1}(\eta)))\leqslant F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta))$, which implies that $\mathrm{cl}(F^{-1}(\eta))\leqslant F^{-1}F(\mathrm{cl}(F^{-1}(\eta)))\leqslant F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta))$. Thus $\mathrm{cl}(F^{-1}(\eta))\leqslant F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta))$.

(iii) $\Longrightarrow$ (iv): Let $\eta\in L^Y$, then $\mathrm{cl}(F^{-1}(\eta'))\leqslant F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta'))$ by (iii). Since $\mathrm{cl}(F^{-1}(\eta'))=(\mathrm{int}(F^{-1}(\eta)))'$ and $F^{-1}(\mathrm{HC}\cdot\mathrm{cl}(\eta'))=(F^{-1}(\mathrm{HC}\cdot\mathrm{int}(\eta)))'$, so $(\mathrm{int}(F^{-1}(\eta)))'\leqslant(F^{-1}(\mathrm{HC}\cdot\mathrm{int}(\eta)))'$ and taking the complement, $\mathrm{int}(F^{-1}(\eta))\geqslant F^{-1}(\mathrm{HC}\cdot\mathrm{int}(\eta))$.

(iv) $\Longrightarrow$ (v): Let $\varrho\in L^Y$ be an HC-open set. By (iv), $F^{-1}(\mathrm{HC}\cdot\mathrm{int}(\varrho))\leqslant\mathrm{int}(F^{-1}(\varrho))$, so $F^{-1}(\varrho)\leqslant\mathrm{int}(F^{-1}(\varrho))$. Thus $F^{-1}(\varrho)\in\tau$.

(v) $\Longrightarrow$ (vi): Let $\lambda\in L^Y$ be an HC-closed set. By (v), $F^{-1}(\lambda')\in\tau$. Then $(F^{-1}(\lambda))'=F^{-1}(\lambda')\in\tau$. So $F^{-1}(\lambda)\in\tau'$.

(vi) $\Longrightarrow$ (i): Let η be an almost N-compact closed set in L^Y. So by Theorem 3.4 (ii) we obtain that η is an HC-closed set in L^Y. By (vi), $F^{-1}(\eta)\in\tau'$. Hence F is HL-continuous. □

Theorem 5.4. *Suppose the mapping $F\colon(L^X,\tau)\to(L^Y,\Delta)$ from an L-ts (L^X,τ) into an* LT$_2$-*space (L^Y,Δ) is L-valued Zadeh* HL-*continuous. Then the L-valued Zadeh mapping $F|^{F(X)}\colon(L^X,\tau)\to(L^{F(X)},\Delta_{F(X)})$ is also* HL-*continuous.*

Proof. It is similar to that of Theorem 3.8 in [4]. □

Theorem 5.5. *If $F\colon(L^X,\tau)\to(L^Y,\Delta)$ is an L-valued Zadeh* HL-*continuous mapping and $A\subseteq X$, then the L-valued Zadeh mapping $F|_A\colon(L^A,\tau_A)\to(L^Y,\Delta)$ is* HL-*continuous.*

Proof. Let $\eta\in L^Y$ be an almost N-compact and closed. Since F is HL-continuous, so $F^{-1}(\eta)\in\tau'$ and $(F|_A)^{-1}(\eta)=F^{-1}(\eta)\wedge 1_A\in\tau'_A$. Hence $F|_A\colon(L^A,\tau_A)\to(L^Y,\Delta)$ is HL-continuous. □

It is easy to show that the composition of two HL-continuous mappings need not be HL-continuous. However, we have the following result.

Theorem 5.6. *If $F\colon (L^X,\tau_1) \to (L^Y,\tau_2)$ is L-valued Zadeh continuous and $G\colon (L^Y,\tau_2) \to (L^Z,\tau_3)$ is L-valued Zadeh* HL*-continuous, then the L-valued Zadeh mapping $G\circ F\colon (L^X,\tau_1) \to (L^Z,\tau_3)$ is* HL*-continuous.*

P r o o f. Straighforward. □

Theorem 5.7. *If (L^X,τ) and (L^Y,Δ) are L-ts's and $1_X = 1_A \vee 1_B$, where 1_A and 1_B are closed sets in L^X and $F\colon (L^X,\tau) \to (L^Y,\Delta)$ is an L-valued Zadeh mapping such that $F|_A$ and $F|_B$ are* HL*-continuous, then F is* HL*-continuous.*

P r o o f. Let $1_A, 1_B \in \tau'$. Let $\mu \in L^Y$ be an almost N-compact and closed. Then $(F|_A)^{-1}(\mu) \vee (F|_B)^{-1}(\mu) = (F^{-1}(\mu) \wedge 1_A) \vee (F^{-1}(\mu) \wedge 1_B) = F^{-1}(\mu) \wedge (1_A \vee 1_B) = F^{-1}(\mu) \wedge 1_X = F^{-1}(\mu)$. Hence $F^{-1}(\mu) = (F|_A)^{-1}(\mu) \vee (F|_B)^{-1}(\mu) \in \tau'$. So $F\colon (L^X,\tau) \to (L^Y,\Delta)$ is HL-continuous. □

Theorem 5.8. *If $F\colon (L^X,\tau) \to (L^Y,\Delta)$ is an injective L-valued Zadeh* HL*-continuous mapping and (L^Y,Δ) is an N-compact* LT_1*-space* [8]*, then (L^X,τ) is an* LT_1*-space.*

P r o o f. Let $x_\alpha, y_\beta \in M(L^X)$ be such that $x \neq y$. Since F is injective, so $F(x_\alpha)$ and $F(y_\beta)$ are in $M(L^Y)$ with $F(x) \neq F(y)$. Since (L^Y,Δ) is an LT_1-space, so $F(x_\alpha)$ and $F(y_\beta)$ are closed sets in (L^Y,Δ). Also, since (L^Y,Δ) is N-compact, so $F(x_\alpha)$ and $F(y_\beta)$ are N-compact and closed sets, hence $F(x_\alpha)$ and $F(y_\beta)$ are almost N-compact and closed sets. Now, since F is HL-continuous, so $F^{-1}F(x_\alpha) = x_\alpha$ and $F^{-1}F(y_\beta) = y_\beta$ are closed in (L^X,τ). Hence (L^X,τ) is an LT_1-space. □

Theorem 5.9. *Let $F\colon (L^X,\tau) \to (L^Y,\Delta)$ be an L-valued Zadeh mapping. Then the following conditions are equivalent:*

(i) *F is* HL*-continuous.*

(ii) *For each $x_\alpha \in M(L^X)$ and each L-net S in L^X, $F(S)\xrightarrow{\mathrm{HC}}F(x_\alpha)$ if $S \to x_\alpha$ and F is onto.*

(iii) *$F(\lim(S)) \leqslant \mathrm{HC}\cdot\lim(F(S))$, for each L-net S in L^X.*

P r o o f. (i) $\Longrightarrow$ (ii): Let $x_\alpha \in M(L^X)$ and let $S = \{x^n_{\alpha_n}; n \in D\}$ be an L-net in L^X which converges to x_α. Let $\eta \in \mathrm{HC}\,R_{F(x_\alpha)}$, then by (i), $F^{-1}(\eta) \in R_{x_\alpha}$. Since $S \to x_\alpha$, there exists $n \in D$ such that for each $m \in D$ and $m \geqslant n$, $S(m) \notin F^{-1}(\eta)$. Then $F(S(m)) \notin FF^{-1}(\eta) = \eta$, thus $F(S(m)) \notin \eta$. Hence $F(S)\xrightarrow{\mathrm{HC}}F(x_\alpha)$.

(ii) $\Longrightarrow$ (iii): Let $x_\alpha \in \mathrm{HC}\cdot\lim(S)$, then $F(x_\alpha) \in F(\mathrm{HC}\cdot\lim(S))$ and by (ii) also $F(x_\alpha) \in \mathrm{HC}\cdot\lim(F(S))$. Thus $F(\mathrm{HC}\cdot\lim(S)) \leqslant \mathrm{HC}\cdot\lim(F(S))$.

(iii) $\Longrightarrow$ (i): Let $\eta \in L^Y$ be HC-closed and let $x_\alpha \in M(L^X)$ with $x_\alpha \in \mathrm{cl}(F^{-1}(\eta))$. Then by Theorem 2.8 in [14], there exists an L-net S in $F^{-1}(\eta)$ which converges to x_α. Since $x_\alpha \in \lim(S)$, hence $F(x_\alpha) \in F(\lim(S))$. By (iii), $F(x_\alpha) \in F(\lim(S)) \leqslant \mathrm{HC}\cdot\lim(F(S))$ and so $F(S) \xrightarrow{\mathrm{HC}} F(x_\alpha)$. Since S is an L-net in $F^{-1}(\eta)$, we have $S(n) \in F^{-1}(\eta)$ for each $n \in D$. Thus $F(S(n)) \in FF^{-1}(\eta) \leqslant \eta$. So $F(S(n)) \in \eta$ for each $n \in D$. Hence $F(S)$ is an L-net in η. Since $F(S) \xrightarrow{\mathrm{HC}} F(x_\alpha)$ and $F(S)$ is an L-net in η, so by Theorem 4.4, $F(x_\alpha) \in \mathrm{HC}\cdot\mathrm{cl}(\eta)$. But since η is HC-closed, so $\eta = \mathrm{HC}\cdot\mathrm{cl}(\eta)$. Thus $F(x_\alpha) \in \eta$. Hence $x_\alpha \in F^{-1}(\eta)$. So $\mathrm{cl}(F^{-1}(\eta)) \leqslant F^{-1}(\eta)$. Hence $F^{-1}(\eta) \in \tau'$. Consequently, F is HL-continuous. □

Theorem 5.10. *Let $F\colon (L^X, \tau) \to (L^Y, \Delta)$ be an L-valued Zadeh mapping. Then the following conditions are equivalent:*

(i) *F is* HL-*continuous.*

(ii) *For each $x_\alpha \in M(L^X)$ and each L-ideal $\mathcal{L}$ which converges to x_α in L^X, $F^*(\mathcal{L})$* HC-*converges to $F(x_\alpha)$.*

(iii) *$F(\lim(\mathcal{L})) \leqslant \mathrm{HC}\cdot\lim(F^*(\mathcal{L}))$ for each L-ideal $\mathcal{L}$ in L^X.*

Proof. Follows directly from Theorems 4.20 and 5.9. □

6. Comparison of L-valued Zadeh mappings

Definition 6.1. An L-valued Zadeh mapping $F\colon (L^X, \tau) \to (L^Y, \Delta)$ is said to be:

(i) almost L-continuous iff $F^{-1}(\eta) \in \tau'$ for each regular closed set $\eta \in L^Y$,

(ii) CL-continuous iff $F^{-1}(\eta) \in \tau'$ for each N-compact and closed set $\eta \in L^Y$.

Theorem 6.2. *Every* HL-*continuous mapping is* CL-*continuous. The converse is true if the codomain of the mapping is an LR_2-space.*

Proof. Let $F\colon (L^X, \tau) \to (L^Y, \Delta)$ be L-valued Zadeh HC-continuous and let η in L^Y be an N-compact and closed set. Since every N-compact set is almost N-compact, hence η is almost N-compact and closed. By HL-continuity of F we have $F^{-1}(\eta) \in \tau'$. So F is CL-continuous. Conversely; let $F\colon (L^X, \tau) \to (L^Y, \Delta)$ be L-valued Zadeh CL-continuous and let (L^Y, Δ) be an LR_2-space. Let $\eta \in L^Y$ be an almost N-compact closed set, then by Theorem 3.10 in [6] η is N-compact closed. By CL-continuity of F we have $F^{-1}(\eta) \in \tau'$. So F is an HL-continuous mapping. □

Theorem 6.3. *Every L-continuous mapping is* HL-*continuous.*

Proof. Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be an L-valued Zadeh L-continuous mapping and $\eta\in L^Y$ an almost N-compact closed set. Then $\eta\in\Delta'$, so by L-continuity of F we have $F^{-1}(\eta)\in\tau'$. Thus F is HL-continuous. $\square$

The following example shows that not every HL-continuous mapping is L-continuous.

Example 6.4. If $L=[0,1]$, then the mapping defined in Example 3.6 in [4] is HL-continuous but not L-continuous.

Theorem 6.5. *If $F\colon (L^X,\tau)\to(L^Y,\Delta)$ is an L-valued Zadeh almost L-continuous, bijective mapping and (L^Y,Δ) is a fully stratified* LT_2-*space, then $F^{-1}\colon (L^Y,\Delta)\to(L^X,\tau)$ is* HL-*continuous.*

Proof. Let $\mu\in L^X$ be an almost N-compact and closed set. Since F is almost L-continuous so by Theorem 4.2 in [6], $F(\mu)$ is almost N-compact in (L^Y,Δ). Also, since (L^Y,Δ) is a fully stratified LT_2-space, so $F(\mu)\in\Delta'$. Thus $F(\mu)$ is almost N-compact closed and $(F^{-1})^{-1}(\mu)=F(\mu)\in\Delta'$. Hence $F^{-1}\colon (L^Y,\Delta)\to(L^X,\tau)$ is HL-continuous. $\square$

The following theorem shows that under some reasonable conditions HL-continuity and L-continuity are equivalent.

Theorem 6.6. *Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be L-valued Zadeh* HL-*continuous and let (L^Y,Δ) be a fully stratified* LT_2-*space. If $F(1_X)$ is an L-fuzzy set of an almost N-compact set of L^Y, then F is L-continuous.*

Proof. Let $\lambda\in\Delta'$ and let $\eta\in L^Y$ be an almost N-compact set containing $F(1_X)$. Since $\eta\in L^Y$ is almost N-compact and (L^Y,Δ) is a fully stratified LT_2-space, so $\eta\in\Delta'$. Thus $\eta\wedge\lambda\in\Delta'$. Hence by Theorem 2.5 (ii), $\eta\wedge\lambda$ is almost N-compact. Thus $\eta\wedge\lambda\in L^Y$ is an almost N-compact and closed set. Since F is HL-continuous, we have $F^{-1}(\eta\wedge\lambda)\in\tau'$. But $F^{-1}(\eta\wedge\lambda)=F^{-1}(\eta)\wedge F^{-1}(\lambda)=1_X\wedge F^{-1}(\lambda)=F^{-1}(\lambda)$, so $F^{-1}(\lambda)\in\tau'$. Hence F is L-continuous. $\square$

Corollary 6.7. *Let (L^X,τ) be an almost N-compact space and (L^Y,Δ) a fully stratified* LT_2-*space. If $F\colon (L^X,\tau)\to(L^Y,\Delta)$ is a bijective L-valued Zadeh L-continuous mapping, then F is an L-homeomorphism* [7].

Proof. By Theorem 6.5, F^{-1} is HL-continuous and by Theorem 6.6, F^{-1} is L-continuous. $\square$

Theorem 6.8. *For an L-valued Zadeh mapping $F\colon (L^X,\tau)\to(L^Y,\Delta)$ the following assertions hold:*

(i) *$F\colon (L^X,\tau)\to(L^Y,\Delta)$ is* HL*-continuous iff $F^*\colon (L^X,\tau)\to(L^Y,\Delta_{\mathrm{HC}})$ is L-continuous.*

(ii) *$F\colon (L^X,\tau)\to(L^Y,\Delta)$ is* CL*-continuous iff $F^*\colon (L^X,\tau)\to(L^Y,\Delta_{\mathrm{NC}})$ is L-continuous.*

(iii) *The identity mappings $I_Y\colon (L^Y,\Delta)\to(L^Y,\Delta_{\mathrm{HC}})$ and ${I_Y}^*\colon (L^Y,\Delta_{\mathrm{HC}})\to(L^Y,\Delta_{\mathrm{NC}})$ are L-continuous.*

(iv) *$I_Y^{-1}\colon (L^Y,\Delta_{\mathrm{HC}})\to(L^Y,\Delta)$ is* HL*-continuous and ${I_Y}^{*-1}\colon (L^Y,\Delta_{\mathrm{NC}})\to(L^Y,\Delta_{\mathrm{HC}})$ is* CL*-continuous.*

Proof. Straightforward. □

Theorem 6.9. *Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be an L-valued Zadeh* HL*-continuous mapping. If $F^*\colon (L^X,\tau)\to(L^Y,\Delta_{\mathrm{HC}})$ is an L-closed (L-open) mapping, then so is F.*

Proof. Let μ be a closed set in (L^X,τ). By hypothesis, $F^*(\mu)$ is a closed set in $(L^Y,\Delta_{\mathrm{HC}})$. By Theorem 6.8 (iii), the identity map $I_Y\colon (L^Y,\Delta)\to(L^Y,\Delta_{\mathrm{HC}})$ is L-continuous, so $I_Y^{-1}(F^*(\mu))$ is a closed set in (L^Y,Δ). But $I_Y^{-1}\circ F^*=F$, so $I_Y^{-1}(F^*(\mu))=F(\mu)$ is a closed set in (L^Y,Δ). Thus F is an L-closed mapping. The proof for the case in the parentheses is similar. □

Corollary 6.10. *If $F\colon (L^X,\tau)\to(L^Y,\Delta)$ is a bijective L-valued Zadeh* HL*-continuous mapping and $F^*\colon (L^X,\tau)\to(L^Y,\Delta_{\mathrm{HC}})$ is an L-valued Zadeh L-closed (or L-open) mapping, then F^{-1} is L-continuous.*

Proof. Let F^* be a L-closed (L-open) mapping and μ a closed (open) set in (L^X,τ). Then by Theorem 6.9, F is a L-closed (open) mapping, so $F(\mu)$ is a closed (open) set in (L^Y,Δ). But $F(\mu)=(F^{-1})^{-1}(\mu)$. Thus F^{-1} is L-continuous. □

Theorem 6.11. *Let (L^X,τ) be an L-ts. If (L^X,τ_{HC}) is an* LT_2*-space, then (L^X,τ) is an almost N-compact space.*

Proof. Let $\Phi=\{\eta_j\colon j\in J\}\subset\tau'$ be an α-RF of 1_X. Since (L^X,τ_{HC}) is an LT_2-space and ${\tau_{\mathrm{HC}}}'\subset\tau'$, there exist almost N-compact closed sets μ and λ with $\mu\vee\lambda=1_X$. Since μ and λ are almost N-compact sets, there exist $\Phi_k=\{\eta_{j_k}\colon k=1,2,\dots,n\}\in 2^{(\Phi)}$ and $\Phi_h=\{\eta_{j_h}\colon h=1,2,\dots,m\}\in 2^{(\Phi)}$ with Φ_k and Φ_h are almost $\overline{\alpha}$-RF of μ and λ, respectively. Thus for each $x_{\gamma_1}\in\mu$ there exists $\eta_{j_k}\in\Phi_k$ with $\eta_{j_k}\in R_{x_{\gamma_1}}$ and also for each $x_{\gamma_2}\in\lambda$ there exists $\eta_{j_h}\in\Phi_h$ with $\eta_{j_h}\in R_{x_{\gamma_2}}$, where $\gamma_1,\gamma_2\in\beta^*(\alpha)$. Now, since $\Phi_k\vee\Phi_h\in 2^{(\Phi)}$, so for each $x_{(\gamma_1\vee\gamma_2)}\in\mu\vee\lambda=1_X$ there exists $\eta_{j_l}\in(\Phi_k\vee\Phi_h)$ with $\eta_{j_l}\in R_{(x_{\gamma_1\vee\gamma_2})}$. Hence (L^X,τ) is an almost N-compact space. □

Theorem 6.12. *Let $F\colon (L^X,\tau)\to(L^Y,\Delta)$ be an L-valued Zadeh* HL*-continuous mapping. If $(L^Y,\Delta_{\mathrm{HC}})$ is a fully stratified* LT_2*-space, then F is L-continuous.*

Proof. Follows from Theorems 6.6 and 6.11. □

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