OPERATORS ON LORENTZ SEQUENCE SPACES

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Abstract. Description of multiplication operators generated by a sequence and composition operators induced by a partition on Lorentz sequence spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ is presented.

Keywords: composition operator, distribution function, Fredholm operator, Lorentz space, Lorentz sequence space, multiplication operator, non-increasing rearrangement

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1. Introduction

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X,\mathcal{A},\mu)$. For $s \geqslant 0$, define the *distribution function* μ_f of f as

$$\mu_f(s) = \mu\{x \in X\colon\ |f(x)| > s\}.$$

By f^* we mean the *non-increasing rearrangement* of f given as

$$f^*(t) = \inf\{s > 0\colon\ \mu_f(s) \leqslant t\}, \quad t \geqslant 0.$$

The *Lorentz space* $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is the set of all complex-valued measurable functions f on X such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left\{ \dfrac{q}{p} \displaystyle\int_0^\infty (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 1 < p \leqslant \infty,\ q = \infty. \end{cases}$$

$L(p,q)$ spaces are linear spaces and $\|\cdot\|^*_{pq}$ is a quasi-norm which is a norm for $1 \leqslant q < p < \infty$. For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s.$$

Now the functional defined as

$$\|f\|_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{\mathrm{d}t}{t}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^{**}(t), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$

is equivalent to $\|\cdot\|^*_{pq}$ and $L(p,q)$ is a normed linear space with respect to $\|\cdot\|_{pq}$. The $L(p,q)$ space is moreover a Banach space. The L^p-spaces for $1 < p \leqslant \infty$ are equivalent to the spaces $L(p,p)$. For more details on Lorentz spaces one can refer to [2], [7] and [8] and references therein. For $X = \mathbb{N}$ with $\mathcal{A} = 2^{\mathbb{N}}$, the power set of X, and $\mu =$ counting measure, the distribution function of any complex-valued function $a = \{a(n)\}_{n\geqslant 1}$ can be written as

$$\mu_a(s) = \mu\{n \in \mathbb{N}\colon\ |a(n)| > s\}, \quad s \geqslant 0.$$

The *non-increasing rearrangement* a^* of a is given as

$$a^*(t) = \inf\{s > 0\colon\ \mu_a(s) \leqslant t\}, \quad t \geqslant 0.$$

We can interpret the non-increasing rearrangement of a with $\mu_a(s) < \infty$, $s > 0$, as a sequence $\{a^*(n)\}$ if we define for $n-1 \leqslant t < n$

$$a^*(n) = a^*(t) = \inf\{s > 0\colon\ \mu_a(s) \leqslant n-1\}.$$

Then the sequence $a^* = \{a^*(n)\}$ is obtained by permuting $\{|a(n)|\}_{n\in S}$, $S = \{n\colon\ a(n) \neq o\}$, in the decreasing order with $a^*(n) = 0$ for $n > \mu(S)$ if $\mu(S) < \infty$.

The *Lorentz sequence space* $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is the set of all complex sequences $a = \{a(n)\}$ such that $\|a\|_{(p,q)} < \infty$, where

$$\|a\|_{(p,q)} = \begin{cases} \left\{\displaystyle\sum_{n=1}^\infty (n^{1/p} a^*(n))^q n^{-1}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{n\geqslant 1} n^{1/p} a^*(n), & 1 < p \leqslant \infty,\ q = \infty. \end{cases}$$

The *Lorentz sequence space* $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is a linear space and $\|\cdot\|_{(p,q)}$ is a quasi-norm. Moreover, $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is complete with respect to the quasi-norm $\|\cdot\|_{(p,q)}$ and $l(p,q)$, $1 \leqslant q \leqslant p < \infty$ is a complete normed linear space with respect to $\|\cdot\|_{(p,q)}$. Throughout this paper we consider the spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, with respect to $\|\cdot\|_{(p,q)}$. Such spaces $l(p,q)$ fall in the category of $L(p,q)$ spaces [8] as well as in the category of functional Banach spaces [7]. The l^p-spaces for $1 < p \leqslant \infty$ are equivalent to the spaces $l(p,p)$. In [7], [9], a description of the duals, isomorphic l^p-subspaces of *Orlicz-Lorentz sequence spaces* $L_{\varphi,w}$ is given and in [12] isomorphic properties of Orlicz-Lorentz sequence spaces are discussed.

The *Lorentz sequence space* $l(p,q)$ coincides with $L_{\varphi,w}$ when $\varphi(t) = t^q$ and the weight sequence is $w(n) = n^{(q/p)-1}$. In the case of the Lorentz sequence space $l(p,q)$ one can have a better feeling of the behavior of multiplication, composition operators and the inducing sequences while in the case of the abstract Lorentz space $L(p,q)$ as well as the Banach function spaces [6] it becomes difficult. Multiplication and composition operators are studied in various function spaces in [1], [3], [5], [6], [13] and [14]. In [15], Singh studied these operators on the weak Lebesgue space l^p.

Let $u = \{u(n)\}$ be a complex sequence. We define a linear transformation M_u on the Lorentz sequence space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, into the linear space of all complex sequences by

$$M_u(a) = ua = \{u(n)a(n)\}, \text{ where } a = \{a(n)\}.$$

If M_u is bounded with range in $l(p,q)$, then it is called a multiplication operator on $l(p,q)$. For a mapping $T\colon \mathbb{N} \to \mathbb{N}$ we define a linear transformation C_T on the Lorentz sequence space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, into the linear space of all complex sequences by

$$C_T(a) = a \circ T = \{a(T(n))\}, \text{ where } a = \{a(n)\}.$$

If C_T is bounded with range in $l(p,q)$, then it is called a composition operator on $l(p,q)$. By $\mathcal{B}(l(p,q))$ we mean the algebra of all bounded linear operators on $l(p,q)$. An operator $A \in \mathcal{B}(l(p,q))$ is said to be *Fredholm* if it has closed range, $\dim(\mathrm{Ker}(A))$ and $\operatorname{codim}(R(A))$ are finite, where $\dim(\mathrm{Ker}(A))$ is the dimension of the kernel of A and $\operatorname{codim}(R(A))$ is the co-dimension of the range of A, namely the dimension of any subspace complementary to the range of A.

In this paper we are interested in the study of compactness, Fredholmness, invertibility etc. of multiplication and composition operators on the Lorentz sequence spaces $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. It is shown in this paper that there exists a plenty of compact multiplication operators on $l(p,q)$. Multiplication and composition operators having closed ranges are also characterized.

2. Characterizations: Multiplication operators

The section is devoted to the study of multiplication operators M_u on the space $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, induced by a sequence $u = \{u(n)\}$. It follows immediately from [6] Theorem 2.4 that the only compact multiplication operator on the non-atomic Lorentz space is the zero operator. In the case of the Lorentz sequence space we show the existence of plenty of compact non-zero multiplication operators on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and compact multiplication operators are characterized.

Theorem 2.1. *Let $u = \{u(n)\}$ be a complex sequence. Then M_u induced by u is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, if and only if $\{u(n)\}$ is bounded.*

Proof. If M_u is a bounded operator, then there exists $K > 0$ such that

$$\|M_u a\|_{(p,q)} \leqslant K\|a\|_{(p,q)} \text{ for all } a = \{a(n)\} \in l(p,q).$$

For each $n \in \mathbb{N}$ and $e_n = \{e_n(m)\}_m$ in $l(p,q)$, where

$$e_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad e_n^*(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\|e_n\|_{(p,q)} = 1$ and so

$$\|M_u e_n\|_{(p,q)}^q \leqslant K^q \|e_n\|_{(p,q)}^q.$$

This gives, for $1 < p < \infty$, $1 \leqslant q < \infty$,

$$\sum_{m=1}^{\infty} ((ue_n)^*(m))^q m^{(q/p)-1} \leqslant K^q \sum_{m=1}^{\infty} (e_n^*(m))^q m^{(q/p)-1}$$
$$\Rightarrow (ue_n)^*(1) \leqslant K e_n^*(1), \text{ that is, } |u(n)| \leqslant K,$$

and for $q = \infty$, $1 < p \leqslant \infty$,

$$\sup_{m \geqslant 1} m^{1/p}((ue_n)^*(m)) \leqslant K \sup_{m \geqslant 1} m^{1/p}(e_n^*(m))$$
$$\Rightarrow (ue_n)^*(1) \leqslant K e_n^*(1), \text{ that is, } |u(n)| \leqslant K.$$

Thus in any case $\{u(n)\}$ is a bounded sequence.

Conversely, if $u = \{u(n)\}$ satisfies $|u(n)| \leqslant K$ for all $n \in \mathbb{N}$ and some $K > 0$, then for any $a = \{a(n)\}$ in $l(p,q)$, $ua = \{u(n)a(n)\}$ satisfies

$$|u(n)a(n)| \leqslant K|a(n)|.$$

This gives $(ua)^*(n) \leqslant Ka^*(n)$ for each $n \in \mathbb{N}$, and so we obtain

$$\|M_u a\|_{(p,q)} = \begin{cases} \left\{\sum_{n=1}^{\infty}((ua)^*(n))^q n^{(q/p)-1}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup_{n\geqslant 1} n^{1/p}(ua)^*(n), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$
$$\leqslant K\|a\|_{(p,q)}.$$

Thus M_u is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. □

Theorem 2.2. *Let $M_u \in \mathcal{B}(l(p,q))$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$. Then M_u is invertible if and only if there is $\delta > 0$ such that*

$$|u(n)| \geqslant \delta \quad \textit{for all } n \in \mathbb{N}.$$

Proof. If M_u is invertible then we find $\delta > 0$ satisfying

$$\|M_u a\|_{(p,q)} \geqslant \delta\|a\|_{(p,q)} \quad \text{for all } a \in l(p,q).$$

In particular, for $e_n = \{e_n(m)\}$ this gives $|u(n)| \geqslant \delta$.

Conversely, if $|u(n)| \geqslant \delta$ for all $n \in \mathbb{N}$ and some $\delta > 0$, then define another sequence $v = \{v(n)\}$ where $v(n) = 1/u(n)$. Clearly, in view of Theorem 2.1, M_v is bounded on $l(p,q)$ and $M_v = M_u^{-1}$. □

Theorem 2.3. *Let $M_u \in \mathcal{B}(l(p,q))$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then M_u has closed range if and only if for some $\delta > 0$,*

$$|u(n)| \geqslant \delta \quad \textit{for all } n \in S,$$

where $S = \{n \in \mathbb{N} \colon\ u(n) \neq 0\}$.

Proof. Suppose $|u(n)| \geqslant \delta$ for all $n \in S$ and some $\delta > 0$. We claim that $M_u|_{l_{pq}(S)}$ has closed range where

$$l_{pq}(S) = \{a = \{a(n)\} \in l(p,q) \colon\ a(n) = 0 \text{ for } n \in \mathbb{N} \setminus S\}.$$

Let $f, f_k \in l_{pq}(S)$ where $f = \{f(n)\}$ and for each $k \geqslant 1$, $f_k = \{f_k(n)\}$ are such that $M_u f_k \to f$ as $k \to \infty$. Then we have, as $n, m \to \infty$,

$$\|M_u f_n - M_u f_m\|_{(p,q)} \to 0.$$

Put $a_{nm} = f_n - f_m$, then for each $s > 0$,

$$\{k \in \mathbb{N} \colon\ |u(k)a_{nm}(k)| > s\} \supseteq \{k \in \mathbb{N} \colon\ |a_{nm}(k)| > s/\delta\}.$$

This gives $\delta a^*_{nm}(k) \leqslant (ua_{nm})^*(k)$ for each $k \in \mathbb{N}$. Therefore

$$\begin{aligned}
\|ua_{nm}\|_{(p,q)} &= \|M_u f_n - M_u f_m\|_{(p,q)} \\
&= \begin{cases} \left\{ \sum\limits_{k \in S} ((ua_{nm})^*(k))^q k^{(q/p)-1} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{k \in S} k^{1/p} (ua^*_{nm}(k), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&\geqslant \begin{cases} \left\{ \sum\limits_{k \in S} \delta^q ((a_{nm})^*(k))^q k^{(q/p)-1} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{k \in S} k^{1/p} \delta(a^*_{nm}(k), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&= \delta \|a_{nm}\|_{(p,q)}.
\end{aligned}$$

Since $\|ua_{nm}\|_{(p,q)} \to 0$ as $n, m \to \infty$, this implies $a_{nm} \to 0$ as $n, m \to \infty$. This means $\{f_k\}$ is a Cauchy sequence in $l_{pq}(S)$, which is a closed subspace of $l(p,q)$.

Hence we can find $g \in l_{pq}(S)$ such that $f_k \to g$ as $k \to \infty$. By virtue of the continuity of M_u, $M_u f_k \to M_u g$. Hence $f = M_u g$ and thus $M_u|_{l_{pq}(S)}$ has closed range. Since $\mathrm{Ker}(M_u) = l_{pq}(\mathbb{N} \setminus S)$, we find that M_u has closed range.

Conversely, if the condition does not hold, then for each $n \in \mathbb{N}$ we can find $k_n \in S$ satisfying

$$|u(k_n)| < 1/n.$$

For each n, the sequence $e_{k_n} = \{e_{k_n}(m)\}$, where

$$e_{k_n}(m) = \begin{cases} 1 & \text{if } m = k_n, \\ 0 & \text{otherwise}, \end{cases}$$

satisfies $\|e_{k_n}\|_{(p,q)} = 1$ and

$$\begin{aligned}
\|M_u e_{k_n}\|_{(p,q)} &= \|ue_{k_n}\|_{(p,q)} \\
&= \begin{cases} \left\{ \sum\limits_{m=1}^{\infty} ((ue_{k_n})^*(m))^q m^{(q/p)-1} \right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup\limits_{m \geqslant 1} m^{1/p} (ue_{k_n})^*(m), & 1 < p \leqslant \infty,\ q = \infty \end{cases} \\
&= (ue_{k_n})^*(1) = |u(k_n)| < \frac{1}{n} \|e_{k_n}\|_{(p,q)}.
\end{aligned}$$

Thus M_u is not bounded away from zero, a contradiction. Hence the result. $\square$

Theorem 2.4. *Let $M_u \in \mathcal{B}(l(p,q)), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. A necessary and sufficient condition for M_u to be compact is that $|u(n)| \to 0$ as $n \to \infty$.*

Proof. Suppose $u(n)$ does not tend to 0 as $n \to \infty$. Then $|u(n)| \geqslant \delta$ for infinitely many values of n and some $\delta > 0$. Let

$$A = \{n \in \mathbb{N}\colon\ |u(n)| \geqslant \delta\} \quad \text{and} \quad B = \{e_k = \{e_k(n)\}\colon\ k \in A\}.$$

Then B is a bounded set in $l(p,q)$. Moreover, for each $n, k, l \in A$,

$$|(ue_k - ue_l)(n)| \geqslant \delta|(e_k - e_l)(n)|$$

and so

$$(ue_k - ue_l)^*(n) \geqslant \delta(e_k - e_l)^*(n).$$

Thus

$$\|M_u e_k - M_u e_l\|_{(p,q)} \geqslant \delta\|e_k - e_l\|_{(p,q)}$$

or

$$\|M_u e_k - M_u e_l\|_{(p,q)} \geqslant \delta \quad \text{for } k \neq l,$$

which shows that M_u is not compact.

Conversely, if $u(n) \to 0$ as $n \to \infty$, we can find $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $|u(n)| < \delta$ for all $n \geqslant n_0$. For each $n \in \mathbb{N}$, define $u_n \equiv \{u_n(k)\}$, where

$$u_n(k) = \begin{cases} u(k) & \text{if } k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{u_n(k)\}$ is a bounded sequence so that M_{u_n} is bounded on $l(p,q)$. Moreover, each M_{u_n} is compact and one can check that $M_{u_n} \to M_u$ uniformly. This yields that M_u is compact. □

As one can easily find that if $\mathbb{N} \setminus S$ is a finite set then $\operatorname{Ker}(M_u)$ and range of M_u are subspaces generated by $\{e_n\colon\ n \in \mathbb{N} \setminus S\}$ and $\{e_m\colon\ m \in S\}$ respectively, we have

Theorem 2.5. *Let $M_u \in \mathcal{B}(l(p,q)), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then M_u is Fredholm if and only if $\mathbb{N} \setminus S$ is finite and there exists $\delta > 0$ such that*

$$|u(n)| \geqslant \delta \quad \text{for all } n \in \mathbb{N}.$$

3. Characterizations: Composition operators

In this section, isometric and Fredholm composition operators are characterized. The study of boundedness, compactness and closed range of composition operators on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is also included.

Theorem 3.1. *A mapping $T\colon \mathbb{N}\to\mathbb{N}$ induces a bounded composition operator*

$$C_T\colon\ a\mapsto a\circ T$$

on $l(p,q)$, $1<p\leqslant\infty$, $1\leqslant q\leqslant\infty$, if and only if there exists $M>0$ such that

$$\mu T^{-1}(\{n\})\leqslant M \text{ for all } n\in\mathbb{N}.$$

Proof. In case C_T is bounded, we have for some $R>0$

$$\|C_Ta\|_{(p,q)}\leqslant R\|a\|_{(p,q)} \text{ for all } a\in l(p,q).$$

Let $n\in\mathbb{N}$ be such that $T^{-1}(\{n\})$ is not empty.
Then $e_n=\{e_n(k)\}\in l(p,q)$ and hence

$$\|C_Te_n\|_{(p,q)}\leqslant R\|e_n\|_{(p,q)}=R,$$

that is,

$$\|e_{T^{-1}(\{n\})}\|_{(p,q)}\leqslant R.$$

However, $e_{T^{-1}(\{n\})}=\{e_{T^{-1}(\{n\})}(k)\}$ where

$$e_{T^{-1}(\{n\})}(k)=\begin{cases}1 & \text{if } k\in T^{-1}(\{n\}),\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$e^*_{T^{-1}(\{n\})}(k)=\begin{cases}1 & \text{if } k=1,2,\ldots,\mu T^{-1}(\{n\}),\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\begin{aligned}
R&\geqslant\|e_{T^{-1}(\{n\})}\|_{(p,q)}\\
&=\begin{cases}\left\{\displaystyle\sum_{k=1}^{\mu T^{-1}(\{n\})}k^{(q/p)-1}\right\}^{1/q}, & 1<p<\infty,\ 1\leqslant q<\infty,\\ \displaystyle\sup_{k=1,2,\ldots,\mu T^{-1}(\{n\})}k^{1/p}e^*_{T^{-1}(\{n\})}(k), & 1<p\leqslant\infty, q=\infty\end{cases}\\
&=\begin{cases}\left\{(1)+\Big(\dfrac{1}{2^{1-q/p}}\Big)+\ldots+\Big(\dfrac{1}{(\mu T^{-1}(\{n\}))^{1-q/p}}\Big)\right\}^{1/q}, & 1<p<\infty,\ 1\leqslant q<\infty,\\ (\mu T^{-1}(\{n\})), & 1<p\leqslant\infty,\ q=\infty\end{cases}\\
&\geqslant\begin{cases}\left\{(\mu T^{-1}(\{n\})\Big(\dfrac{1}{(\mu T^{-1}(\{n\}))^{1-q/p}}\Big)\right\}^{1/q}, & 1\leqslant q<p<\infty,\\ (\mu T^{-1}(\{n\})^{1/q}, & 1<p\leqslant q<\infty,\\ (\mu T^{-1}(\{n\})^{1/p}, & 1<p\leqslant\infty,\ q=\infty\end{cases}\\
&=\begin{cases}(\mu T^{-1}(\{n\})^{1/p}, & 1\leqslant q<p<\infty \text{ or } 1<p\leqslant\infty,\ q=\infty,\\ (\mu T^{-1}(\{n\})^{1/q}, & 1<p\leqslant q<\infty.\end{cases}
\end{aligned}$$

Hence in any case we can find $M > 0$ such that $\mu T^{-1}(\{n\}) \leqslant M$ for each $n \in \mathbb{N}$.

Conversely, if $\mu T^{-1}(\{n\}) \leqslant M$ for some $M \in \mathbb{N}$ then for any $a = \{a(n)\}$ in $l(p,q)$ and $a \circ T = \{(a \circ T)(n)\}$ we have for all $t > 0$

$$(a \circ T)^*(Mt) \leqslant a^*(t),$$

and so for all $k \in \mathbb{N} \cup \{0\}$ and $m = 1, 2, \ldots, M$ we have

$$(a \circ T)^*(kM + m) \leqslant a^*(k+1).$$

Hence, for $1 < p < \infty$, $1 \leqslant q < \infty$, taking $r = 1 - q/p$ we obtain

$$\begin{aligned}
&\|a \circ T\|^q_{(p,q)} \\
&\quad = \sum_{k=1}^{\infty} ((a \circ T)^*(k))^q k^{(q/p)-1} \\
&\quad = \Big[((a \circ T)^*(1))^q + ((a \circ T)^*(2))^q \frac{1}{2^r} + \ldots + ((a \circ T)^*(M))^q \frac{1}{M^r}\Big] \\
&\qquad + \Big[((a \circ T)^*(M+1))^q \frac{1}{(M+1)^r} + \ldots + ((a \circ T)^*(2M))^q \frac{1}{(2M)^r}\Big] + \ldots \\
&\quad \leqslant \Big[1 + \frac{1}{2^r} + \ldots + \frac{1}{M^r}\Big](a^*(1))^q + \Big[\frac{1}{(M+1)^r} + \ldots + \frac{1}{(2M)^r}\Big](a^*(2))^q \\
&\qquad + \Big[\frac{1}{(2M+1)^r} + \ldots + \frac{1}{(3M)^r}\Big](a^*(3))^q + \ldots \\
&\quad \leqslant \begin{cases} M\Big[(a^*(1))^q + \dfrac{1}{2^r}(a^*(2))^q + \dfrac{1}{3^r}(a^*(3))^q + \ldots\Big], & 1 \leqslant q < p < \infty, \\ M^{(1-r)}\Big[(a^*(1))^q + \dfrac{1}{2^r}(a^*(2))^q + \dfrac{1}{3^r}(a^*(3))^q + \ldots\Big], & 1 < p \leqslant q < \infty \end{cases} \\
&\quad = \begin{cases} M\|a\|^q_{(p,q)}, & 1 \leqslant q < p < \infty, \\ M^{q/p}\|a\|^q_{(p,q)}, & 1 < p \leqslant q < \infty \end{cases}
\end{aligned}$$

and for $q = \infty$, $1 < p \leqslant \infty$ we have

$$\|a \circ T\|^q_{(p,q)} \leqslant M\|a\|^q_{(p,q)}.$$

Thus C_T is bounded on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. □

Theorem 3.2. *Let C_T be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then the following conditions are equivalent:*
(1) *T is invertible,*
(2) *C_T is invertible,*
(3) *C_T is an isometry.*

Proof. The proofs of (1) $\Leftrightarrow$ (2) follow the lines of the proof given in [15] in the case of l^p, which is independent of any other result except Theorem 3.1. Here we just prove the equivalence of (1) and (3). In case (1) holds, then for every $E \subseteq \mathbb{N}$

$$\mu\{T^{-1}(E)\} = \mu(E).$$

Then for each $a = \{a(n)\}$ in $l(p,q)$ and $a \circ T = \{(a \circ T)(n)\}$ we have for all $s > 0$

$$\mu_{a\circ T}(s) = \mu_a(s) \Rightarrow (a \circ T)^*(n) = a^*(n) \quad \text{for all } n \in \mathbb{N}.$$

Hence $\|C_T\|_{(p,q)} = \|a\|_{(p,q)}$ so that C_T is an isometry.

Conversely, if C_T is an isometry, then for each $n \in \mathbb{N}$ we have

$$\|C_T e_n\|_{(p,q)} = \|e_n\|_{(p,q)} = 1.$$

This implies $\mu T^{-1}(\{n\}) = 1$. Thus $T^{-1}(\{n\})$ is a singleton for each $n \in \mathbb{N}$. Hence T is invertible. $\square$

Theorem 3.3. *Let C_T be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then C_T is Fredholm if and only if both $\{n \in \mathbb{N}\colon \mu T^{-1}(\{n\}) \geqslant 2\}$ and $\mathbb{N} \setminus T(\mathbb{N})$ are finite.*

Proof. Suppose C_T is Fredholm. If $E = \{n \in \mathbb{N}\colon \mu T^{-1}(\{n\}) \geqslant 2\}$ is not finite, then for each $k \in E$ let $n_k, m_k \in \mathbb{N}$ be such that $T(n_k) = T(m_k), n_k \neq m_k$. For each $k \in E$, define $f_k = \{f_k(m)\}$ where

$$f_k(m) = \begin{cases} 1 & \text{if } m = n_k, \\ -1 & \text{if } m = m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_k lies in $l(p,q)$ but not in range of C_T. Moreover, $\{f_k\colon k \in E\}$ being linearly independent implies $l(p,q) \setminus R(C_T)$ is infinite dimensional, a contradiction. Thus the set E must be finite. Similarly, $\mathbb{N} \setminus T(\mathbb{N})$ being an infinite set implies that $\mathrm{Ker}(C_T)$ is infinite dimensional, a contradiction.

The converse is easy to prove. Hence the result follows. $\square$

Along the lines of the proof carried out in [15] for l_p-spaces, we arrive at the following results:

(1) Let C_T be a bounded linear composition operator on $l(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. Then C_T has closed range but not a compact one.

(2) An operator A on $l(p,q), 1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, is a composition operator if and only if there exists a partition $\{P_n\}$ of $\mathbb{N}$ such that

$$A(e_n) = \sum_{m \in P_n} e_m.$$

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