POSITIVE SOLUTIONS OF THIRD ORDER DAMPED NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. We study solutions tending to nonzero constants for the third order differential equation with the damping term

$$(a_1(t)(a_2(t)x'(t))')' + q(t)x'(t) + r(t)f(x(\varphi(t))) = 0$$

in the case when the corresponding second order differential equation is oscillatory.

Keywords: third order differential equation, damping term, second order oscillatory equation, positive solution, asymptotic properties

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1. Introduction

The aim of this paper is to investigate the third order nonlinear damped differential equation with deviating argument

$$(a_1(t)(a_2(t)x'(t))')' + q(t)x'(t) + r(t)f(x(\varphi(t))) = 0. \tag{1.1}$$

The following assumptions will be assumed:

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hp$_1$) a_1, a_2, q, are continuous differentiable functions for $t \geqslant 0$, $a_i(t) > 0$, $i = 1, 2$ and

$$\inf_{[0,\infty)} q(t) = q_\infty > 0;$$

hp$_2$) r, φ are continuous functions for $t \geqslant 0$ such that $r(t) > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$;
hp$_3$) f is a continuous function satisfying $f(u)u > 0$ for $u \neq 0$.

In this paper we will restrict our attention to solutions x of (1.1) which are defined in a neighborhood of infinity and sup $\{|x(s)|:\ s > t\} > 0$ for any t from this neighborhood. As usual, a solution of (1.1) is said to be *oscillatory* if it has a sequence of zeros converging to infinity; otherwise it is said to be *nonoscillatory*.

When $q \equiv 0$, the oscillation of the two-term equation

$$(a_1(t)(a_2(t)x')')' + r(t)f(x(\varphi(t))) = 0 \tag{1.2}$$

has been considered in many papers, see, e.g., [2], [12], [13], [14] and references therein. Here oscillation means that any solution x of (1.2) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

When q is not identically zero, the special case

$$x''' + q(t)x'(t) + r(t)f(x(\varphi(t))) = 0 \tag{1.3}$$

has been recently investigated, especially as regards the possible types of nonoscillatory solutions, see, e.g., [3], [6], [9]. In these papers the study is accomplished by assuming that the corresponding second order linear equation

$$y'' + q(t)y = 0 \tag{1.4}$$

is nonoscillatory. This assumption plays a crucial role, because it is known that, in this case, (1.3) can be written in the disconjugate form, i.e. as a two-term equation, see [6].

For the more general equation (1.1) the oscillation and the asymptotic behavior of nonoscillatory solutions is examined in [1], [15], [16] by assuming that the second order equation

$$(a_1(t)y')' + \frac{q(t)}{a_2(t)}y = 0 \tag{1.5}$$

is nonoscillatory. This equation plays an analogous role to that (1.4) plays for (1.3), as the following lemma shows.

Lemma 1.1. *Assume* (1.5) *is nonoscillatory and let h be its solution, $h(t) > 0$ for $t \geqslant t_0$. Then we have for $t \geqslant t_0$*

$$(a_1(t)(a_2(t)x')')' + q(t)x' = \frac{1}{h(t)}\Big(a_1(t)h^2(t)\Big(\frac{a_2(t)}{h(t)}x'\Big)'\Big)'.$$

Proof. For the sake of simplicity we omit the independent variable t. Denote by L the operator

$$Ly \equiv (a_1y')' + \frac{q}{a_2}y.$$

Then

$$hL(hu) = h(a_1h'u + a_1hu')' + \frac{q}{a_2}h^2u = (a_1h')'hu + a_1hh'u' + h(a_1hu')' + \frac{q}{a_2}h^2u.$$

Since h is a solution of (1.5) and $(a_1h^2u')' = a_1hh'u' + (a_1hu')'h$, we obtain

$$hL(hu) = (a_1h')'hu + (a_1h^2u')' + \frac{q}{a_2}h^2u = (a_1h^2u')'.$$

Setting $u = a_2x'/h$ we get

$$h(a_1(a_2x')')' + hqx' = \Big(a_1h^2\Big(\frac{a_2}{h}x'\Big)'\Big)',$$

which gives the assertion. □

Hence, when equation (1.5) is nonoscillatory, equation (1.1) can be also written in the disconjugate form, i.e. without the damping term, and the existence of various types of nonoscillatory solutions of (1.1) can be obtained from results for the equation with the disconjugate operator, see e.g. [7], [12].

Very little is known when (1.4) or (1.5) is oscillatory. We refer only to [10] in which necessary and sufficient conditions for the oscillation of (1.3) are given, and to [4], [5] in which the asymptotic behavior of nonoscillatory solutions of (1.3) is examined.

Our aim here is to give a sufficient condition for the existence of solutions of (1.1) which tend to a nonzero constant, in the case when the corresponding second order differential equation (1.5) is oscillatory. This result extends to (1.1) a previous one given in [4].

2. Preliminaries

Let x be a solution of (1.1) and let us denote its quasiderivatives

$$x^{[1]}(t) = a_2(t)x'(t), \quad x^{[2]}(t) = a_1(t)(x^{[1]}(t))'.$$

The following lemmas will be useful.

Lemma 2.1. *Let x be a solution (for large t) of the integral equation*

$$x(t) = c - \int_t^\infty \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\varphi(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau \tag{2.1}$$

where $c \in \mathbb{R}$ and u,v are linearly independent solutions of (1.5) *with the Wronskian*

$$W = a_1(t)(u(t)v'(t) - u'(t)v(t)) \equiv 1.$$

Then x is a solution of (1.1) *satisfying*

$$\lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} x^{[i]}(t) = 0,\ i = 1, 2. \tag{2.2}$$

Proof. Let x be a solution of equation (2.1). By a standard calculation we obtain

$$x^{[1]}(t) = a_2(t)x'(t) = \int_t^\infty r(s) f(x(\varphi(s)))(u(s)v(t) - u(t)v(s))\,\mathrm{d}s.$$

Thus, differentiating and using the fact that u, v satisfy (1.5) with Wronskian $W \equiv 1$, we get

$$(a_2(t)x'(t))' = \int_t^\infty r(s) f(x(\varphi(s)))(u(s)v'(t) - u'(t)v(s))\,\mathrm{d}s,$$

and

$$x^{[2]}(t) = \int_t^\infty r(s) f(x(\varphi(s)))a_1(t)(u(s)v'(t) - u'(t)v(s))\,\mathrm{d}s. \tag{2.3}$$

Thus

$$(a_1(t)(a_2(t)x'(t))')' = -r(t) f(x(\varphi(t))) - q(t)x'(t),$$

i.e., we have the required assertion. □

Lemma 2.2. *Assume that all solutions y of* (1.5) *and their quasiderivatives defined as $y^{[1]}(t) = a_1(t)y'(t)$ are bounded for $t \geqslant 0$. Fix $T > t_0 \geqslant 0$, let $\bar\varphi$ be a continuous function such that $\bar\varphi(t) \geqslant t_0$ for $t \geqslant t_0$. Let u, v be two linearly independent solutions of equation* (1.5) *with the Wronskian $W \equiv 1$. Denote*

$$h(s, t) = u(s)v'(t) - u'(t)v(s) \tag{2.4}$$

and

$$G_x(t) = -\int_t^T \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\bar\varphi(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau, \tag{2.5}$$

where x is a continuous function on $[t_0,\infty)$ such that $r(\cdot)f(x(\bar\varphi(\cdot)))\in L^1[t_0,\infty)$. Then for $t\in I=[t_0,T]$ we have

$$q(t)\biggl(G_x(t)+\int_t^T q'(s)G_x(s)\,\mathrm{d}s\biggr)=\int_t^T r(s)f(x(\bar\varphi(s))(1-a_1(t)h(s,t))\,\mathrm{d}s$$
$$+\int_T^\infty r(s)f(x(\bar\varphi(s))\,(a_1(T)h(s,T)-a_1(t)h(s,t))\,\mathrm{d}s.$$

3. Main theorem

Our main result is the following.

Theorem 3.1. *Assume that all solutions y of* (1.5) *and their quasiderivatives defined as $y^{[1]}(t)=a_1(t)y'(t)$ are bounded for $t\geqslant 0$. If*

$$\int_0^\infty r(t)\,\mathrm{d}t<\infty,\qquad\int_0^\infty|q'(t)|\,\mathrm{d}t<\infty,\tag{3.1}$$

then for any $c\in\mathbb{R}\setminus\{0\}$ there exists a solution x of (1.1) *satisfying*

$$\lim_{t\to\infty}x(t)=c,\qquad\lim_{t\to\infty}x^{[i]}(t)=0,\ i=1,2.\tag{3.2}$$

Proof. We prove the existence of solutions of (1.1) satisfying (3.2) for $c=1$.

Let u and v be two linearly independent solutions of (1.5) with the Wronskian $W\equiv 1$ and consider the function h given by (2.4). Since all solutions of (1.5) and their quasiderivatives are bounded, there exists $M>0$ such that for any $(s,t)\in[0,\infty)\times[0,\infty)$

$$a_1(t)|h(s,t)|\leqslant M.$$

Put

$$\beta=\max_{1/2\leqslant u\leqslant 3/2}f(u)\tag{3.3}$$

and let k_1 be such that $1+2M<k_1/\beta$. Hence

$$2M\leqslant k_1/\beta,\quad 1+M\leqslant k_1/\beta.\tag{3.4}$$

Let t_0 be so large that

$$\frac{2k_1}{q_\infty}\int_{t_0}^\infty r(t)\,\mathrm{d}t\leqslant\frac12,\qquad\int_{t_0}^\infty|q'(s)|\,\mathrm{d}s<\frac12 q_\infty,\tag{3.5}$$

and choose $\bar{t} \geqslant t_0$ such that $\varphi(t) \geqslant t_0$ for $t \geqslant \bar{t}$. Define

$$\bar{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \geqslant \bar{t}, \\ \varphi(\bar{t}) & \text{if } t_0 \leqslant t \leqslant \bar{t}. \end{cases}$$

In the Fréchet space $C[t_0, \infty)$ of all continuous functions on $[t_0, \infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$, consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \Big\{ x \in C[t_0, \infty) \colon \ \frac{1}{2} \leqslant x(t) \leqslant \frac{3}{2} \Big\}.$$

Fix $T \in [t_0, \infty)$ and let $I = [t_0, T]$. For any $x \in \Omega$, consider the "cut" function given by (2.5). Applying Lemma 2.2, in view of (3.4) and (3.5) we obtain for $t \in I$

$$|G_x(t)| \leqslant \frac{1}{q_\infty} \Big(k_1 \int_t^\infty r(s)\,\mathrm{d}s + \frac{1}{2} q_\infty \max_{t \leqslant s \leqslant T} |G_x(s)| \Big),$$

or

$$\begin{aligned} |G_x(t)| &= \Big| \int_t^T \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\bar{\varphi}(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau \Big| \\ &\leqslant \max_{t \leqslant \sigma \leqslant T} |G_x(\sigma)| \ \leqslant \frac{2k_1}{q_\infty} \int_t^\infty r(s)\,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Hence, using the Cauchy criterion, the limit

$$\lim_{T \to \infty} \int_t^T \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\bar{\varphi}(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

exists and it is finite for any fixed t and

$$\Big| \int_t^\infty \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\bar{\varphi}(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau \Big| \leqslant \frac{2k_1}{q_\infty} \int_t^\infty r(s)\,\mathrm{d}s. \tag{3.7}$$

This fact means that the operator

$$\mathcal{T}(x)(t) = 1 - \int_t^\infty \frac{1}{a_2(\tau)} \int_\tau^\infty r(s) f(x(\bar{\varphi}(s)))(u(s)v(\tau) - u(\tau)v(s))\,\mathrm{d}s\,\mathrm{d}\tau$$

is well defined for any $x \in \Omega$. Clearly

$$\mathcal{T}'(x)(t) = \frac{1}{a_2(t)} \int_t^\infty r(s) f(x(\bar{\varphi}(s))(u(s)v(t) - u(t)v(s))\,\mathrm{d}s.$$

Hence, this together (3.7) yields

$$(3.8)\qquad |\mathcal{T}(x)(t)-1| = \left|\int_t^\infty \mathcal{T}'(x)(\tau)\,\mathrm{d}\tau\right| \leqslant \frac{2k_1}{q_\infty}\int_t^\infty r(\tau)\,\mathrm{d}\tau.$$

So, in view of (3.5), $\mathcal{T}$ maps Ω into itself. Moreover, for any $x \in \Omega$ we have

$$(3.9)\qquad |\mathcal{T}'(x)(t)| \leqslant \frac{2m^2\beta}{a_2(t)}\int_t^\infty r(s)\,\mathrm{d}s,$$

where $m = \sup\limits_{t\geqslant t_0}\{|u(t)|, |v(t)|\}$, and so $\mathcal{T}(\Omega)$ is relatively compact, i.e. $\mathcal{T}(\Omega)$ consists of functions which are equibounded and equicontinuous on every compact subinterval of $[t_0,\infty)$.

Now we prove the continuity of $\mathcal{T}$ on Ω. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence in Ω which uniformly converges on every compact subinterval of $[t_0,\infty)$ to $x \in \Omega$. Because $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\{\mathcal{T}(x_n)\}$ admits a subsequence, denoted again by $\{\mathcal{T}(x_n)\}$ for the sake of simplicity, which is convergent to $\overline{x} \in \Omega$. In virtue of (3.9), by the Lebesgue dominated convergence theorem, the sequence $\{G_{x_n}(t)\}$ pointwise converges to $G_x(t)$ on $I = [t_0, T]$, i.e.

$$(3.10)\qquad \lim_{n\to\infty} G_{x_n}(t) = G_x(t).$$

Choosing a sufficiently large T, in view of (3.8) we obtain

$$\begin{aligned}
|\mathcal{T}(x_n)(t)-\mathcal{T}(x)(t)| &= \left|\int_t^\infty \mathcal{T}'(x_n)(\tau)\,\mathrm{d}\tau - \int_t^\infty \mathcal{T}'(x)(\tau)\,\mathrm{d}\tau\right| \\
&\leqslant \left|\int_T^\infty \mathcal{T}'(x_n)(\tau)\,\mathrm{d}\tau\right| + \left|\int_T^\infty \mathcal{T}'(x)(\tau)\,\mathrm{d}\tau\right| + |G_{x_n}(t)-G_x(t)| \\
&\leqslant \frac{4k_1}{q_\infty}\int_T^\infty r(\tau)\,\mathrm{d}\tau + |G_{x_n}(t)-G_x(t)|.
\end{aligned}$$

Hence the sequence $\{\mathcal{T}(x_n)\}$ pointwise converges to $\mathcal{T}(x)$. In view of the uniqueness of the limit, $\mathcal{T}(x) = \overline{x}$ is the only cluster point of the compact sequence $\{\mathcal{T}(x_n)\}$, which means the continuity of $\mathcal{T}$ in Ω. Applying the Tychonov fixed point theorem, there exists a solution x of the integral equation

$$x(t) = \mathcal{T}(x)(t)$$

which, in view of Lemma 2.1, is a solution of (1.1) with the required properties. From (3.8) we get that $\lim\limits_{t\to\infty} x(t) = 0$. Moreover, from (3.9) we have $\lim x^{[1]}(t) = 0$ and in a similar way from (2.3) the conclusion for $\lim x^{[2]}(t) = 0$ follows. $\square$

The following result gives sufficient conditions for the boundedness of each solution of (1.5) together with its quasiderivative.

Lemma 3.1. *If*

$$\log q(t)\frac{a_1(t)}{a_2(t)} \text{ is of bounded variation on } [0,\infty), \tag{3.11}$$

then for each solution y of (1.5), *y and $y^{[1]}$ are bounded.*

P r o o f. Consider equation (1.5) in the form

$$y'' + \frac{a_1'(t)}{a_1(t)}y' + \frac{q(t)}{a_1(t)a_2(t)}y = 0.$$

Then for each solution y of (1.5) the function

$$E(t) = \frac{q(t)}{a_1(t)a_2(t)}y^2 + y'^2$$

satisfies

$$E'(t) \leqslant \frac{p'(t)}{p(t)}E(t), \quad p(t) = \frac{q(t)}{a_1(t)a_2(t)},$$

see [11, Theorem 3] and its proof. Hence we obtain

$$\frac{q(t)}{a_1(t)a_2(t)}y^2(t) + y'^2(t) \leqslant \overline{k}\frac{q(t)}{a_1(t)a_2(t)},$$

where $\overline{k}$ is a positive constant. Thus y is bounded and

$$(y^{[1]}(t))^2 \leqslant \overline{k}q(t)\frac{a_1(t)}{a_2(t)}.$$

In virtue of (3.11), the function qa_1/a_2 is bounded and the assertion follows. □

R e m a r k. Lemma 3.1 follows also from [8, Theorem 4] with minor changes.

From Theorem 3.1 and Lemma 3.1 we obtain the following.

Theorem 3.2. *If* (3.1) *holds and*

$$\log\frac{a_1(t)}{a_2(t)} \text{ is of bounded variation on } [0,\infty),$$

then for any $c \in \mathbb{R}\setminus\{0\}$ there exists a solution x of (1.1) *satisfying* (3.2).

P r o o f. In virtue of (3.1), we have

$$\int_0^\infty \frac{|q'(t)|}{q(t)}\,\mathrm{d}t \leqslant \frac{1}{q_\infty}\int_0^\infty |q'(t)|\,\mathrm{d}t$$

and so $\log q(t)$ is of bounded variation on $[0,\infty)$. Since

$$\log q(t)\frac{a_1(t)}{a_2(t)} = \log q(t) + \log\frac{a_1(t)}{a_2(t)},$$

applying Lemma 3.1 and Theorem 3.1 we get the assertion. □

E x a m p l e. Consider the equation

$$(a(t)(a(t)x'(t))')' + x'(t) + r(t)f(x(\varphi(t))) = 0,$$

where $\int_0^\infty r(s)\,\mathrm{d}s < \infty$. By Theorem 3.2 this equation has solutions satisfying (3.2) for any $c \neq 0$. If $a(t) \equiv 1$, we get the result from [10].

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