ON EXTENSIONS OF PRIMARY ALMOST TOTALLY PROJECTIVE ABELIAN GROUPS

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(Received October 18, 2006)

Abstract. Suppose G is a subgroup of the reduced abelian p-group A. The following two dual results are proved:

($*$) If A/G is countable and G is an almost totally projective group, then A is an almost totally projective group.

($**$) If G is countable and nice in A such that A/G is an almost totally projective group, then A is an almost totally projective group.

These results somewhat strengthen theorems due to Wallace (J. Algebra, 1971) and Hill (Comment. Math. Univ. Carol., 1995), respectively.

Keywords: totally projective group, almost totally projective group, countable group, extension

MSC 2000: 20K10, 20K25, 20K27, 20K40

I. Introduction and preliminaries

In his remarkable investigation [6], Hill introduced the class of the so-called *almost direct sums of cyclic p-groups* and established two results concerning them. Specifically, the following two theorems hold true:

Theorem (Hill, 1995). *If A is a separable abelian p-group which is the countable extension of an almost direct sum of cyclic groups, then A itself is an almost direct sum of cyclic groups.*

Theorem (Hill, 1995). *If A is a separable abelian p-group which is the extension of a countable group by an almost direct sum of cyclic groups, then A itself is an almost direct sum of cyclic groups.*

Note that the first assertion alluded to above is of the so-termed "type of Wallace" who was the first to obtain such a theorem for totally projective primary groups (see [9]).

On the other hand, Hill jointly with Ullery defined in [7] a generalization of the aforementioned class of separable groups to the class of the so-called *almost totally projective p-groups*. Their definition reads like this:

Definition (Hill and Ullery, 1996). A (reduced) p-primary abelian group H is said to be almost totally projective provided it possesses a collection $\mathcal{C}$ of nice subgroups of H which satisfies the following three conditions:

(0) $\{0\} \in \mathcal{C}$;

(1) $\mathcal{C}$ is closed with respect to unions of ascending chains, that is, if $H_i \in \mathcal{C}$ are such that $H_i \subseteq H_j$ whenever $i \leqslant j$ with $i, j \in I$, an index set, then $\bigcup_{i \in I} H_i \in \mathcal{C}$;

(2) if C is a countable subgroup of H, then there exists $B \in \mathcal{C}$ such that $C \subseteq B$ and B is countable.

Recall that if H is separable, then it is called an almost direct sum of cyclic groups.

It is the purpose of this paper to initiate a study of these major extensions of totally projective p-torsion groups by showing that the preceding two theorems of Hill remain valid for them. It is worthwhile noticing that similar investigations in this direction were done in [1]–[4] as well as our Theorem 1 stated and proved below was already established in ([8], Theorem 6) but we have not seen this article.

Any other terms or concepts which are not defined explicitly here can be found in [5].

II. Main results

We proceed by proving our main theorems. Before doing that, we need the following useful technicality on niceness, which can be of general interest as well, since our technique requires a set-theoretical gymnastic with intersections of certain groups. It will be necessary for establishing the property of niceness for some subgroups, although we can also successfully copy the idea from [9] which is based on homological methods and especially on commutative diagrams. Nevertheless, our scheme of proof of such a niceness is more global and "purely algebraic" than the cited one. Note also that the technique used by Hill in [6] cannot be used in our first theorem.

So, we are prepared to prove

Lemma. *Suppose G is an abelian p-group with a nice subgroup N and M is an abelian p-group such that $G \cap M \subseteq N$. Then $N + M$ is nice in $G + M$.*

P r o o f. First of all, we shall show by transfinite induction on the ordinal α that $p^\alpha(G+M) \subseteq p^\alpha G + M + N$ for every ordinal number α; actually we shall show a little more. We consider two cases.

C a s e 1: α is non-limit. Thus $p^\alpha(G+M) = p(p^{\alpha-1}(G+M)) \subseteq p(p^{\alpha-1}G + M + N) \subseteq p(p^{\alpha-1}G) + M + N = p^\alpha G + M + N$ by applying the induction hypothesis.

C a s e 2: α is limit. We have $p^\alpha(G+M) = \bigcap\limits_{\tau<\alpha} p^\tau(G+M) \subseteq \bigcap\limits_{\tau<\alpha} (p^\tau G + M + N)$ by the induction hypothesis. We take an arbitrary element x lying in the last intersection. Hence, we may write $x = g_{\tau_1} + m_1 + n_1 = g_{\tau_2} + m_2 + n_2 = \ldots$, where $g_{\tau_1} \in p^{\tau_1}G$, $m_1 \in M$, $n_1 \in N$; $g_{\tau_2} \in p^{\tau_2}G$, $m_2 \in M$, $n_2 \in N$; $\tau_1 < \tau_2 < \alpha$. Since $G \cap M \subseteq N$, we obtain that $m_2 \in m_1 + N$. Therefore, because the sums are finite while the number of equalities is not affected by the infinite cardinality of the limit ordinals, $g_{\tau_1} + n_1 \in \bigcap\limits_{\tau<\alpha}(N + p^\tau G) = N + p^\alpha G$ owing to the niceness of N in G. As a final conclusion, $x \in p^\alpha G + M + N$. This allows us to infer that $\bigcap\limits_{\tau<\alpha}(p^\tau G + M + N) \subseteq p^\alpha G + M + N$, and we are done.

Further, to deduce the desired niceness, one must show that $\bigcap\limits_{\delta<\beta}(N + M + p^\delta(G + M)) = N + M + p^\beta(G+M)$ for each limit ordinal number β. Indeed, by what we have already shown above, we conclude that $\bigcap\limits_{\delta<\beta}(N + M + p^\delta(G+M)) = \bigcap\limits_{\delta<\beta}(N + M + p^\delta G) = N + M + p^\beta G = N + M + p^\beta(G+M)$. The proof is complete. □

And so, we are ready to prove

Theorem 1. *Let A be a reduced abelian p-group with a subgroup G such that A/G is countable. If G is an almost totally projective group, then A is an almost totally projective group.*

P r o o f. Write $A = G + C$ where C is countable. Clearly $G \cap C \leqslant G$ is at most countable because C is countable. Since G is almost totally projective, it has a collection $\mathcal{C}_G$ consisting of nice subgroups of G and satisfying points (0), (1) and (2) quoted above. Thus there exists a countable subgroup L of G that belongs to $\mathcal{C}_G$ and $G \cap C \subseteq L$. Observe that $C + L$ is again countable. Moreover, in virtue of the classical modular law from [5], we obtain that $(C+L) \cap G = L + C \cap G = L$. Furthermore, $A/(C+L) = (G+C)/(L+C) \cong G/(G \cap (L+C)) = G/L$. Since L is countable, G is almost totally projective and $L \in \mathcal{C}_G$, it is a straightforward argument, which we leave to the reader, that G/L is again almost totally projective. We only suggest that $\mathcal{C}_{G/L} = \{K/L\colon\ L \subseteq K \in \mathcal{C}_G\}$. That is why so does $A/(C+L)$

and consequently it has a collection $\mathcal{C}_{A/(C+L)}$ of nice subgroups of $A/(C+L)$ such that conditions (0), (1) and (2) are fulfilled.

Henceforth, we put

$$\mathcal{C}_A = \{0\} \cup \{B \leqslant A | B/(C+L) \in \mathcal{C}_{A/(C+L)}\}.$$

First of all, we derive that B is nice in A. In fact, $B/(C+L)$ is nice in $A/(C+L)$, and moreover according to the foregoing Lemma the sum $C+L$ is nice in $C+G=A$; it is worthwhile to noting that the method in [9] for proving the wanted niceness also work, but however our approach is more direct and general. Consequently, applying ([5], p. 92, Lemma 79.3) we deduce that B is really a nice subgroup of A.

Next, we shall verify the truthfulness of (0), (1) and (2). That the zero subgroup of A lies in $\mathcal{C}_A$ follows by definition. Suppose now that $(B_i)_{i\in I}$ is an ascending chain of members of $\mathcal{C}_A$. Since $B_i/(C+L) \in \mathcal{C}_{A/(C+L)}$ is also ascending for each index $i \in I$, we find that $\bigcup\limits_{i\in I}(B_i/(C+L)) = \Bigl(\bigcup\limits_{i\in I} B_i\Bigr)/(C+L) \in \mathcal{C}_{A/(C+L)}$, hence $\bigcup\limits_{i\in I} B_i \in \mathcal{C}_A$ in view of $\bigcup\limits_{i\in I} B_i \leqslant A$.

Finally, assume that P is a countable subgroup of A. Thus $(P+C+L)/(C+L) \cong P/(P\cap(C+L))$ is at most a countable subgroup of $A/(C+L)$ (finite or infinite) and so there is $B/(C+L) \in \mathcal{C}_{A/(C+L)}$ such that $(P+C+L)/(C+L) \subseteq B/(C+L)$ and such that $B/(C+L)$ is countable. We therefore elementarily see that $P+C+L \subseteq B$, hence $P \subseteq B$, where $B \in \mathcal{C}_A$ and B is countable because as already pointed out so is $C+L$.

Thereby, the validity of the three conditions is proved. $\square$

Next, we come to the other major statement.

Theorem 2. *Let A be a reduced abelian p-group with a nice countable subgroup G such that A/G is almost totally projective. Then A is an almost totally projective group.*

P r o o f. Since A/G is an almost totally projective group, it has a collection $\mathcal{L}$ of nice subgroups which possesses properties (0), (1) and (2). What suffices to verify is that these conditions (0), (1) and (2) are also satisfied for an appropriate selected collection $\mathcal{C}$ of nice subgroups of A. Indeed, we set

$$\mathcal{C} = \{K \leqslant A\colon\ K/G \in \mathcal{L}\} \cup \{0\}.$$

First, if we assume that $0 \neq K \in \mathcal{C}$, then $K/G \in \mathcal{L}$ and hence K/G is nice in A/G. Bearing in mind that G is nice in A, it follows from ([5], p. 92, Lemma 79.3) that K is nice in A. Thus $\mathcal{C}$ consists of nice subgroups of A.

Because (0) is trivially fulfilled by construction, what remains to show is the validity of (1) and (2). And so, suppose $(H_i)_{i\in I}$ is an ascending chain of non-zero members of $\mathcal{C}$. We therefore have that the chain $(H_i/G)_{i\in I}$ of elements of $\mathcal{L}$ also ascends, whence $\bigcup\limits_{i\in I}(H_i/G) = \Big(\bigcup\limits_{i\in I} H_i\Big)/G \in \mathcal{L}$. Consequently, in accordance with the definition, $\bigcup\limits_{i\in I} H_i \in \mathcal{C}$ because $\bigcup\limits_{i\in I} H_i \leqslant A$. Hence (1) follows. Furthermore, letting C be an arbitrary countable subgroup of A, we obtain that $(C+G)/G \cong C/(C\cap G)$ is a countable (finite or infinite) subgroup of A/G and there is $B/G \in \mathcal{L}$ such that $(C+G)/G \subseteq B/G$ and B/G is countable. Thus $B \in \mathcal{C}$, $C+G \subseteq B \leqslant A$, hence $C \subseteq B$ and B is countable because by hypothesis so is G. Notice that when $C \subseteq G$, we can take $B = G$ since $G \in \mathcal{C}$ and G is countable. $\square$

Under some length restrictions on the whole group, we yield the following immediate consequences.

Corollary 1. *Suppose that G is a subgroup of the reduced abelian p-group A with length not exceeding Ω. If G is either an almost direct sum of countable groups such that A/G is countable, or G is countable and nice in A such that A/G is an almost direct sum of countable groups, then A is an almost direct sum of countable groups.*

Proof. Applying Theorems 1 or 2, respectively, for length$(A) \leqslant \Omega$, we are done. $\square$

Corollary 2 ([6]). *Suppose that G is a subgroup of the separable abelian p-group A. If G is either an almost direct sum of cyclic groups such that A/G is countable, or G is countable such that A/G is an almost direct sum of cyclic groups, then A is an almost direct sum of cyclic groups.*

Proof. Employing Theorems 1 or 2, respectively, for length$(A) \leqslant \omega$, we complete the proof. $\square$

III. Left-open problems

First of all, we ask whether the restriction on the subgroup G in Theorem 2 to be nice in the full group A can be omitted. Moreover, in what direction the two theorems listed above may be united; that is, letting A/G be an almost totally projective group, find a suitable restriction on G which leads to a necessary and sufficient condition for A to be almost totally projective.

Secondly, does it follow that quasi-complete almost direct sums of cyclic groups are bounded, same as the pure-complete almost direct sums of cyclic groups are direct sums of cyclic groups?

R e m a r k. We would like to note that the proof of Proposition 3 in [7] is erroneous. In fact, Hill and Ullery state that [sic]: "Recall that N is a nice subgroup of G if and only if $p^\alpha N$ is a nice subgroup of $p^\alpha G$ and $(N + p^\alpha G)/p^\alpha G$ is a nice subgroup of $G/p^\alpha G$."

On the other hand, the same Hill proved thirty years ago that (see, e.g., [5], Chapter 79, p. 93, Exercise 10) N is nice in G if and only if $N \cap p^\alpha G$ is nice in $p^\alpha G$ and $(N + p^\alpha G)/p^\alpha G$ is nice in $G/p^\alpha G$, which is actually the true statement.

Certainly, these two assertions are not equivalent as simple examples show when, for instance, N is not isotype in G.

Indeed, let G be an unbounded direct sum of cycles with a subgroup N so that $pN = 0$ but $N \cap pG \neq 0$. Thus N is not pure in G. We shall show that N is even not nice in G. To this aim, we shall demonstrate that G/N is not separable.

And so, first observe that

1) $pN = 0$ is ever nice in pG;

2) $G/(N+pG)$ is ever bounded, hence $N+pG$ is ever nice in G; even $(N+pG)/pG$ is a subgroup of the bounded G/pG, whence it is nice in G/pG and consequently $N + pG$ is nice in G since pG is nice in G.

Thus all conditions in the Hill-Ullery's claim are trivially satisfied, whereas $N \cap pG$ may not be nice in pG, i.e. not nice in G, hence N is not nice in G. Notice that $pG/(N \cap pG) \cong (N + pG)/N \subseteq G/N$ both have equal first Ulm subgroups since $G/N/(N + pG)/N \cong G/(N + pG)$ is bounded.

A c k n o w l e d g e m e n t. The author would like to express his gratitude to the anonymous referee for his helpful suggestions.

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