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MAXIMAL REGULARITY FOR ABSTRACT PARABOLIC PROBLEMS WITH INHOMOGENEOUS BOUNDARY DATA IN L_p-SPACES

Jan Prüss, Halle

Abstract. Several abstract model problems of elliptic and parabolic type with inhomogeneous initial and boundary data are discussed. By means of a variant of the Dore-Venni theorem, real and complex interpolation, and trace theorems, optimal L_p-regularity is shown. By means of this purely operator theoretic approach, classical results on L_p-regularity of the diffusion equation with inhomogeneous Dirichlet or Neumann or Robin condition are recovered. An application to a dynamic boundary value problem with surface diffusion for the diffusion equation is included.

Keywords: maximal regularity, sectorial operators, interpolation, trace theorems, elliptic and parabolic initial-boundary value problems, dynamic boundary conditions

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1. Introduction

The operator-sum method as developed by Da Prato and Grisvard [1], Dore and Venni [2], and recently by Kalton and Weis [7] has been employed successfully to solve abstract Cauchy problems of the form

$$\dot{u}(t) + Au(t) = f(t),\ t > 0, \quad u(0) = 0,$$

where A denotes the generator of a bounded analytic C_0-semigroup in the Banach space X. It is the goal of this paper to show how this method can be used to obtain maximal L_p-regularity for a variety of abstract elliptic and parabolic problems with *inhomogeneous* initial and boundary values. Such problems arise in the real pde world as model problems. Results on such model problems can be transferred to elliptic and parabolic boundary value problems on domains with smooth boundary by well-known techniques like localization, perturbation and coordinate transformation.

As an example of this strategy let us consider the diffusion equation with a dynamic boundary condition involving *surface diffusion*. So let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\Gamma = \partial\Omega$ of class C^2, such that Γ decomposes into three disjoint parts $\Gamma = \Gamma_0 \cup \Gamma_n \cup \Gamma_d$, where Γ_j is open and closed in Γ. Consider the problem

$$
\begin{aligned}
&\partial_t u(t,x) - \Delta u(t,x) = f(t,x), \quad t > 0,\ x \in \Omega,\\
&\quad u(t,x) = \varphi(t,x),\ x \in \Gamma_0, \quad \partial_n u(t,x) = \psi(t,x),\ x \in \Gamma_n,\ t > 0,\\
&\partial_t u(t,x) + \partial_n u(t,x) - \Delta_\Gamma u(t,x) = h(t,x),\ x \in \Gamma_d,\ ;\ t > 0,\\
&\quad u(0,x) = u_0(x),\ x \in \Omega,
\end{aligned}
\tag{1.1}
$$

where ∂_n means the normal derivative at the boundary of Ω, and Δ_Γ denotes the Laplace-Beltrami operator on the manifold Γ. By means of localization, transformation and perturbation this problem can be reduced to four model problems, namely the diffusion equation on $\mathbb{R}^n$, and the diffusion equation on the half-space $\mathbb{R}^n_+$ with Dirichlet or Neumann or dynamic surface diffusion condition on the boundary of $\mathbb{R}^n_+$. This paper deals with the abstract version of these model problems. To show the strength of our results we have included surface diffusion, which plays the role of an unbounded Robin condition.

As an application, let us present the L_p-maximal regularity result for (1.1) which follows from Theorem 4.3 below.

Theorem. *Let $1 < p < \infty$, $p \neq 3/2, 3$, and let $J = [0,a]$. Then problem* (1.1) *admits a unique solution*

$$u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$$

such that

$$u|_{\Gamma_d} \in W_p^{3/2-1/2p}(J; L_p(\Gamma_d)) \cap L_p(J; W_p^{3-1/p}(\Gamma_d)),$$

if and only if $f \in L_p(J \times \Omega)$, $u_0 \in W_p^{2-2/p}(\Omega)$,

$$
\begin{aligned}
\varphi &\in W_p^{1-1/2p}(J; L_p(\Gamma_0)) \cap L_p(J; W_p^{2-1/p}(\Gamma_0)),\\
\psi &\in W_p^{1/2-1/2p}(J; L_p(\Gamma_n)) \cap L_p(J; W_p^{1-1/p}(\Gamma_n)),\\
h &\in W_p^{1/2-1/2p}(J; L_p(\Gamma_d)) \cap L_p(J; W_p^{1-1/p}(\Gamma_d)),
\end{aligned}
$$

and the compatibility conditions

$$
\begin{aligned}
u_0|_{\Gamma_0} &= \varphi|_{t=0} \quad \text{for } p > 3/2,\\
\partial_n u_0|_{\Gamma_n} &= \psi|_{t=0} \quad \text{for } p > 3,\\
u_0|_{\Gamma_d} &\in W_p^{3-3/p}(\Gamma_d) \quad \text{for } p > 1
\end{aligned}
$$

are satisfied.

Specializing the results proved below, we are able to recover classical results on parabolic initial-boundary value problems. So e.g. in case $Y = L_p(\mathbb{R}^n)$ and $\mathcal{B} = -\Delta$, $D = 0$, Theorems 4.1 and 4.2 reduce to the famous results of Ladyzhenskaya, Solonnikov and Ural'tseva [9] on maximal L_p-regularity of second order parabolic initial-boundary value problems. Many other problems fit into the framework of our approach. So for example, in (1.1) we can allow for a Robin condition like $\partial_n u - \Delta_\Gamma u = \psi$ instead of the Neumann condition on Γ_n, and surface diffusion can be dropped in the dynamic boundary condition on Γ_d.

In this paper we only employ a theorem of Dore-Venni type, [2], [10], real and complex interpolation, and trace theorems. We would like to point out that our techniques have been applied successfully also to the study of free boundary value problems like the *Stefan problem with surface tension*, see Escher, Prüss and Simonett [3] or the *one and two phase free boundary value problems for the Navier-Stokes equation*, see Escher, Prüss and Simonett [4].

Some of our results may be generalized using a more recent theorem of Kalton and Weis [7]. For example, in Theorems 3.1 and 3.3, the involved operators A and F need only to be $\mathcal{R}$-sectorial instead of class $\mathcal{BIP}$, and their power angles can be replaced by their $\mathcal{R}$-angles. In Theorem 3.4, say, D needs only to be $\mathcal{R}$-sectorial if in addition F is assumed to admit an $\mathcal{H}^\infty$-calculus and $\varphi_F^\infty + \varphi_D^R < \pi$. However, extensions in this direction are not clear for Theorems 4.1 to 4.3.

2. Preliminaries

To introduce some notation used below recall that a Banach space X belongs to the class $\mathcal{HT}$ if the Hilbert transform is bounded on $L_2(\mathbb{R}; X)$. Recall also that a closed linear operator A in X is called *nonnegative* or *pseudo-sectorial* if $(-\infty, 0)$ is contained in the resolvent set of A and the resolvent estimate

$$t|(t+A)^{-1}|_{\mathcal{B}(X)} \leqslant M_0, \ t > 0,$$

holds for some constant $M_0 > 0$. If in addition the domain $\mathcal{D}(A)$ and the range $\mathcal{R}(A)$ of A are dense in X then A is called *sectorial*. We emphasize that for (pseudo-) sectorial operators the Dunford functional calculus is available and in particular, the fractional powers A^z, $z \in \mathbb{C}$ are well-defined closed linear operators in X; see e.g. Komatsu [8]. We also recall that in case X is reflexive and A is pseudo-sectorial then the space X decomposes according to $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, where $\mathcal{N}(A)$ designates the kernel of A. Thus in such a situation A is sectorial on $\overline{\mathcal{R}(A)}$.

We say that an operator A in X belongs to $\mathcal{BIP}(X)$ if A is sectorial and the imaginary powers $A^{\mathrm{i}s}$ of A form a bounded C_0-group on X. The type θ_A of this group will be called the power angle of A. We begin by repeating a variant of the Dore-Venni theorem; see Dore and Venni [2] and Prüss and Sohr [10].

Theorem 2.1. *Suppose X belongs to the class $\mathcal{H}T$, assume $A, B \in \mathcal{B}IP(X)$ commute and satisfy the strong parabolicity condition $\theta_A + \theta_B < \pi$, and let $t > 0$. Then*

(i) *$A + tB$ is closed and sectorial;*

(ii) *$A + tB \in \mathcal{B}IP(X)$ with $\theta_{A+tB} \leqslant \max\{\theta_A, \theta_B\}$;*

(iii) *there is a constant $C > 0$, independent of $t > 0$, such that*

$$|Ax| + t|Bx| \leqslant C|Ax + tBx|, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B). \tag{2.1}$$

In particular, if A or B is invertible, then $A + tB$ is invertible as well.

Some consequences of this result concerning complex interpolation are contained in the next corollary. By $[X, Z]_\theta$ we denote the complex interpolation spaces between X and Z.

Corollary 2.2. *Suppose X is a Banach space of class $\mathcal{H}T$, $A, B \in \mathcal{B}IP(X)$ are commuting in the resolvent sense, and their power angles satisfy the parabolicity condition $\theta_A + \theta_B < \pi$. Let A or B be invertible and $\alpha \in (0, 1)$. Then*

(a) *$A^\alpha(A + B)^{-\alpha}$ and $B^\alpha(A + B)^{-\alpha}$ are bounded in X;*

(b) *$\mathcal{D}((A + B)^\alpha) = [X, \mathcal{D}(A + B)]_\alpha = [X, \mathcal{D}(A)]_\alpha \cap [X, \mathcal{D}(B)]_\alpha = \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha)$.*

The proof of this result is not difficult, but due to the limited space we refer to the forthcoming monograph Hieber and Prüss [6].

The following result is due to Grisvard [5], even in a more general context. We denote the real interpolation spaces between X and Z by $(X, Z)_{\alpha,p}$.

Proposition 2.3. *Suppose A, B are sectorial linear operators in a Banach space X commuting in the resolvent sense, and let $\alpha \in (0, 1)$, $p \in [1, \infty]$. Then*

$$(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\alpha,p} = (X, \mathcal{D}(A))_{\alpha,p} \cap (X, \mathcal{D}(B))_{\alpha,p}.$$

The next result is known as the *mixed derivative theorem* and is due to Sobolevskii [11].

Proposition 2.4. *Suppose A and B are sectorial linear operators in a Banach space X with spectral angles $\varphi_A + \varphi_B < \pi$, which commute and are coercively* *positive, i.e.* $A + tB$ *with natural domain* $\mathcal{D}(A + tB) = \mathcal{D}(A) \cap \mathcal{D}(B)$ *is closed for each* $t > 0$ *and there is a constant* $M > 0$ *such that*

$$|Ax|_X + t|Bx|_X \leqslant M|Ax + tBx|_X \quad \text{for all } x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ t > 0.$$

Then there is a constant $C > 0$ *such that*

$$|A^\alpha B^{1-\alpha} x|_X \leqslant C|Ax + Bx|_X \quad \text{for all } x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \alpha \in [0, 1].$$

In particular, $A^\alpha B^{1-\alpha}(A + B)^{-1}$ *is bounded in* X *for each* $\alpha \in [0, 1]$.

Observe that Proposition 2.4 applies in particular to the situation of the Dore-Venni theorem, Theorem 2.1. The next result deals with traces of vector-valued functions, where we employ the standard notation $D_A(\alpha, p) = (X, X_A)_{\alpha,p}$ for the real interpolation spaces, where X_A means the Banach space $\mathcal{D}(A)$ equipped with the graph norm of A.

Proposition 2.5. *Let* Y *be a Banach space of class* $\mathcal{H}T$, A *a sectorial operator in* Y *which belongs to* $\mathcal{B}IP(Y)$ *with power angle* $\theta_A < \pi/s$, $1 < p < \infty$, $s > 1/p$, *and let* J *be an interval. Then*

$$H^s_p(J; Y) \cap L_p(J; \mathcal{D}(A^s)) \hookrightarrow BUC^n(J; D_A(s - n - 1/p, p))$$

and also

$$W^s_p(J; Y) \cap L_p(J; D_A(s, p)) \hookrightarrow BUC^n(J; D_A(s - n - 1/p, p)),$$

where n *means any integer smaller than* $s - 1/p$.

Here, as usual, $H^s_p(J; Y)$ and $W^s_p(J; Y)$ mean respectively the vector-valued Bessel potential spaces and Sobolev-Slobodecky spaces on an open interval $J \subset \mathbb{R}$, where $1 < p < \infty$ and $s > 0$. For the case $\theta_A = 0$ a proof is given in Escher, Prüss and Simonett [3], the general case is proved in Hieber and Prüss [6].

The next result is a direct consequence of the definition of the real interpolation spaces $D_A(\alpha, p)$, which nevertheless is very useful.

Proposition 2.6. *Let* $1 < p < \infty$, $1/p < \alpha < 1$, *suppose* A *is an invertible pseudo-sectorial operator in* X *with* $\varphi_A < \pi/2$, *and set* $u(t) = \mathrm{e}^{-At}x$, $x \in X$.

Then the following statements are equivalent.

(i) $x \in D_A(\alpha - 1/p, p)$;

(ii) $u \in L_p(\mathbb{R}_+; D_A(\alpha, p))$;

(iii) $u \in W_p^\alpha(\mathbb{R}_+; X)$.

Observe that $u \in W_p^\alpha(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A(\alpha, p))$ holds for all $x \in X$ provided $0 \leqslant \alpha < 1/p$.

3. Abstract equations on the halfline

We consider now the following abstract theorem on evolution equations

$$\dot{u} + Au = f, \quad t > 0, \quad u(0) = u_0, \tag{3.1}$$

in a Banach space Y. The main result on maximal L_p-regularity for (3.1) is a well-known consequence of Theorem 2.1.

Theorem 3.1. *Suppose Y is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, let $A \in \mathcal{BIP}(Y)$ be invertible with power angle $\theta_A < \pi/2$, and let D_A denote the domain $\mathcal{D}(A)$ of A equipped with the graph norm of A.*

Then (3.1) *has precisely one solution in $Z := H_p^1(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_A)$ if and only if*

$$f \in X := L_p(\mathbb{R}_+; Y) \quad \text{and} \quad u_0 \in D_A(1 - 1/p, p).$$

A result much easier to obtain—but nevertheless useful—is the following

Proposition 3.2. *Let $1 < p < \infty$, $1/p < \alpha < 1$, suppose A is an invertible pseudo-sectorial operator in X with $\varphi_A < \pi/2$, let $f \in L_p(\mathbb{R}_+; X)$ and $u_0 \in X$.*

Then the following statements for the solution u of (3.1) *are equivalent.*

(i) $u \in W_p^{1+\alpha}(\mathbb{R}_+; X) \cap W_p^\alpha(\mathbb{R}_+; \mathcal{D}(A))$;

(ii) $f \in W_p^\alpha(\mathbb{R}_+; X)$, $u_0 \in \mathcal{D}(A)$, $Ax - f(0) \in D_A(\alpha - 1/p, p)$.

Our next theorem concerns the abstract second order problem with Dirichlet condition

$$\begin{aligned} -u''(y) + F^2 u(y) &= g(y), \quad y > 0, \\ u(0) &= \varphi, \end{aligned} \tag{3.2}$$

in $L_p(\mathbb{R}_+; Y)$.

Theorem 3.3. *Suppose Y is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, let $F \in \mathcal{BIP}(Y)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $\mathcal{D}(F^j)$ of F^j equipped with its graph norm, $j = 1, 2$.*

Then (3.2) *has precisely one solution* u *in* $Z := H_p^2(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_F^2)$ *if and only if*

$$g \in X := L_p(\mathbb{R}_+;Y) \quad \text{and} \quad \varphi \in D_F(2-1/p,p).$$

If this is the case we have in addition $u \in H_p^1(\mathbb{R}_+;D_F^1)$.

Proof. Apply Theorem 2.1 in $X = L_p(\mathbb{R}_+;Y)$ to $A = F^2$ and $B = -\,\mathrm{d}^2/\,\mathrm{d}y^2$ with domain $\mathcal{D}(B) = H_p^2(\mathbb{R}_+;Y) \cap {}_0H_p^1(\mathbb{R}_+;Y)$ to see that (3.2) admits a unique solution $u \in Z$ for each $g \in X$, $\varphi = 0$. It is given explicitly by the formula

$$u(y) = \frac{1}{2}F^{-1}\int_0^\infty [\mathrm{e}^{-F|y-s|} - \mathrm{e}^{-F(y+s)}]g(s)\,\mathrm{d}s, \quad t > 0.$$

On the other hand, if $-u'' + F^2u = 0$ and u is bounded then $u' + Fu = 0$. Hence the unique solution of (3.2) in X with $g = 0$ is given by

$$u(y) = \mathrm{e}^{-Fy}\varphi,$$

which belongs to Z if and only if $\varphi \in D_F(2-1/p,p)$. In fact, by Theorem 3.1, $F\varphi \in D_F(1-1/p,p)$ is equivalent to $v(y) := \mathrm{e}^{-Fy}F\varphi \in H_p^1(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_F^1)$, which by invertibility of F is in turn equivalent to $u \in Z$. The last assertion follows from the mixed derivative theorem with $\alpha = 1/2$. $\square$

There is a companion result for the abstract second order problem with abstract Robin condition which for $D = 0$ becomes the Neumann condition:

$$\begin{aligned} -u''(y) + F^2u(y) &= g(y), \quad y > 0, \\ -u'(0) + Du(0) &= \psi. \end{aligned} \tag{3.3}$$

For this problem the maximal regularity result in $L_p(\mathbb{R}_+;Y)$ reads as follows.

Theorem 3.4. *Suppose* Y *is a Banach space of class* $\mathcal{HT}$, $1 < p < \infty$, *let* $F \in \mathcal{BIP}(Y)$ *be invertible with power angle* $\theta_F < \pi/2$, *and let* D_F^j *denote the domain* $\mathcal{D}(F^j)$ *of* F^j *equipped with its graph norm,* $j = 1, 2$. *Suppose that* D *is pseudo-sectorial, belongs to* $\mathcal{BIP}(\overline{\mathcal{R}(D)})$, *commutes with* F, *and is such that* $\theta_F + \theta_D < \pi$.

Then (3.3) *has precisely one solution in* $Z := H_p^2(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_F^2)$ *with* $u(0) \in \mathcal{D}(D)$ *and* $Du(0) \in D_F(1-1/p,p)$ *if and only if*

$$g \in X := L_p(\mathbb{R}_+;Y) \quad \text{and} \quad \psi \in D_F(1-1/p,p).$$

If this is the case we have in addition $u \in H_p^1(\mathbb{R}_+;D_F^1)$.

Proof. Since $\mathcal{N}(D) \oplus \overline{\mathcal{R}(D)} = Y$ by reflexivity of Y, and D and F commute, Theorem 2.1 implies that $F+D$ with domain $\mathcal{D}(F+D) = \mathcal{D}(F) \cap \mathcal{D}(D)$ is invertible. The solution u of (3.3) can be written explicitly as will be shown below:

$$\begin{aligned} u(y) &= \mathrm{e}^{-Fy}(F+D)^{-1}\psi \\ &\quad + \frac{1}{2}F^{-1}\int_0^\infty [\mathrm{e}^{-F|y-s|} + (F-D)(F+D)^{-1}\mathrm{e}^{-F(y+s)}]g(s)\,\mathrm{d}s. \end{aligned}$$

This time we proceed as follows. Apply Theorem 2.1 to $A = F^2$ and $B = -\,\mathrm{d}^2/\,\mathrm{d}y^2$ in $X = L_p(\mathbb{R}; Y)$, i.e. for the problem on the entire line. This way, for each $g \in X$, we obtain a unique solution $v \in H_p^2(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_F^2)$. It is given by the formula

$$v(y) = \frac{1}{2}F^{-1}\int_{-\infty}^\infty \mathrm{e}^{-F|y-s|}g(s)\,\mathrm{d}s, \quad y \in \mathbb{R}.$$

We write $v = (A+B)^{-1}g$. Now suppose first $\psi = 0$ and let $g \in L_p(\mathbb{R}_+; Y)$. Let $E_0\colon L_p(\mathbb{R}_+; Y) \to L_p(\mathbb{R}; Y)$ denote the operator of extension by 0, i.e.

$$(E_0 f)(y) = f(y), \quad y > 0, \quad (E_0 f)(y) = 0, \quad y < 0,$$

let $P_+\colon L_p(\mathbb{R}; Y) \to L_p(\mathbb{R}_+; Y)$ be the restriction to $\mathbb{R}_+$, and $R\colon L_p(\mathbb{R}; Y) \to L_p(\mathbb{R}; Y)$ the reflection at 0, i.e.

$$(Rf)(y) = f(-y), \quad y \in \mathbb{R}.$$

These operators are all bounded with norm 1. The solution formula for u may then be rewritten as

$$u = P_+(A+B)^{-1}E_0 g + (F-D)(F+D)^{-1}P_+R(A+B)^{-1}E_0 g,$$

which shows that the solution u belongs to the maximal regularity space $Z = H_p^2(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_F^2)$. The trace of u at $y = 0$ exists and equals

$$u(0) = (F+D)^{-1}\int_0^\infty \mathrm{e}^{-Fs}g(s)\,\mathrm{d}s.$$

This shows $u(0) \in \mathcal{D}(D)$ and

$$Du(0) = D(F+D)^{-1}\int_0^\infty \mathrm{e}^{-Fs}g(s)\,\mathrm{d}s \in D_F(1-1/p, p),$$

since F and D commute by assumption, $D(F+D)^{-1}$ is bounded and is leaving $D_F(1-1/p,p)$ invariant. In the case $g=0$ the solution u is given by

$$u(y) = \mathrm{e}^{-Fy}(F+D)^{-1}\psi,$$

hence $u(0) = (F+D)^{-1}\psi \in \mathcal{D}(D)$ as well as

$$Du(0) = D(F+D)^{-1}\psi \in D_F(1-1/p,p),$$

but also $u(0) \in D_F(2-1/p,p)$, hence u belongs to Z.

Conversely, if a function $u \in Z$ with $u(0) \in \mathcal{D}(D)$, $Du(0) \in D_F(1-1/p,p)$ is given then $g := -u'' + F^2 u \in L_p(\mathbb{R}_+;Y)$, hence the condition on g is necessary. Now solve the problem with right hand side g and homogeneous boundary condition to obtain $v \in Z$. Then $w := u - v$ satisfies $-w'' + F^2 w = 0$ on $\mathbb{R}_+$, hence is given by $w(y) = \mathrm{e}^{-Fy}w(0)$. This implies $w(0) \in D_F(2-1/p,p)$, $w'(0) \in D_F(1-1/p,p)$ as well as $w(0) \in \mathcal{D}(D)$, hence

$$\psi = -w'(0) + Dw(0) = -u'(0) + Du(0) \in D_F(1-1/p,p).$$

Therefore the condition on ψ is also necessary, and in particular the solutions are unique. The last statement follows again from the mixed derivative theorem. □

It is worthwhile to imagine which kind of operators F and D are covered by Theorem 3.4. To examine this, we let $Y = L_p(\mathbb{R}^n)$ and $F = (1+D_n)^{1/2}$, where D_n means the negative Laplacian on $\mathbb{R}^n$. For D we may choose an oblique derivative operator of the form $Du = a\cdot\nabla u + bu$, where $a \in \mathbb{R}^n$ and $b \geqslant 0$. This covers the case of a second order elliptic equation with oblique boundary condition. We even may put more generally $Du = D_n u + a\cdot\nabla u + bu$.

On the other hand, we may consider in $Y = L_p(\mathbb{R}; L_p(\mathbb{R}_n))$ an elliptic equation, e.g. $F = (1+D_n)^{1/2}$ canonically extended to Y and a dynamic boundary condition like $Du = \partial_t u + bu$ where $b \geqslant 0$, or more generally $Du = \partial_t u + D_n u + a\cdot\nabla u + bu$, where $b > 0$.

Third, let again $Y = L_p(\mathbb{R}; L_p(\mathbb{R}^n))$ but $F = (\partial_t + D_n)^{1/2}$ this time, i.e. the underlying equation is second order parabolic. In virtue of $\theta_F = \pi/4$ we may also choose oblique boundary conditions $Du = a\cdot\nabla u + bu$ as well as $Du = D_n u + a\cdot\nabla u + bu$, and dynamic boundary conditions like $Du = \partial_t u + bu$ and $Du = \partial_t u + D_n u + bu$ are still allowed.

If we consider $Y = L_p(\mathbb{R}_+; L_p(\mathbb{R}^n))$ then the above choices are also possible, but we have to add homogeneous initial conditions at $t = 0$ whenever the operators F or D contain ∂_t.

We now consider the vector-valued problem

$$
\begin{aligned}
\partial_t u - \partial_y^2 u + \mathcal{B}u &= f, \quad t, y > 0,\\
u(t,0) &= \varphi(t), \quad t > 0,\\
u(0,y) &= \psi(y), \quad y > 0.
\end{aligned}
\tag{4.1}
$$

Here $\mathcal{B}$ denotes an invertible sectorial operator in a Banach space Y which belongs to the class $\mathcal{HT}$, and the data f, φ, and ψ are given. We are interested in solutions u which belong to the maximal regularity class of type L_p, i.e.

$$u \in H_p^1(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H_p^2(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B)),$$

where D_B denotes the space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. The main result reads as follows.

Theorem 4.1. *Suppose Y is a Banach space of class $\mathcal{HT}$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$ and let $p \in (1,\infty)$, $p \neq 3/2$. Let D_B denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. Then the problem* (4.1) *has exactly one solution*

$$u \in Z := H_p^1(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H_p^2(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B))$$

if and only if the data f, φ, ψ satisfy the following conditions:

1. $f \in X := L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y))$;
2. $\varphi \in W_p^{1-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1-1/2p, p))$;
3. $\psi \in W_p^{2-2/p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1-1/p, p))$;
4. $\varphi|_{t=0} = \psi|_{y=0}$ *provided* $p > 3/2$.

If this is the case then $\varphi|_{t=0} = \psi|_{y=0} \in D_B(1-3/2p, p)$ for $p > 3/2$.

P r o o f. Suppose u is a solution of (4.1). Then we evidently have $f = \partial_t u - \partial_y^2 u + \mathcal{B}u \in X$, i.e. the first condition is necessary. Let B denote the natural extension of $\mathcal{B}$ to $E := L_p(\mathbb{R}_+; Y)$ with domain $\mathcal{D}(B) = L_p(\mathbb{R}_+; D_B)$. Then B is again invertible, sectorial, and $B \in \mathcal{BIP}(E)$ with power angle $\theta_B < \pi/2$. Let $G = -\partial_y^2$ with domain $\mathcal{D}(G) = H_p^2(\mathbb{R}_+; Y) \cap {}_0H_p^1(\mathbb{R}_+; Y)$; then G is sectorial and belongs to $\mathcal{BIP}(E)$ with power angle $\theta_G = 0$. Since both operators commute, Theorem 2.1 yields that $A := G + B$ with domain $\mathcal{D}(A) = \mathcal{D}(G) \cap \mathcal{D}(B)$ is sectorial, belongs to $\mathcal{BIP}(E)$ with power angle $\theta_A < \pi/2$. Then by Theorem 3.1 we obtain the function $u_0 := \mathrm{e}^{-At} * f \in Z$, hence $v := u - u_0$ satisfies (4.1) with $f = 0$ and the traces of u and v for $y = 0$ and also for $t = 0$ coincide. Thus we may assume $f = 0$ in the sequel.

Next we extend u and φ for $t < 0$ by symmetry, i.e. $u(t, y) = u(-t, y)$ and $\varphi(t) = \varphi(-t)$ for $t < 0$. Then u satisfies a problem of the form

$$\begin{aligned} \partial_t u - \partial_y^2 u + \mathcal{B} u &= g, \quad y > 0,\ t \in \mathbb{R}, \\ u(t, 0) &= \varphi(t), \quad t \in \mathbb{R}, \end{aligned} \tag{4.2}$$

with $g \in L_p(\mathbb{R}; L_p(\mathbb{R}_+; Y))$. Define B in $L_p(\mathbb{R}; Y)$ again by pointwise extension, and let $D = \partial_t$ with domain $\mathcal{D}(D) = H_p^1(\mathbb{R}; Y)$. Both operators are sectorial, they commute and belong to $\mathcal{BIP}(L_p(\mathbb{R}; Y))$, their power angles satisfy $\theta_D + \theta_B \leqslant \pi/2 + \theta_B < \pi$. Therefore, by Theorem 2.1, $D + B$ with domain $\mathcal{D}(D + B) = \mathcal{D}(B) \cap \mathcal{D}(D)$ is invertible, sectorial, belongs to $\mathcal{BIP}(L_p(\mathbb{R}; Y))$, with power angle smaller than $\pi/2$. Now we are in position to apply Theorem 3.3 to $F := \sqrt{D + B}$, to the result $\varphi \in D_F(2 - 1/p, p)$. By means of Corollary 2.2 we have

$$\mathcal{D}(F) = \mathcal{D}((D + B)^{1/2}) = \mathcal{D}(D^{1/2}) \cap \mathcal{D}(B^{1/2}),$$

and Proposition 2.3 gives

$$D_F(\alpha, p) = D_{D^{1/2}}(\alpha, p) \cap D_{B^{1/2}}(\alpha, p).$$

The reiteration theorem yields $D_{B^{1/2}}(\alpha, p) = D_B(\alpha/2, p)$, similarly for D, hence we get

$$D_F(2 - 1/p, p) = W_p^{1-1/2p}(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1 - 1/2p, p)).$$

Therefore by restriction to $\mathbb{R}_+$ we see that the second condition in Theorem 4.1 is necessary.

In a similar way we prove necessity of the third condition. This time we extend u w.r.t. y to all of $\mathbb{R}$, say e.g. by $u(t, y) = 3u(t, -y) - 2u(t, -2y)$. The resulting function belongs to

$$H_p^1(\mathbb{R}_+; L_p(\mathbb{R}; Y)) \cap L_p(\mathbb{R}_+; H_p^2(\mathbb{R}; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}; D_B)).$$

This time we let $A = B + G$ in $L_p(\mathbb{R}; Y)$, where $\mathcal{D}(G) = H_p^2(\mathbb{R}; Y)$, and apply Theorem 2.1 and Theorem 3.1 to the result $u|_{t=0} \in D_A(1 - 1/p, p)$. Proposition 2.3 yields

$$\begin{aligned} D_A(1 - 1/p, p) &= D_G(1 - 1/p, p) \cap D_B(1 - 1/p, p) \\ &= W_p^{2-2/p}(\mathbb{R}; Y) \cap L_p(\mathbb{R}; D_B(1 - 1/p, p)), \end{aligned}$$

and so after restriction to $y \in \mathbb{R}_+$ we obtain necessity of the third condition.

Last but not least, by the mixed derivative theorem, Proposition 2.4, we have

$$u \in H_p^s(\mathbb{R}_+; H_p^{2-2s}(\mathbb{R}_+; Y))$$

for each $s \in [0,1]$. This space embeds into $BUC(\mathbb{R}_+^2; Y)$ if $1/p < s$ and $1/p < 2-2s$, i.e. if $1/p < s < 1-1/2p$. This shows that the compatibility condition is necessary for $p > 3/2$. Taking the trace of φ at $t = 0$, Proposition 2.5 yields $\varphi(0) \in D_B(1-3/2p, p)$, the last assertion of Theorem 4.1.

Obviously the solution is unique, since $-A$ generates an analytic C_0-semigroup in $L_p(\mathbb{R}_+; Y)$ as was observed before.

Conversely, let the data f, φ, ψ be given such that the compatibility condition holds in the case $p > 3/2$. Then with $\varphi_0 = \varphi(0)$ we write the solution in the following way:

$$\begin{aligned} u = \mathrm{e}^{-At} * f + \mathrm{e}^{-At}[\psi - \mathrm{e}^{-\mathcal{B}^{1/2}y/\sqrt{2}}\varphi_0] + \mathrm{e}^{-Fy}[\varphi - \mathrm{e}^{-\mathcal{B}t/2}\varphi_0] \\ + \mathrm{e}^{-\mathcal{B}t/2}\mathrm{e}^{-\mathcal{B}^{1/2}y/\sqrt{2}}\varphi_0. \end{aligned} \tag{4.3}$$

Here A is defined as above in $L_p(\mathbb{R}_+; Y)$ and F as above in $L_p(\mathbb{R}_+; Y)$ where now $\mathcal{D}(G) = {}_0H_p^1(\mathbb{R}_+; Y) \cap H_p^2(\mathbb{R}_+; Y)$. This formula is written for the case $p > 3/2$; for $p < 3/2$ simply set $\varphi(0) = 0$. Observe that the last term v satisfies $\partial_t v - \partial_y^2 v + \mathcal{B}v = 0$ and has traces $\mathrm{e}^{-\mathcal{B}^{1/2}y/\sqrt{2}}\varphi(0)$ at $t = 0$ and $\mathrm{e}^{-\mathcal{B}t/2}\varphi(0)$ at $y = 0$. According to Theorems 3.1 and 3.3 each term in (4.3) belongs to the space Z, which completes the proof. □

There is an analogous result for the problem

$$\begin{aligned} \partial_t u - \partial_y^2 u + \mathcal{B}u &= f, \quad t, y > 0, \\ -\partial_y u(t,0) + \mathcal{D}u(t,0) &= \varphi(t), \quad t > 0, \\ u(0,y) &= \psi(y), \quad y > 0. \end{aligned} \tag{4.4}$$

It reads as follows:

Theorem 4.2. *Suppose Y is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $p \neq 3$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$, $\mathcal{D}$ pseudo-sectorial in Y, $\mathcal{D} \in \mathcal{BIP}(\overline{\mathcal{R}(\mathcal{D})})$ with $\theta_{\mathcal{D}} < 3\pi/4$, and suppose $\mathcal{B}$ and $\mathcal{D}$ commute in the resolvent sense. Let D_B denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$. Then the problem* (4.4) *has exactly one solution*

$$u \in Z := H_p^1(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H_p^2(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B))$$

with
$$\mathcal{D}u(\cdot,0) \in W_p^{1/2-1/2p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2-1/2p,p))$$
if and only if the data f, φ, ψ satisfy the following conditions.

1. $f \in X := L_p(\mathbb{R}_+;L_p(\mathbb{R}_+;Y))$;
2. $\varphi \in W_p^{1/2-1/2p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2-1/2p,p))$;
3. $\psi \in W_p^{2-2/p}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1-1/p,p))$;
4. $-\partial_y\psi(0) + \mathcal{D}\psi(0) = \varphi(0) \in Y$ *provided* $p > 3$.

If this is the case then $\varphi(0) \in D_B(1/2-3/2p,p)$ and $\psi(0) \in \mathcal{D}(\mathcal{D}) \cap D_B(1-3/2p,p)$ for $p > 3/2$, $\mathcal{D}\psi(0) \in D_B(1/2-3/2p,p)$ for $p > 3$.

Proof. Splitting again the space Y if necessary we may assume $\mathcal{N}(\mathcal{D}) = 0$. Suppose that u is a solution with the stated properties. Extend u in t to $\mathbb{R}$ by symmetry, i.e. $u(t) = u(-t)$ for $t \leqslant 0$. Then

$$u \in H_p^1(\mathbb{R};L_p(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R};H_p^2(\mathbb{R}_+;Y)) \cap L_p(\mathbb{R};L_p(\mathbb{R}_+;D_B)).$$

Define A in $X := L_p(\mathbb{R};Y)$ by means of

$$Av = \partial_t v + \mathcal{B}v, \quad \mathcal{D}(A) = H_p^1(\mathbb{R};Y) \cap L_p(\mathbb{R};D_B),$$

and let $F = A^{1/2}$. Then (4.4) is equivalent to (3.2). Hence employing Theorem 3.3 we obtain $\partial_t u + \mathcal{B}u \in L_p(\mathbb{R};L_p(\mathbb{R};Y))$ as well as $\partial_y u|_{y=0} \in D_F(1-1/p,p)$. Restricting to $t > 0$ this yields 1. and 2., since by the results of Section 3 we have

$$D_F(1-1/p,p) = W_p^{1/2-1/2p}(\mathbb{R};Y) \cap L_p(\mathbb{R};D_B(1/2-1/2p,p)).$$

Similarly extend u w.r.t. y in the same regularity class, and consider $A = -\partial_y^2 + \mathcal{B}$ in $X = L_p(\mathbb{R};Y)$. By Theorem 2.1, A is sectorial, admits bounded imaginary powers with power angle $\theta_A < \pi/2$, in particular A generates a bounded analytic C_0-semigroup in X. Then appply Theorem 3.1 to obtain $\psi \in D_A(1-1/p,p)$. Corollary 2.2 and Proposition 2.3 yield

$$D_A(1-1/p,p) = W_p^{2-2/p}(\mathbb{R};Y) \cap L_p(\mathbb{R};D_B(1-1/p,p)),$$

hence we obtain 3. by restriction of y to $\mathbb{R}_+$.

The compatibility condition follows from the embedding

$$W_p^{1/2-1/2p}(J;Y) \cap L_p(J;D_B(1/2-1/2p,p)) \hookrightarrow C(J;D_B(1/2-3/2p,p)),$$

which is valid for $p > 3$ by Proposition 2.5.

Conversely, if the data f, φ, ψ are given, then as in the proof of Theorem 3.4, the solution can be written in the form

$$\begin{aligned} u &= \mathrm{e}^{-At}\psi_1 + u_1 + \mathrm{e}^{-Fy}(F+D)^{-1}\varphi_1 + \\ &+ \frac{1}{2}F^{-1}\int_0^\infty [\mathrm{e}^{-F|y-s|} + (F-D)(F+D)^{-1}\mathrm{e}^{-F(y+s)}]f(s)\,\mathrm{d}s, \end{aligned} \tag{4.5}$$

where D denotes the canonical extension of $\mathcal{D}$ to $L_p(\mathbb{R}_+;Y)$, Here $F = (\partial_t + B)^{1/2}$ in $L_p(\mathbb{R}_+;Y)$, $\mathcal{D}(F) = {}_0H_p^{1/2}(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;\mathcal{D}(\mathcal{B}^{1/2}))$, by Theorem 2.1 and Corollary 2.2, is subject to the assumptions of Theorem 3.4. The function u_1 is defined by

$$u_1(t,y) = \mathrm{e}^{-\mathcal{B}t/2}\mathrm{e}^{-\mathcal{B}^{1/2}y/\sqrt{2}}\psi_0,\ \ \psi_0 := \psi|_{y=0},$$

and ψ_1 by

$$\psi_1 = \psi - \mathrm{e}^{-\mathcal{B}^{1/2}y/\sqrt{2}}\psi_0.$$

Further, $A = -\partial_y^2 + \mathcal{B}$ in $L_p(\mathbb{R}_+;Y)$ with domain

$$\mathcal{D}(A) = H_p^2(\mathbb{R}_+;Y) \cap {}_0H_p^1(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;\mathcal{D}(\mathcal{B})).$$

Finally, the function φ_1 is given by

$$\varphi_1 = \varphi - (D + \mathcal{B}^{1/2}/\sqrt{2})\mathrm{e}^{-\mathcal{B}t/2}\psi_0 + \partial_y \mathrm{e}^{-At}\psi_1|_{y=0}.$$

Note that the traces of ψ_1 and of φ_1 at $t = 0$ are zero by construction. It is now easy to check by means of Theorems 3.1 and 3.4 that each term in this decomposition of u belongs to Z; cf. Theorem 4.1. $\square$

The last result we want to discuss here is for the problem

$$\begin{aligned} \partial_t u - \partial_y^2 u + \mathcal{B}u &= f, \quad t,y>0, \\ \partial_t u|_{y=0} - \partial_y u|_{y=0} + \mathcal{D}u|_{y=0} &= \varphi, \quad t>0, \\ u(0,y) &= \psi(y), \quad y \geqslant 0. \end{aligned} \tag{4.6}$$

It reads as follows:

Theorem 4.3. *Suppose Y is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $p \neq 3$, let $\mathcal{B} \in \mathcal{BIP}(Y)$ be invertible with power-angle $\theta_{\mathcal{B}} < \pi/2$, $\mathcal{D}$ pseudo-sectorial, $\mathcal{D} \in \mathcal{BIP}(\overline{\mathcal{R}(\mathcal{D})})$ with $\theta_{\mathcal{D}} < \pi/2$, and suppose $\mathcal{B}$ and $\mathcal{D}$ commute in the resolvent sense. Let D_B denote the Banach space $\mathcal{D}(\mathcal{B})$ equipped with the graph norm of $\mathcal{B}$, and let $Y_\alpha = D_B(\alpha,p)$, $\alpha = 1/2 - 1/2p$.*

Then the problem (4.6) has exactly one solution

$$u \in Z := H^1_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}_+; Y)) \cap L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; D_B))$$

with

$$\mathcal{D}u|_{y=0} \in W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p))$$

and

$$u|_{y=0} \in W_p^{3/2-1/2p}(\mathbb{R}_+; Y) \cap H^1_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)),$$

if and only if the data f, φ, ψ satisfy the following conditions:

1. $f \in X := L_p(\mathbb{R}_+; L_p(\mathbb{R}_+; Y))$;
2. $\varphi \in W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p))$;
3. $\psi \in W_p^{2-2/p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1 - 1/p, p))$,
4. $\psi_0 := \psi(0) \in D_B(1 - 1/p, p)$ *and* $\psi_0 \in (Y_\alpha, D(\mathcal{D}|_{Y_\alpha}))_{1-1/p,p}$;
5. $\mathcal{D}\psi_0 \in D_B(1/2 - 3/2p, p)$ *and* $\psi_1 := \varphi(0) + \partial_y\psi(0) - \mathcal{D}\psi_0 \in D_{\mathcal{D}}(1/2 - 3/2p, p)$ *provided* $p > 3$.

P r o o f. Suppose $u \in Z$ is a solution of (4.6). Then by Theorem 4.2 we have 1. and 3. as well as

$$\partial_y u|_{y=0} \in V := W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)).$$

The additional properties $\mathcal{D}u|_{y=0} \in V$ and $\partial_t u|_{y=0} \in V$ yield 2. Since the trace of u at $y = 0$ satisfies

$$u|_{y=0} \in H^1_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p)),$$

we obtain 4. and 5. by taking the traces of $u|_{y=0}$, $\partial_t u$ and $\mathcal{D}u$ at $t = 0$, according to Proposition 2.5.

Conversely, let the data f, φ, and ψ with properties 1., 2., 3., 4. and 5. be given. As before we let F denote the operator $F = (\partial_t + \mathcal{B})^{1/2}$ in $L_p(\mathbb{R}_+; Y)$. Then

$$\mathcal{D}(F) = {}_0H_p^{1/2}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; \mathcal{D}(\mathcal{B}^{1/2}))$$

and

$$D_F(1 - 1/p, p) = {}_0W_p^{1/2-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1/2 - 1/2p, p)).$$

Suppose

$$w := u|_{y=0} \in W_p^{1-1/2p}(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_B(1 - 1/2p, p))$$

is already known. Then by Theorem 4.1 we obtain a unique solution u of (4.1), with φ replaced by w, which belongs to Z. Theorem 4.2 then yields $\partial_y u|_{y=0} \in V$.

Denoting the canonical extension of $\mathcal{D}$ to $L_p(\mathbb{R}_+;Y)$ by D, we may write $w = v + z$ where v has traces zero and z solves the problem

$$\dot{z} + (D + B^{1/2})z = \mathrm{e}^{-Bt}(\psi_1 + (D + B^{1/2})\psi_0), \quad z(0) = \psi_0.$$

Then u is of the form $u = u_1 + \mathrm{e}^{-Fy}v$, where u_1 is determined by the data. This then yields

$$\varphi + \partial_y u|_{y=0} = \varphi + \partial_y u_1|_{y=0} - Fv = \varphi_1 - Fv.$$

Inserting this identity into the dynamic boundary condition we obtain

$$\begin{aligned}\partial_t v + Fv + Dv &= \varphi_1 - [\partial_t + D]z \\ &= \varphi_1 + B^{1/2}z - \mathrm{e}^{-Bt}(\psi_1 + (D + B^{1/2})\psi_0 =: \varphi_2,\end{aligned}$$

which means that we have reduced the problem to the integro-differential equation

$$\partial_t v + Fv + Dv = \varphi_2, \quad t > 0,\ v(0) = 0. \tag{4.7}$$

Here the function φ_2 is determined by the data of the problem and has zero trace, by construction.

Defining $G = \partial_t$ in $L_p(\mathbb{R}_+;Y)$ with domain $\mathcal{D}(G) = {}_0H^1_p(\mathbb{R}_+;Y)$ and applying Theorem 2.1 twice we see that the operator $G + F + D$ is invertible in $D_F(\alpha, p)$, for each α. Therefore (4.7) admits a unique solution

$$v \in {}_0W^{3/2-1/2p}_p(\mathbb{R}_+;Y) \cap {}_0H^1_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+;D_B(1 - 1/2p, p)),$$

and $\mathcal{D}v \in {}_0W^{1/2-1/2p}_p(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p))$ holds once we know $\varphi_2 \in D_F(1 - 1/p, p)$.

Now, $\varphi \in V$ and, according to Proposition 2.5, by 2. and 3. we have

$$\psi_2 := \psi_1 + (D + B^{1/2})\psi_0 = \varphi|_{t=0} + \partial_y\psi|_{y=0} + \mathcal{B}^{1/2}\psi_0 \in D_B(1/2 - 3/2p, p),$$

hence, by Proposition 2.6, $\mathrm{e}^{-Bt}\psi_2 \in V$. Thus we obtain $\varphi_2 \in D_F(1-1/p, p)$ provided $z \in W^{1-1/2p}_p(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1-1/2p, p))$. Therefore we have reduced our task to showing that z enjoys the regularity claimed for $w = u|_{y=0}$.

To prove this regularity of z, we employ once more Proposition 2.6, but also Proposition 3.2. Using Proposition 2.6, by 4. and 5. we have $\psi_0 \in D_{\mathcal{D}+\mathcal{B}^{1/2}}(1-1/p, p)$ and $\mathcal{D}\mathrm{e}^{-Bt}\psi_2 \in V$. Therefore Proposition 3.2 yields

$$z \in {}_0W^{3/2-1/2p}_p(\mathbb{R}_+;Y) \cap {}_0H^1_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p)) \cap L_p(\mathbb{R}_+;D_B(1 - 1/2p, p))$$

as well as $\mathcal{D}z \in W^{1/2-1/2p}_p(\mathbb{R}_+;Y) \cap L_p(\mathbb{R}_+;D_B(1/2 - 1/2p, p))$.

Since w is unique, the solution $u \in Z$ of problem (4.6) is also unique, by Theorem 4.1. □

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Author's address: *Jan Prüss*, Fachbereich Mathematik und Informatik, Martin-Luther-Universität Halle-Wittenberg, Theodor-Lieser Str. 5, D-60120 Halle, Germany, e-mail: `anokd@volterra.mathematik.uni-halle.de`.