SMOOTH BIFURCATION FOR A SIGNORINI PROBLEM ON A RECTANGLE

Jan Eisner, České Budějovice, Milan Kučera, Praha, Lutz Recke, Berlin

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Abstract. We study a parameter depending semilinear elliptic PDE on a rectangle with Signorini boundary conditions on a part of one edge and mixed (zero Dirichlet and Neumann) boundary conditions on the rest of the boundary. We describe smooth branches of smooth nontrivial solutions bifurcating from the trivial solution branch in eigenvalues of the linearized problem. In particular, the contact sets of these nontrivial solutions are intervals which change smoothly along the branch. The main tools of the proof are first a certain local equivalence of the unilateral BVP to a system consisting of a corresponding classical BVP and of two scalar equations (which determine the ends of the contact intervals), and secondly an application of the classical Crandall-Rabinowitz type local bifurcation techniques (scaling and application of the Implicit Function Theorem) to that system.

Keywords: Signorini problem, smooth bifurcation, variational inequality, boundary obstacle, Crandall-Rabinowitz type theorem

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1. Introduction

Let $l > 0$, $\Omega := (0,1) \times (0,l)$, $\Gamma_D := (\{0\} \times (0,l)) \cup (\{1\} \times (0,l))$, $\Gamma_U := ((\gamma_1, \gamma_2) \times \{0\}) \subset ((0,1) \times \{0\})$ with $0 < \gamma_1 < \gamma_2 < 1$, and $\Gamma_N := \partial\Omega \setminus (\Gamma_D \cup \Gamma_U)$. We study the Signorini boundary value problem

$$\Delta u + \lambda u + g(\lambda, u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$u \leqslant 0, \quad \frac{\partial u}{\partial \nu} \leqslant 0, \quad u\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_U, \tag{1.3}$$

where λ is a real parameter and $g\colon \Lambda \times \mathbb{R} \to \mathbb{R}$ is a C^2-smooth function, Λ is an open interval containing a given eigenvalue λ_0 of the (nonlinear) eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{1.4}$$

with (1.2), (1.3). We assume that $g(\lambda, 0) = 0$, $\partial g/\partial u(\lambda, 0) = 0$ for all $\lambda \in \Lambda$, and

$$\Big|\frac{\partial g}{\partial \lambda}(\lambda, u)\Big| + \Big|\frac{\partial g}{\partial u}(\lambda, u)\Big| \leqslant C(1 + |u|^q) \text{ for all } (\lambda, u) \in \Lambda \times \mathbb{R}$$

with some $C > 0$ and $q > 2$. Finally, we assume that the contact set

$$\mathcal{A}(u) := \{x \in (\gamma_1, \gamma_2)\colon\ u(x, 0) = 0\}$$

of the eigenfunction $u = u_0$ of the problem (1.4), (1.2), (1.3) corresponding to λ_0 is an interval $\mathcal{A}(u_0) = [\alpha_0, \beta_0]$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$.

Our main result states that, under natural assumptions, there is a smooth branch of nontrivial solutions to the problem (1.1)–(1.3) bifurcating at $(\lambda_0, 0)$ from the branch of trivial solutions and that there are no other nontrivial solutions close to $(\lambda_0, 0)$. Moreover, the contact sets $\mathcal{A}(u)$ of the nontrivial solutions u on this branch are intervals changing C^1-smoothly along the bifurcating branch.

In this contribution we will explain the main ideas of the proofs only, the results with all details and full generality will be published in [2].

2. Main results

Let us introduce a real Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$ by

$$H := \{u \in W^{1,2}(\Omega)\colon\ u = 0 \text{ on } \Gamma_D\},\ \langle u, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}y,$$

and a closed convex subset K of H by

$$K := \{u \in H\colon\ u \leqslant 0 \text{ on } \Gamma_U\}.$$

The weak formulations of the problems (1.1)–(1.3) and (1.4), (1.2), (1.3) are the variational inequalites

$$u \in K\colon \int_\Omega \nabla u \cdot \nabla(\varphi - u) - [\lambda u + g(\lambda, u)](\varphi - u) \,\mathrm{d}x\,\mathrm{d}y \geqslant 0 \quad \text{for all } \varphi \in K \tag{2.1}$$

and

$$u \in K\colon \int_\Omega \nabla u \cdot \nabla(\varphi - u) - \lambda u(\varphi - u) \,\mathrm{d}x\,\mathrm{d}y \geqslant 0 \text{ for all } \varphi \in K, \tag{2.2}$$

respectively. Besides the unilateral boundary value problem (1.1)–(1.3) we consider the "corresponding" non-unilateral boundary value problem (1.1), (1.2),

$$u = 0 \text{ on } I_{\alpha,\beta}, \quad \partial_y u = 0 \text{ on } E_{\alpha,\beta}, \tag{2.3}$$

where $\gamma_1 < \alpha < \beta < \gamma_2$ will be properly chosen later and

$$\begin{aligned} I_{\alpha,\beta} &:= \{(x,0) \in \Gamma_U \colon \ \alpha < x < \beta\} = (\alpha,\beta) \times \{0\}, \\ E_{\alpha,\beta} &:= \{(x,0) \in \Gamma_U \colon \ \gamma_1 < x < \alpha \text{ or } \beta < x < \gamma_2\} = \Gamma_U \setminus \overline{I_{\alpha,\beta}}. \end{aligned}$$

Let us fix a couple $(\alpha_0, \beta_0) \in \mathbb{R}^2$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$, and define $\delta := \frac{1}{3} \min\{\alpha_0 - \gamma_1, \beta_0 - \alpha_0, \gamma_2 - \beta_0\}$ and

$$D := \{(\alpha,\beta) \colon \ |\alpha - \alpha_0| < \delta, \, |\beta - \beta_0| < \delta\}.$$

For $(\alpha,\beta) \in D$ we introduce diffeomorphisms $(x,y) \mapsto (\xi_{\alpha,\beta}(x), y)$ of $\overline{\Omega}$ onto itself which map $I_{\alpha,\beta}$ onto I_{α_0,β_0} and $E_{\alpha,\beta}$ onto E_{α_0,β_0}. The corresponding transformations

$$(\Phi_{\alpha,\beta} u)(x,y) := u(\xi_{\alpha,\beta}(x), y)$$

of functions transform the boundary value problem (1.1), (1.2), (2.3), which has (α,β)-independent coefficients in the equation but (α,β)-dependent boundary conditions, into a boundary value problem, which has (α,β)-dependent coefficients in the equation but (α,β)-independent boundary conditions.

Let us take a C^∞-smooth function $\chi \colon \ [0,\infty) \to [0,1]$ such that

$$\chi(r) = 1 \text{ for } 0 \leqslant r \leqslant \delta/2, \quad \chi(r) = 0 \text{ for } r \geqslant \delta.$$

Further, define functions $X^{(-1/2)}, Y^{(-1/2)} \colon \ \overline{\Omega} \to \mathbb{R}$ by

$$\begin{aligned} X^{(-1/2)}(x,y) &= X^{(-1/2)}(\alpha_0 + r\cos\omega, r\sin\omega) := \chi(r) r^{-1/2} \sin\frac{\omega}{2}, \\ Y^{(-1/2)}(x,y) &= Y^{(-1/2)}(\beta_0 + r\cos\omega, r\sin\omega) := \chi(r) r^{-1/2} \sin\frac{\omega}{2}, \end{aligned} \tag{2.4}$$

where r is the distance of $(x,y) \in \overline{\Omega}$ from $(\alpha_0, 0)$ or $(\beta_0, 0)$, respectively, ω is the angle measured anticlockwise or clockwise from the segments $\overline{(x,y),(\alpha_0,0)}$ or $\overline{(x,y),(\beta_0,0)}$, respectively, to I_{α_0,β_0}.

Let $X_{\alpha,\beta}, Y_{\alpha,\beta} \in W^{1,2}(\Omega)$ be the weak solutions to the boundary value problems $-\Phi^*_{\alpha,\beta} \Delta \Phi_{\alpha,\beta} u = f$ in Ω with $f = \Delta X^{(-1/2)}$ and $f = \Delta Y^{(-1/2)}$, respectively, and with the boundary conditions $u = 0$ on $\Gamma_D \cup I_{\alpha_0,\beta_0}$, $\partial_\nu u = 0$ on $\Gamma_N \cup E_{\alpha_0,\beta_0}$. Further, denote $\overline{X}_{\alpha,\beta} := X_{\alpha,\beta} + X^{(-1/2)}$, $\overline{Y}_{\alpha,\beta} := Y_{\alpha,\beta} + Y^{(-1/2)}$ and

$$\begin{aligned}
a_{11} &:= \int_\Omega \overline{X}_{\alpha_0,\beta_0} \partial_x u_0 \partial_\alpha \xi_{\alpha_0,\beta_0} + u_0 \left(\partial_\alpha X_{\alpha,\beta} - \overline{X}_{\alpha_0,\beta_0} \partial_\alpha \xi'_{\alpha_0,\beta_0}\right) \,\mathrm{d}x \,\mathrm{d}y, \\
a_{12} &:= \int_\Omega \overline{X}_{\alpha_0,\beta_0} \partial_x u_0 \partial_\beta \xi_{\alpha_0,\beta_0} + u_0 \left(\partial_\beta X_{\alpha,\beta} - \overline{X}_{\alpha_0,\beta_0} \partial_\beta \xi'_{\alpha_0,\beta_0}\right) \,\mathrm{d}x \,\mathrm{d}y, \\
a_{21} &:= \int_\Omega \overline{Y}_{\alpha_0,\beta_0} \partial_x u_0 \partial_\alpha \xi_{\alpha_0,\beta_0} + u_0 \left(\partial_\alpha Y_{\alpha,\beta} - \overline{Y}_{\alpha_0,\beta_0} \partial_\alpha \xi'_{\alpha_0,\beta_0}\right) \,\mathrm{d}x \,\mathrm{d}y, \\
a_{22} &:= \int_\Omega \overline{Y}_{\alpha_0,\beta_0} \partial_x u_0 \partial_\beta \xi_{\alpha_0,\beta_0} + u_0 \left(\partial_\beta Y_{\alpha,\beta} - \overline{Y}_{\alpha_0,\beta_0} \partial_\beta \xi'_{\alpha_0,\beta_0}\right) \,\mathrm{d}x \,\mathrm{d}y.
\end{aligned}$$

Our main result is the following

Theorem 2.1. *Let (λ_0, u_0) satisfy (2.2), $\mathcal{A}(u_0) = [\alpha_0, \beta_0]$, $\|u_0\| = 1$. Assume that there is $d > 0$ such that*

$$\partial_y u_0 > 0 \text{ on } I_{\alpha_0,\beta_0} \cup ((0,1) \times (0,d)), \tag{2.5}$$

λ_0 is simple as an eigenvalue of the BVP (1.4), (1.2), (2.3) with $(\alpha, \beta) = (\alpha_0, \beta_0)$,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0. \tag{2.6}$$

Then there exist $s_0 > 0$ and mappings $\hat{\lambda}, \hat{\alpha}, \hat{\beta} \colon [0, s_0) \to \mathbb{R}$ and $\hat{u} \colon [0, s_0) \to H$ with $\hat{\lambda}(0) = \lambda_0$, $\hat{u}(0) = 0$, $\hat{\alpha}(0) = \alpha_0$ and $\hat{\beta}(0) = \beta_0$ such that the following assertions hold:

(i) *For all $s \in (0, s_0)$ the pair $(\lambda, u) = (\hat{\lambda}(s), \hat{u}(s))$ is a solution to (2.1) with $\mathcal{A}(\hat{u}(s)) = [\hat{\alpha}(s), \hat{\beta}(s)]$, $\hat{u}(s) \in W^{2,p}(\Omega)$ for all $p \geqslant 2$, and there exists $\varepsilon > 0$ such that*

$$\partial_y \hat{u}(s) > 0 \text{ on } I_{\hat{\alpha}(s),\hat{\beta}(s)} \cup ((0,1) \times (0,\varepsilon)). \tag{2.7}$$

(ii) *There exists a C^1-smooth map $\hat{v} \colon [0, s_0) \to H$ such that $\hat{v}(0) = 0$ and*

$$\hat{u}(s) = s\Phi_{\hat{\alpha}(s),\hat{\beta}(s)} (u_0 + \hat{v}(s)) \text{ for all } s \in (0, s_0).$$

(iii) *The functions $\hat{\lambda}$, $\hat{\alpha}$, $\hat{\beta}$ are C^1-smooth from $[0, s_0)$ into $\mathbb{R}$ and the map $\hat{u}$ is continuous from $[0, s_0)$ into H and C^1-smooth from $[0, s_0)$ into $L^2(\Omega)$.*

(iv) *There exists $\eta > 0$ such that for any solution $(\lambda, u) \in \Lambda \times (H \setminus \{0\})$ to (2.1) with $|\lambda - \lambda_0| + \|u\| + \|u/\|u\| - u_0\| < \eta$ there is $s \in (0, s_0)$ with $u = \hat{u}(s)$ and $\lambda = \hat{\lambda}(s)$.*

It is possible to find simple examples where the assumption (2.5) is fulfilled (see Fig. 1 and [2, Example 2.6] for details). The assumption (2.6) is generically fulfilled, but in concrete situations it must be verified numerically.

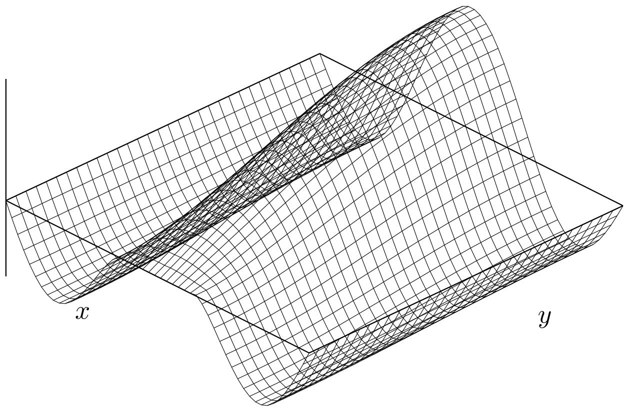

Figure 1. The eigenfunction u_0 with $l = 0.27$, $\lambda_0 = 99.8$, $\alpha_0 = 0.38$ and $\beta_0 = 0.62$.

3. Sketch of the proof

The main idea of the proof of Theorem 2.1 is to show that the variational inequality (2.1) is equivalent in a neighbourhood of the bifurcation point $(\lambda_0, 0)$ to a C^1-smooth operator equation, to use a scaling and a Liapunov-Schmidt reduction and to apply the Implicit Function Theorem to the scaled equation.

Define a mapping $F\colon \mathbb{R} \times H \to H$ by

$$\langle F(\lambda, u), \varphi\rangle := -\int_\Omega \nabla u \cdot \nabla\varphi - [\lambda u + g(\lambda, u)]\varphi \,\mathrm{d}x\,\mathrm{d}y \text{ for all } \varphi \in H.$$

Further, denote

$$H_0 := \{u \in H\colon\ u = 0 \text{ in } I_{\alpha_0,\beta_0}\}, \quad H_1 := \{u \in H_0\colon\ \langle u, u_0\rangle = 0\},$$
$$v_{\alpha,\beta} := \Phi_{\alpha,\beta}(X_{\alpha,\beta} + X^{(-1/2)}),\ w_{\alpha,\beta} := \Phi_{\alpha,\beta}(Y_{\alpha,\beta} + Y^{(-1/2)}) \text{ for } (\alpha,\beta) \in D.$$

It is possible to show (see [2, Theorem 3.1] for details) that for any $\eta > 0$ there exists $\zeta > 0$ such that for any couple $(\lambda, u) \in \Lambda \times H$ satisfying (2.1), $\|u\| \neq 0$ and $\|u\| + \|u/\|u\| - u_0\| + |\lambda - \lambda_0| < \zeta$, there exists $(s, v, \alpha, \beta) \in \mathbb{R} \times H_1 \times D$ with $s > 0$,

$s + \|v\| + |\alpha - \alpha_0| + |\beta - \beta_0| < \eta$ such that $\mathcal{A}(u) = [\alpha, \beta]$ and (s, v, α, β) satisfies

$$\langle F(\lambda, s\Phi_{\alpha,\beta}(u_0 + v)), \Phi_{\alpha,\beta}\varphi\rangle = 0 \quad \text{for any } \varphi \in H_0, \tag{3.1}$$

$$\int_\Omega [\lambda s\Phi_{\alpha,\beta}(u_0 + v) + g(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))]v_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{3.2}$$

$$\int_\Omega [\lambda s\Phi_{\alpha,\beta}(u_0 + v) + g(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))]w_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

$$u = s\Phi_{\alpha,\beta}(u_0 + v). \tag{3.3}$$

And vice versa, for any $(s, \lambda, v, \alpha, \beta) \in \mathbb{R}^2 \times H_1 \times D$ satisfying (3.1), (3.2), $s > 0$, $s + \|v\| + |\lambda - \lambda_0| + |\alpha - \alpha_0| + |\beta - \beta_0| < \zeta$, the couple (λ, u) with u from (3.3) satisfies (2.1), $\|u - u_0\| < \eta$ and $\mathcal{A}(u) = [\alpha, \beta]$.

Roughly speaking, the structure of the system (3.1), (3.2) is as follows: Because the codimension of H_1 in H_0 is one, there is a hope to solve (3.1) with respect to λ and v. Putting this solution, which depends on α, β and s, into (3.2), one can hope to solve the resulting two scalar equations with respect to α and β.

Let us explain where the two scalar equations (3.2) come from. It is known that solutions u to (2.1) need to be C^1-smooth on $\Omega \cup \Gamma_U$, contrary to those to (3.1). Hence, the conditions (3.2) should imply this additional smoothness of u and v, respectively. In fact, they choose the proper α, β such that the solution v to (3.1) (and hence also u given by (3.3)) is C^1. In order to explain this in more detail, let us define, for any fixed $(\alpha, \beta) \in D$, functions $X_\alpha^{(1/2)}, Y_\beta^{(1/2)} \colon \overline{\Omega} \to \mathbb{R}$ by

$$X_\alpha^{(1/2)}(\alpha + r\cos\omega, r\sin\omega) := \chi(r)r^{1/2}\sin\frac{\omega}{2},$$
$$Y_\beta^{(1/2)}(\beta + r\cos\omega, r\sin\omega) := \chi(r)r^{1/2}\sin\frac{\omega}{2},$$

similarly to (2.4). Let u be a weak solution of the boundary value problem $-\Delta u = f$ in Ω with (1.2), (2.3), $f \in L^p(\Omega)$, $p > 2$ and $p \neq 4$. It follows from [4, Theorem 2] that

$$u = \tilde{u} + K^1_{\alpha,\beta}(f)X_\alpha^{(1/2)} + K^2_{\alpha,\beta}(f)Y_\beta^{(1/2)},$$

where $\tilde{u} \in W^{2,p}(\Omega)$. The so-called stress intensity coefficients $K^1_{\alpha,\beta}(f)$ and $K^2_{\alpha,\beta}(f)$ can be calculated as

$$K^1_{\alpha,\beta}(f) = -\frac{2}{\pi}\int_\Omega f v_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y,\ K^2_{\alpha,\beta}(f) = -\frac{2}{\pi}\int_\Omega f w_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y$$

([2, Lemma 3.6]). Let us emphasize that the functions $X_\alpha^{(1/2)}$ and $Y_\beta^{(1/2)}$ belong neither to $W^{2,2}(\Omega)$ nor to $C^1(\overline{\Omega})$, because of the singularity in the first derivatives at $(\alpha, 0)$ or $(\beta, 0)$, respectively. In particular,

$$\partial_x X_\alpha^{(1/2)}(\alpha-, 0) = -\infty, \quad \partial_x Y_\beta^{(1/2)}(\beta+, 0) = +\infty.$$

Therefore we have $u \in C^1(\overline{\Omega})$ if and only if $K^1_{\alpha,\beta}(f) = K^2_{\alpha,\beta}(f) = 0$. Putting $f = \lambda u + g(\lambda, u)$ we get $u \in C^1(\overline{\Omega})$ if and only if

$$\int_\Omega (\lambda u + g(\lambda, u)) v_{\alpha,\beta} \,\mathrm{d}x \,\mathrm{d}y = \int_\Omega (\lambda u + g(\lambda, u)) w_{\alpha,\beta} \,\mathrm{d}x \,\mathrm{d}y = 0.$$

This is the condition (3.2), if (3.3) holds.

The proof of Theorem 2.1 consists of several steps. One step is to show that the solutions to (2.1) which are sufficiently close to a bifurcation point have as their contact set an interval ([2, Lemma 3.13]). Another step is to show that the smooth solutions to (3.1) satisfy the sign conditions (1.3). In fact, it is possible to prove that those solutions satisfy (2.7) ([2, Lemma 3.14]). Further, one has to show that the problem (3.1), (3.2) is equivalent to an operator equation with a C^1-smooth operator from $\mathbb{R}^4 \times H_1$ (where $(\lambda, \alpha, \beta, s, v)$ belongs) into $H_0 \times \mathbb{R}^2$ ([2, Theorem 3.1]). The final step is to divide (3.1), (3.2) by s and solve the resulting system with respect to $(\lambda, \alpha, \beta, v)$ by means of the Implicit Function Theorem (close to its solution $\lambda = \lambda_0, \alpha = \alpha_0, \beta = \beta_0, s = 0, v = 0$). To this end one has to use (2.6) in order to show that the linearized with respect to $(\lambda, \alpha, \beta, v)$ system generates an isomorphism between $\mathbb{R}^3 \times H_1$ and $H_0 \times \mathbb{R}^2$.

Let us mention that regularity properties of solutions to variational inequalities and boundary value problems (see e.g. [3]–[7]) play an essential role in the complete proof given in [2].

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Authors' addresses: *Jan Eisner*, Institute of Mathematics and Biomathematics, Faculty of Science, University of South Bohemia, Branišovská 31, 370 05 České Budějovice, Czech Republic, e-mail: `jeisner@prf.jcu.cz`; *Milan Kučera*, Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `kucera@math.cas.cz`; *Lutz Recke*, Institute of Mathematics, Humboldt University of Berlin, Unter den Linden 6, 10099 Berlin, Germany, e-mail: `recke@mathematik.hu-berlin.de`.