On the weak solutions to the compressible Euler and Euler/Fourier systems

Eduard Feireisl
joint work with E.Chiodaroli (Lausanne) and O.Kreml (Praha)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Recent Advances in PDE's and Applications, Levico Terme, February 16 - 21, 2014

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Equations of a barotropic gas

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Pressure-density state equation

$$p = p(\varrho), \ p(\varrho) = a\varrho^\gamma$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0$$

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \boxed{\Delta\vartheta} = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Global existence of weak solutions

1-D case

Existence of global-in-time bounded weak solutions via *compensated compactness*

DiPerna [1983], Chen, P.L. Lions, Perthame, Souganidis etc.

2,3-D cases

Existence of *infinitely many* global-in-time bounded weak solutions via *convex integration*

DeLellis, Székelyhidi [2008], Chen, Chiodaroli, Kreml, EF etc.

Admissibility criteria for compressible Euler system

Total energy

$$E(t,x) = \frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho),\ \ H(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2}\ \mathrm{d}z$$

Energy balance (differential form)

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p\mathbf{u}) \boxed{\leq} 0$$

Energy balance (integral form)

$$\partial_t \int_\Omega E\ \mathrm{d}x \leq 0,\ \ \int_\Omega E(t)\ \mathrm{d}x \leq E_0 \text{ for any } t > 0$$

Dissipative solutions

Relative “entropy” (energy)

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)$$

Relative entropy inequality

$$\int_\Omega \mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(\tau, \cdot)\ \mathrm{d}x \leq \int_\Omega \left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(0, \cdot)\ \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\ \mathrm{d}x\ \mathrm{d}t$$

Remainder

Remainder in the relative entropy inequality

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})$$

$$= \left[\varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) + \Big(p(r) - p(\varrho) \Big) \mathrm{div}_x \mathbf{U} \right]$$

$$+ \Big[(r - \varrho) \partial_t H'(r) + (r\mathbf{U} - \varrho \mathbf{u}) \cdot \nabla_x H'(r) \Big]$$

Some properties of weak and dissipative solutions

Weak strong uniqueness

Admissible weak solutions are dissipative - the energy inequality implies the relative energy inequality. Strong solutions are unique in the class of admissible weak solutions - weak and strong solutions emanating from the same initial data coincide as long as the latter exists.

Global existence

For given initial data, there exist (infinitely many) weak solutions. For any density distribution ϱ_0, there is a velocity field $\mathbf{u}_0$ such that the compressible Euler system admits (infinite many) admissible weak solutions.

Riemann problem

Riemann data

$$\varrho_0 = \begin{cases} \varrho_L \text{ for } x_1 \leq 0 \\ \varrho_R \text{ for } x_1 > 0 \end{cases}$$

$$u_0^1 = \begin{cases} u_L^1 \text{ for } x_1 \leq 0 \\ u_R^1 \text{ for } x_1 > 0 \end{cases}$$

Second velocity component

$$u_0^2 \equiv 0$$

Ill posedness - Chiodaroli, DeLellis, Kreml

There exist infinitely many admissible weak solutions for *certain* 2D Riemann problem. There exist infinitely many admissible weak solutions that emanate from *certain* Lipschitz initial data.

Stability of rarefaction waves

Almost regular solutions

$$\varrho,\ \mathbf{u} \in W^{1,\infty}_{\rm loc}((0,T)\times R^N)\ \cap L^\infty(0,T; W^{1,1}_{\rm loc} R^N))$$

Boundedness of the velocity gradient

$$\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \geq -M\mathbb{I}$$

Uniqueness

The solution ϱ, $\mathbf{u}$ is unique in the class of bounded admissible weak solutions. $1-D$ rarefaction waves are unique as solutions of the multi-D Euler system.

Another application of the relative entropy

Problematic term

$$(\mathbf{u} - \mathbf{U}) \cdot (\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \boxed{\geq} 0$$

Pressure convexity

$$(p(\varrho) - p'(r)(\varrho - r) - p(r)) \, \mathrm{div}_x \mathbf{U} \boxed{\geq} 0$$

Maximal dissipation criterion?

Energy dissipation rate (entropy production rate)

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) + p(\varrho) \right) \mathbf{u} \right] = -\sigma$$

$$\sigma \boxed{\geq} 0$$

Criterion à la Dafermos 1974

Admissible solutions should "maximize" the energy dissipation rate σ

Šťastné narozeniny Hugo !