On the well-posedness of certain problems in fluid mechanics

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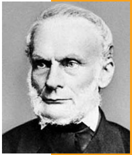
Bialka Tatrzanska, February 24 - 28, 2014

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Thermodynamics



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Well posedness - classical sense

Jacques Hadamard,
[1865 - 1963]

Existence

Given problem is solvable for any choice of (admissible) data

Uniqueness

Solutions are uniquely determined by the data

Stability

Solutions depend continuously on the data

Well posedness - modern way

Jacques-Louis Lions,
[1928 - 2001]

Approximations

Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically

Uniform bounds

Approximate solutions possess uniform bounds depending solely on the data

Stability

The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

Abstract conservation laws

System of equations (conservation laws)

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F}(\mathbf{U}) = 0,$$

$\mathbf{U}$ . . . . state variable
$\mathbb{F}$ . . . . flux

"Entropies"

$$\partial_t E_i(\mathbf{U}) + \mathrm{div}_x \mathbf{F}_{E_i}(\mathbf{U}) = \boxed{(\leq)}\, 0, \ i = 1, 2, \dots$$

E_i . . . . entropy
$\mathbf{F}_i$ . . . . entropy flux

A priori bounds

$$\int E_i(\mathbf{U}) \ \mathrm{d}x \text{ bounded in terms of the initial data}, \ i = 1, 2, \dots$$

Weak vs. strong solutions

Lack of regularity

- bounds available only in L^p (L^∞)
- presence of oscillations
- discontinuities (shocks) appearing in finite time even for initial states

Weak solutions

$$\int_\Omega \mathbf{U} \cdot \varphi(\tau_2, \cdot) - \mathbf{U}\varphi(\tau_1, \cdot) \ \mathrm{d}x$$

$$= \int_{\tau_1}^{\tau_2} \int_\Omega \left[\mathbf{U} \cdot \partial_t \varphi + \mathbb{F}(\mathbf{U}) : \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

Weak continuity

$t \mapsto \mathbf{U}(t, \cdot)$ weakly continuous

Compensated compactness - DiPerna, Tartar

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

$$\partial_t E_i + \mathrm{div}_x \mathbf{F}_i \leq 0, \ i = 1, 2, \ldots$$

Nonlinear constitutive equations

$$\mathbb{F} = \mathbb{F}(\mathbf{U}), \ E_i = E_i(\mathbf{U}), \ \mathbf{F}_i = \mathbf{F}_i(\mathbf{U}), \ i = 1, 2, \ldots$$

Compensated compactness

- linear field equations yield constraints on possible oscillations described by Young measure
- nonlinear constrained imposed by constitutive equations reduce the Young measures to Dirac masses (no oscillations)

Basic ideas of Young measures

Oscillatory sequence - convergence in the sense of averages

$$U_n \to U \text{ weakly-(*)} \Leftrightarrow \int_B U_n \to \int_B U \text{ for any } B$$

Young measure associated to $\{U_n\}$

$$< \sigma_y(U), f >= \lim_{r \to 0} \frac{1}{|B_r(y)|} \left[\lim_{n \to \infty} \int_{B_r} f(U_n) \right]$$

Strong (pointwise a.a.) convergence

$$U_n \to U \text{ for a.a.} \Leftrightarrow \sigma_y(U) = \delta_{U(y)} \text{ for a.a. } y$$

Convex integration

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

Replacing constitutive equation

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \quad \boxed{\text{“implicit”}}$$

$$\Lambda(\mathbf{U}, \mathbb{F}) \text{ convex}, \ \Lambda(\mathbf{U}, \mathbb{F}) \geq E(\mathbf{U})$$

Relaxation of constitutive equation

$$E(\mathbf{U}) \leq \Lambda(\mathbf{U}, \mathbb{F}) \leq e, \ e \text{ given “energy”}$$

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \Leftarrow E(\mathbf{U}) = e$$

Equations of a barotropic gas

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Pressure-density state equation

$$p = p(\varrho), \ p(\varrho) = a\varrho^\gamma$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \varrho\mathbf{u}(0,\cdot) = (\varrho\mathbf{u})_0$$

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \boxed{\Delta\vartheta} = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Global existence of weak solutions

1-D case

Existence of global-in-time bounded weak solutions via *compensated compactness*
DiPerna [1983], Chen, P.L. Lions, Perthame, Souganidis etc.

2,3-D cases

Existence of *infinitely many* global-in-time bounded weak solutions via *convex integration*
DeLellis, Székelyhidi [2008], Chen, Chiodaroli, Kreml, EF etc.

Convex integration - DeLellis and Shékelyhidi

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3}|\mathbf{v}|^2 \mathbb{I} \right) + \nabla_x \Pi = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbf{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3}|\mathbf{v}|^2 \mathbb{I},\ \mathbb{U} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Convex integration continued

Implicit constitutive relation

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]$$

$$\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2$$

$$\boxed{\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}|\mathbf{v}|^2\mathbb{I}$$

Convex integration - subsolutions

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e \text{ in } (0, T)$$

Oscillatory lemma

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0,\ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon,\ \mathbf{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(V)$$

$$\lambda_{\max}\left[(\mathbf{v}+\mathbf{w}_\varepsilon)\otimes(\mathbf{v}+\mathbf{w}_\varepsilon) - (\mathbb{U}+\mathbb{V}_\varepsilon)\right] < e$$

Energy

$$\liminf_{\varepsilon\to 0} \int_V \left(|\mathbf{v}+\mathbf{w}_\varepsilon|^2\right) \geq \int_V |\mathbf{v}|^2 + c\int_V \left(e - \frac{1}{2}|\mathbf{v}|^2\right)^\alpha$$

Non-constant coefficients

Convex set

$$\frac{1}{2}\frac{1}{r(t,x)}|\mathbf{v}+\mathbf{q}(t,x)|^2$$

$$\leq \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{q}(t,x))\otimes(\mathbf{v}+\mathbf{q}(t,x))}{r(t,x)} - \mathbb{U}\right] < e \text{ in } (0,T)$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t(\mathbf{v}+\mathbf{q}) + \mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{q})\otimes(\mathbf{v}+\mathbf{q})}{r}\right) - \frac{1}{3}\nabla_x\left(\frac{|\mathbf{v}+\mathbf{q}|^2}{r}\right) = 0$$

Energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{q}(t,x)|^2}{r(t,x)} = e(t,x)$$

Applications to the Euler system

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + p(\varrho)) = 0$$

Energy

$$e = \chi(t) - H(\varrho) - \frac{3}{2} \partial_t \Psi$$

Admissibility criteria for compressible Euler system

Total energy

$$E(t,x) = \frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho),\ \ H(\varrho) = \varrho\int_1^{\varrho}\frac{p(z)}{z^2}\ \mathrm{d}z$$

Energy balance (differential form)

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p\mathbf{u}) \boxed{\leq} 0$$

Energy balance (integral form)

$$\partial_t \int_\Omega E\ \mathrm{d}x \leq 0,\ \int_\Omega E(t)\ \mathrm{d}x \leq E_0 \text{ for any } t > 0$$

Dissipative solutions

Relative "entropy" (energy)

$$\mathcal{E}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)$$

$$= \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)$$

Relative entropy inequality

$$\int_\Omega \mathcal{E}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)(\tau, \cdot)\ \mathrm{d}x \leq \int_\Omega \left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)(0, \cdot)\ \mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\ \mathrm{d}x\ \mathrm{d}t$$

Remainder

Remainder in the relative entropy inequality

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})$$
$$= \left[\varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) + \Big(p(r) - p(\varrho) \Big) \mathrm{div}_x \mathbf{U} \right]$$
$$+ \Big[(r - \varrho) \partial_t H'(r) + (r\mathbf{U} - \varrho \mathbf{u}) \cdot \nabla_x H'(r) \Big]$$

Some properties of weak and dissipative solutions

Weak strong uniqueness

Admissible weak solutions are dissipative - the energy inequality implies the relative energy inequality. Strong solutions are unique in the class of admissible weak solutions - weak and strong solutions emanating from the same initial data coincide as long as the latter exists.

Global existence

For given initial data, there exist (infinitely many) weak solutions. For any density distribution ϱ_0, there is a velocity field $\mathbf{u}_0$ such that the compressible Euler system admits (infinite many) admissible weak solutions.

Riemann problem

Riemann data

$$\varrho_0 = \begin{cases} \varrho_L \text{ for } x_1 \leq 0 \\ \varrho_R \text{ for } x_1 > 0 \end{cases}$$

$$u_0^1 = \begin{cases} u_L^1 \text{ for } x_1 \leq 0 \\ u_R^1 \text{ for } x_1 > 0 \end{cases}$$

Second velocity component

$$u_0^2 \equiv 0$$

Ill posedness - Chiodaroli, DeLellis, Kreml

There exist infinitely many admissible weak solutions for *certain* 2D Riemann problem. There exist infinitely many admissible weak solutions that emanate from *certain* Lipschitz initial data.

Stability of rarefaction waves

Almost regular solutions

$$\varrho,\ \mathbf{u} \in W^{1,\infty}_{\rm loc}((0,T)\times R^N)\ \cap L^\infty(0,T;W^{1,1}_{\rm loc}R^N))$$

Boundedness of the velocity gradient

$$\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \geq -M\mathbb{I}$$

Uniqueness

The solution ϱ, $\mathbf{u}$ is unique in the class of bounded admissible weak solutions. $1-D$ rarefaction waves are unique as solutions of the multi-D Euler system.

Another application of the relative entropy

Problematic term

$$(\mathbf{u} - \mathbf{U}) \cdot (\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \boxed{\geq} 0$$

Pressure convexity

$$(p(\varrho) - p'(r)(\varrho - r) - p(r)) \, \mathrm{div}_x \mathbf{U} \boxed{\geq} 0$$

Maximal dissipation criterion?

Energy dissipation rate (entropy production rate)

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho) + p(\varrho) \right) \mathbf{u} \right] = -\sigma$$

$$\sigma \boxed{\geq} 0$$

Criterion à la Dafermos 1974

Admissible solutions should "maximize" the energy dissipation rate σ