Stability issues in the theory of complete fluid systems

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The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Do we need analysis?

An example - variable density flow in porous media

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho c(\varrho)) + \mathrm{div}_x(\varrho c(\varrho)\mathbf{u}) - \mathrm{div}_x(\varrho D \nabla_x c(\varrho)) = 0$$

$$\mathbf{u} = \nabla_x p - \varrho \mathbf{g}$$

Compatibility

$$c = \log(\varrho) \ \Rightarrow \Delta p = \Delta \varrho + |\nabla_x \log(\varrho)|^2 + \mathrm{div}_x(\varrho \mathbf{g})$$

periodic boundary conditions $\Rightarrow \ \varrho = \mathrm{const}$!

Navier-Stokes-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Internal energy equation

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + e(\varrho, \vartheta) \right] \ \mathrm{d}x = 0$$

Constitutive relations (weak form)

Newton's law

$$\varrho \mathbb{S}(\nabla_x \mathbf{u}) = \varrho \left[\mu \left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right]$$

Fourier's law

$$\varrho \mathbf{q} = -\varrho \kappa(\vartheta) \nabla_x \vartheta = \varrho \nabla_x K(\vartheta)$$

State equation

$$p(\varrho, \vartheta) = \varrho^\gamma + a \varrho \vartheta$$

Navier-Stokes-Fourier system (weak form)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Thermal energy balance

$$c_v \partial_t(\varrho\vartheta) + c_v \mathrm{div}_x(\varrho\vartheta\mathbf{u}) - \mathrm{div}_x(\kappa \nabla_x \vartheta) \boxed{\geq} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - a\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + c_v \varrho\vartheta \right] \mathrm{d}x \boxed{\leq} 0$$

Renormalization

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Renormalized thermal energy balance

$$c_v \partial_t (\varrho H(\vartheta)) + c_v \mathrm{div}_x (\varrho H(\vartheta)\mathbf{u}) - \mathrm{div}_x \Big(H'(\vartheta)\kappa(\vartheta)\nabla_x \vartheta \Big) \geq$$

$$H'(\vartheta)\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - H''(\vartheta)\kappa(\vartheta)|\nabla_x \vartheta|^2 - aH'(\vartheta)\vartheta\varrho \mathrm{div}_x \mathbf{u}$$

Entropy balance

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta)\mathbf{u}) - \mathrm{div}_x \left(\frac{\kappa(\vartheta)}{\vartheta} \nabla_x \vartheta \right)$$

$$\geq \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla_x \vartheta|^2 \right)$$

Compactness of the density

Density oscillations

$$\partial_t \overline{\varrho \log(\varrho)} + \mathrm{div}_x \left(\overline{\varrho \log(\varrho)} \right) \mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

$$\partial_t (\varrho \log(\varrho)) + \mathrm{div}_x (\varrho \log(\varrho)) \mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

Effective viscous flux

$$0 \leq \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\ \varrho = \overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}$$

Biting limit of the temperature

$$\lim K_\alpha(\vartheta_\varepsilon) = K_\alpha(\vartheta),\ K_\alpha \nearrow K$$

Relative entropy (modulated energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right)$$

$$= \int_{\Omega} \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \Big(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta$$

$$+ \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Weak solutions - summary

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Compatibility

Regular weak solutions are strong solutions

Weak-strong uniqueness

Weak and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

Numerical method

Special choice of boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \ \mathbf{curl}[\mathbf{u}]\times\mathbf{n}|_{\partial\Omega} = 0$$

Nédelec FE's

$$V_h(\Omega) = \left\{\mathbf{v}\ \middle|\ \mathbf{v} = \mathbf{a} + g\mathbf{x},\ \mathbf{v}\in L^2_{\mathrm{div}}(\Omega; R^3)\right\}$$

$$W_h(\Omega) = \left\{\mathbf{w}\ \middle|\ \mathbf{w} = \mathbf{d} + hG(x),\ \nabla_x G + \nabla_x^t G = 0\right\}$$

Upwind discretization of convective terms

$$\langle \mathrm{div}_x(h\mathbf{u});\varphi\rangle_K \approx \int_{\partial K} h(\cdot - \mathbf{u}\cdot\mathbf{n})\mathbf{u}\cdot\mathbf{n}[[\varphi]]\ \mathrm{dS}_x \equiv \int_{\partial K} \mathrm{Up}[h,\mathbf{u}][[\varphi]]\ \mathrm{dS}_x$$

Numerical scheme [Karlsen-Karper], I

Equation of continuity

$$\int_\Omega D_t \varrho_h^k \varphi_h \ \mathrm{d}x \equiv \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \varphi_h \ \mathrm{d}x = \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k] \ [[\varphi_h]] \ \mathrm{dS}_x$$

for all $\varphi_h \in Q_h(\Omega)$

Momentum equation

$$\int_\Omega D_t \left(\varrho_h^k \widehat{\mathbf{u}}_h^k \right) \cdot \varphi_h \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \ \cdot [[\widehat{\varphi}_h]] \mathrm{dS}_x$$

$$- \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \varphi_h \ \mathrm{d}x$$

$$= - \int_\Omega \left(\mu \mathbf{curl}^*[\mathbf{u}_h^k] \cdot \mathbf{curl}^*[\varphi_h] + (\lambda + \mu) \mathrm{div}_x \mathbf{u}_h^k \mathrm{div}_x \varphi_h \right) \ \mathrm{d}x$$

for all $\varphi_h \in V_h(\Omega)$

Numerical scheme [Karlsen-Karper], II

Energy equation

$$\int_\Omega D_t\left(\varrho_h^k \vartheta_h^k\right)\varphi_h \,\mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\vartheta_h^k, \mathbf{u}_h^k]\ [[\varphi_h]]\mathrm{dS}_x$$

$$+\sum_{\Gamma\in\Gamma_h}\int_\Gamma \frac{1}{h}[[K(\vartheta_h^k)]]\ [[\varphi_h]]\mathrm{dS}_x$$

$$= \int_\Omega \left(\mu|\mathbf{curl}^*[\mathbf{u}_h^k]|^2 + (\lambda+\mu)|\mathrm{div}_x\mathbf{u}_h^k|^2\right)\varphi_h \,\mathrm{d}x$$

$$-\int_\Omega \vartheta_h^k \partial_\vartheta p(\varrho_h^k, \vartheta_h^k)\mathrm{div}_x\mathbf{u}_h^k\varphi_h \,\mathrm{d}x$$

for all $\varphi_h \in Q_h(\Omega)$

Synergy analysis - numerics

- The numerical schemes converges to a weak solution of the problem
- Assume that the numerical schemes gives rise to a *bounded* family of solutions $\Rightarrow$ the limit (weak) solution is bounded $\Rightarrow$ the limit weak solution is smooth $\Rightarrow$ the limit weak solution is unique $\Rightarrow$ the numerical scheme converges unconditionally
- The limit solution being smooth, error estimates can be derived