Well posedness of some problems arising in fluid mechanics

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Oscillations in conservation laws

Nonlinear conservation law

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F}(\mathbf{v}) = 0$$

Linear field equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Nonlinear "constitutive" relation

$$\mathbb{F}(\mathbf{v}) = \mathbb{U}$$

Oscillations

$$\int_B \mathbf{v}_\varepsilon \to \int_B \mathbf{v} \text{ for all } B, \ \liminf_{\varepsilon \to 0} \int_B |\mathbf{v}_\varepsilon|^2 \boxed{\geq} \int_B |\mathbf{v}|^2$$

Convex integration

Replacing constitutive relation

$$G(\mathbf{v}, \mathbb{U}) \geq |\mathbf{v}|^2, \ G \text{ convex in } \mathbf{v}, \ \mathbb{U}$$

$$\mathbb{F}(\mathbf{v}) = \mathbb{U} \ \Leftrightarrow \ G(\mathbf{v}, \mathbb{U}) = |\mathbf{v}|^2$$

Subsolutions

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ |\mathbf{v}|^2 \leq G(\mathbf{v}, \mathbb{U}) \boxed{<} e, \ e = \text{“kinetic energy”}$$

Oscillatory lemma

$$\partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0, \ \mathbf{v} + \mathbf{w}_\varepsilon, \ \mathbb{U} + \mathbb{V}_\varepsilon \text{ subsolutions}$$

$$\mathbf{w}_\varepsilon \rightharpoonup 0$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{v} + \mathbf{w}_\varepsilon|^2 \geq \int_B |\mathbf{v}|^2 + \int_B \Lambda \left(e - |\mathbf{v}|^2 \right)$$

Convex integration - DeLellis and Shékelyhidi

Incompressible Euler system in R^3

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \right) + \nabla_x \Pi = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I},\ \mathbb{U} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Convex integration continued

Implicit constitutive relation

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]$$

$$\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2$$

$$\boxed{\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}|\mathbf{v}|^2\mathbb{I}$$

Convex integration - subsolutions

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0,\ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e \text{ in } (0, T)$$

Oscillatory lemma

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0,\ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon,\ \mathbb{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(V)$$

$$\lambda_{\max}\left[(\mathbf{v} + \mathbf{w}_\varepsilon) \otimes (\mathbf{v} + \mathbf{w}_\varepsilon) - (\mathbb{U} + \mathbb{V}_\varepsilon)\right] < e$$

Energy

$$\liminf_{\varepsilon \to 0} \int_V \left(|\mathbf{v} + \mathbf{w}_\varepsilon|^2 \right) \geq \int_V |\mathbf{v}|^2 + c \int_V \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^\alpha$$

Infinitely many solutions

Necessary condition

There exists e such that the set of subsolutions is non-empty

Infinitely many solutions

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) - \boxed{\nabla_x \left(\frac{1}{3} |\mathbf{v}|^2 \right)} = 0$$

Pressure

$$\frac{1}{2} |\mathbf{v}|^2 = e,\ p = -\frac{1}{3} |\mathbf{v}|^2 = -\frac{2}{3} e \text{ in } (0, T)$$

Typical results

Arbitrary (smooth) data

Infinitely many solutions for “any” data; solutions produce energy

Physically admissible solutions

Infinitely many dissipative solutions for “wild” data; existence for special data

Non-constant coefficients

Convex set

$$\frac{1}{2}\frac{1}{\boxed{r(t,x)}}|\mathbf{v}+\boxed{\mathbf{q}(t,x)}|^2$$

$$\leq \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{q}(t,x))\otimes(\mathbf{v}+\mathbf{q}(t,x))}{r(t,x)}+\boxed{\mathbb{W}(t,x)}-\mathbb{U}\right] < e$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{q})\otimes(\mathbf{v}+\mathbf{q})}{r} - \frac{1}{3}\frac{|\mathbf{v}+\mathbf{q}|^2}{r} + \mathbb{W}\right) = 0, \mathbb{W} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{q}(t,x)|^2}{r(t,x)} = \boxed{e(t,x)}$$

Example I: Compressible Euler system

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi,\ \operatorname{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \ \Leftrightarrow\ \partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

$$\Leftrightarrow$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x(\partial_t \Psi + p(\varrho)) = 0$$

Energy

$$e = \chi(t) - \frac{3}{2} p(\varrho) - \frac{3}{2} \partial_t \Psi$$

Example II: Euler-Poisson system

Equation of continuity

$$\partial_t n + \mathrm{div}_x \mathbf{J} = 0$$

Momentum equation

$$\partial_t \mathbf{J} + \mathrm{div}_x \left(\frac{\mathbf{J} \times \mathbf{J}}{n} \right) + \nabla_x p(n) = \pm n \mathbf{V}$$

Poisson equation

$$\mathbf{V} = \nabla_x \Phi, \;\; -\Delta \Phi = n - 1$$

Euler-Poisson system - reformulation

Ansatz

$$\mathbf{J} = \mathbf{v} + \nabla_x \Psi, \ \operatorname{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t n + \Delta \Psi = 0, \ -\Delta \Phi = n - 1$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{n} \pm (\nabla_x \Phi \otimes \nabla_x \Phi) \right]$$

$$+ \nabla_x \left[\partial_t \Psi + p(n) \mp \left(\Phi + \frac{1}{2} |\nabla_x \Phi|^2 \right) \right] = 0$$

Energy

$$e = \chi(t) - \frac{3}{2} \left[p(n) + \partial_t \Psi \mp \left(\Phi + \frac{1}{2} |\nabla_x \Phi|^2 \right) \right]$$

Example III: Quantum hydrodynamics

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{J} = 0$$

Momentum equation

$$\partial_t \mathbf{J} + \mathrm{div}_x \left(\frac{\mathbf{J} \times \mathbf{J}}{\varrho} \right) + \nabla_x p(\varrho) = \frac{h^2}{2} \varrho \nabla_x \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) \pm n \mathbf{V}$$

Poisson equation

$$\mathbf{V} = \nabla_x \Phi, \ -\Delta \Phi = \varrho - c(x)$$

Ansatz

Equation of continuity

$$\mathbf{J} = \mathbf{v} + \nabla_x \Psi,\ \mathrm{div}_x \mathbf{v} = 0,\ \partial_t \varrho + \Delta \Psi = 0$$

"Radiative" term

$$\varrho \nabla_x \left(\frac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right) = -2 \mathrm{div}_x \left[\nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} - \frac{1}{3} |\nabla_x \sqrt{\varrho}|^2 \mathbb{I} \right]$$

$$+ \nabla_x \left[\frac{1}{2} \Delta \varrho - \frac{2}{3} |\nabla_x \sqrt{\varrho}|^2 \right]$$