Stability issues in the theory of complete fluid systems

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Navier-Stokes-Fourier system

Equation of contintuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\ \ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Momentum equation

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}),\ \ \mathbf{u}|_{\partial\Omega} = 0$$

Second law, entropy production

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\right) = \sigma,\ \ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

First law, total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)\ \mathrm{d}x = 0$$

Constitutive relations

Gibbs' law, thermodynamics stability

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right), \quad \frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Viscosity, Newton's law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}$$

Heat conductivity, Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta$$

Second law, entropy production

$$\sigma \boxed{\geq} \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)\cdot \nabla_x \vartheta}{\vartheta}\right)$$

Relative entropy (energy) functional, structure

Relative entropy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \ \Big| \ r, \Theta, \mathbf{U}\right), \ (\varrho, \Theta, \mathbf{U}) \text{ “test functions”}$$

Distance property

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \ \Big| \ r, \Theta, \mathbf{U}\right) \geq 0$$

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \ \Big| \ r, \Theta, \mathbf{U}\right) = 0 \ \Leftrightarrow \ \varrho = r, \ \vartheta = \Theta, \ \mathbf{u} = \mathbf{U}$$

Relative entropy (energy) functional, dynamics

Lyapunov function

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \Big|\ \overline{\varrho},\overline{\vartheta},\overline{\mathbf{u}}\right)\leq 0 \text{ for any equilibrium } \overline{\varrho},\overline{\vartheta},\overline{\mathbf{u}}$$

Gronwall inequality

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \Big|\ \tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}\right)\Big|_{\tau_1}^{\tau_2}\leq\int_{\tau_1}^{\tau_2} h(t)\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \Big|\ \tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}\right)\ \mathrm{d}t$$

$$h\in L^1(0,T)$$

for any smooth solution $\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}$ smooth solution

Relative energy for NSF system

Ballistic free energy [Ericksen]

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$$

Relative NSF energy

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\,\Big|\,r,\Theta,\mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta) \right) \, \mathrm{d}x$$

Relative entropy vs. relative energy

Dafermos [1979] - relative entropy for the full Euler system
factor $\frac{1}{\Theta}$

General RE inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \middle|\ r,\Theta,\mathbf{U}\right)\Big|_{t=\tau_1}^{t=\tau_2} \leq \int_{\tau_1}^{\tau_2} \mathcal{R}(t,\varrho,\vartheta,\mathbf{u},r,\Theta,\mathbf{U})\ \mathrm{d}t$$

Weak solutions

$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho\vartheta)\ \mathrm{d}x$ .... total energy

$\int_\Omega \varrho s(\varrho,\vartheta)\Theta\ \mathrm{d}x$ .... entropy inequality

$\int_\Omega \varrho\mathbf{u}\cdot\mathbf{U}\ \mathrm{d}x$ .... momentum equation

$\int_\Omega \varrho G(r,\Theta,\mathbf{U})\ \mathrm{d}x$ .... equation of continuity

Applications

Weak-strong uniqueness [with A.Novotný ARMA 2012]

Weak and strong solutions to the Navier-Stokes-Fourier system emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions

Conditional regularity [with A.Novotný, Y. Sun ARMA 2014]

A weak solution emanating from smooth initial data is smooth as soon as

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} < \infty$$

Inviscid incompressible limit

Scaled Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho, \vartheta)$$

$$= \boxed{\varepsilon^\alpha} \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \nabla_x F$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^\beta} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+\alpha}} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{\varepsilon^\beta} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Limit system

Path hypotheses [with A.Novotný CMP 2013]

$$0 < \alpha < \frac{10}{3}, \ \beta > 0$$

Euler-Boussinesq system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = -a(\overline{\varrho}, \overline{\vartheta}) \theta \nabla_x F$$

$$c_p(\overline{\varrho}, \overline{\vartheta}) \left(\partial_t \theta + \mathbf{v} \cdot \nabla_x \theta \right) - \overline{\vartheta} a(\overline{\varrho}, \overline{\vartheta}) \mathbf{v} \cdot \nabla_x F = 0$$

Rate of convergence in singular limits

Scaled compressible Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho) = \boxed{\nu} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Large physical space

$$\Omega_M = M\Omega, \ \mathbf{u}|_{\partial \Omega_M} = 0$$

Target system

Path hypotheses

$$\varepsilon \to 0,\ \nu \to 0,\ \varepsilon M(\varepsilon) \to \infty$$

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0 \text{ in } R^3$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Acoustic waves

$$\varepsilon \partial_t Z + \Delta \Phi = 0 \text{ in } R^3$$

$$\varepsilon \partial_t \Phi + p'(\overline{\varrho}) Z = 0 \text{ in } R^3$$

$$Z(0, \cdot) = Z_0,\ \Phi(0, \cdot) = \Phi_0$$

Convergence

Ill prepared initial data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon r_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

Convergence [with Š.Nečasová, Y.Sun 2014]

$$\left\| \sqrt{\varrho}\Big(\mathbf{u} - \mathbf{v} - \nabla_x \Phi\Big)(\tau,\cdot)\right\|_{L^2(\Omega_M;R^3)} + \left\| \left(\frac{\varrho - \overline{\varrho}}{\varepsilon} - Z\right)(\tau,\cdot)\right\|_{L^2 + L^\gamma(\Omega_M;R^3)}$$

$$\leq C \left[\|\mathbf{u}_{0,\varepsilon} - \mathbf{v}_0 - \nabla_x \Phi_0\|_{L^2(\Omega_M;R^3)} + \|r_{0,\varepsilon} - Z_0\|_{L^2(\Omega_M)} \right]$$

$$+ C(\alpha)\left(\nu + \varepsilon^\alpha + \frac{1}{\varepsilon M(\varepsilon)}\right)^{1/2},\ 0 < \alpha < 1$$

Weak solutions of target systems?

Euler-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

$$\frac{3}{2}\left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\right] - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Existence of weak solutions [with E.Chiodaroli, O.Kreml AIHP 2014]

The Euler-Fourier system admits infinitely many global-in-time weak solutions for any smooth initial data ϱ_0, ϑ_0, $\mathbf{u}_0$